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Convolution Conditions for (p, q)-Convexity, (p, q)-Starlikeness and (p, q)-Spirallikeness

J. Mamatha, K. Renuka Devi and M. Manjunatha

ABSTRACT: In the present investigation, we derive the necessary and sufficient conditions for (p, q)-starlike, (p, q)-convex functions of order μ and (p, q)-spirallike, convex (p, q)-spirallike functions using convolution.

Keywords: (p, q)-differential operators, (p, q)-starlike functions, (p, q)-convex functions, (p, q)-spirallike, Hadamard product.

Contents

1. Introduction

Convolution is a powerful tool which forges links between the fundamental properties of operators and beautiful results from the classical theory of analytic functions. The Pólya-Schoenberg conjecture [3], generated a great deal of intrinsic interest in properties of convolutions. Silverman, Silvia and Telage's method [6], described the construction of a function g_φ such that $Re\{\varphi(z)\} > 0$ if and only if $\frac{1}{z}(f * g_\varphi) \neq 0$. Shamsan and Lata [2] introduced a characterization for q-analogue classes of convex, starlike and spirallike functions. Motivated by an emerging idea of (p, q))-analysis as a generalization of q-analysis, in this paper, we extend the idea of q-starlikeness, q-convexity and q-spirallikeness to (p, q)-starlikeness, (p, q)-convexity and (p, q)-spirallikeness, then we will give characterizations for (p, q)-analogue classes of convex, starlike and spirallike functions in terms of convolution. Let $\mathcal{A}$ denote the class of functions of form

$$h(t) = t + \sum_{k=2}^{\infty} a_k t^k, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{t : t \in \mathbb{C} \ \ and \ \ |t| < 1\},$$

and satisfy the normalization conditions $h(0) = h'(0) - 1 = 0$ for every $t \in \mathcal{U}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$. We recall some basic notations and definitions from (p, q)-calculus, which are used in this paper.

In [1], the authors introduced the (p, q)-derivative of the function h as

$$D_{p,q}h(t) = \frac{h(pt) - h(qt)}{(p - q)t}, \quad 0 < q < p \leq 1, \quad t \neq 0, \tag{1.2}$$

and $(D_{p,q}h)(0) = h'(0)$, provided that h is differentiable at 0. Therefore,

$$D_{p,q}h(t) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k t^{k-1},$$

where $\quad [k]_{p,q} = \dfrac{p^k - q^k}{p - q}.$

Also, note that for $p = 1$, the (p, q)-derivative reduces to the q-derivative. The linear (p, q)-derivative operator satisfies the following properties:

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i). $D_{p,q}(ah(t) \pm bg(t)) = aD_{p,q}h(t) \pm bD_{p,q}g(t)$, where a and b any real (or complex) constants

ii). $D_{p,q}(h(t)g(t)) = g(pt)D_{p,q}h(t) + h(qt)D_{p,q}g(t) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}h(t)$

iii). $D_{p,q}\left(\frac{h(t)}{g(t)}\right) = \frac{g(pt)D_{p,q}h(t)-h(pt)D_{p,q}g(t)}{g(pt)g(qt)} = \frac{g(qt)D_{p,q}h(t)-h(qt)D_{p,q}g(t)}{g(pt)g(qt)}$. Also, the (p,q)-integral of the function h on [0,z] is defined as

$$\int_0^z h(t)d_{p,q}t = z(p-q)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}h(\frac{q^k}{p^{k+1}}z).$$

Definition 1.1 *A function $h(t) \in \mathcal{A}$ is said to be (p,q)-starlike of order μ, $0 \leq \mu < 1$, if and only if*

$$\Re\left\{\frac{tD_{p,q}h(t)}{h(t)}\right\} > \mu, \qquad for \quad all \quad t \in \mathcal{U}.$$

We denote by $S^*_{p,q}(\mu)$ the subclass of $\mathcal{A}$ consisting of all (p,q)-starlike functions of order μ in the unit disk $\mathcal{U}$.

Definition 1.2 *A function $h(t) \in \mathcal{A}$ is said to be (p,q)-convex of order μ, $0 \leq \mu < 1$, if and only if*

$$\Re\left\{\frac{D_{p,q}\left(tD_{p,q}h(t)\right)}{D_{p,q}h(t)}\right\} > \mu, \qquad for \quad all \quad t \in \mathcal{U}.$$

We denote by $C_{p,q}(\mu)$ the subclass of $\mathcal{A}$ consisting of all (p,q)-convex functions of order μ in the unit disk $\mathcal{U}$.

Definition 1.3 *The convolution, of two analytic functions*

$$h(t) = t + \sum_{k=2}^{\infty} a_k t^k (|t| < R_1) \ \ and \ g(t) = t + \sum_{k=2}^{\infty} b_k t^k (|t| < R_2),$$

is defined as the power series

$$(h * g)(t) = t + \sum_{k=2}^{\infty} a_k b_k t^k, \qquad |t| < R_1 R_2.$$

It can be easily seen that

$$tD_{p,q}h * g = h * tD_{p,q}g. \tag{1.3}$$

2. Main Results

Theorem 2.1 *The function $h \in C_{p,q}(\mu)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{t}\left[h * \frac{pt + \left(\frac{[2]_{p,q}(\lambda+2\mu-1)q}{[(1-2\mu)p+1]+(1-p)\lambda} + (p+q-1)\right)t^2 + \frac{(p+q-[2]_{p,q})(\lambda+2\mu-1)}{[(1-2\mu)p+1]+(1-p)\lambda}pqt^3}{(1-pt)(1-qt)(p-q^2t)}\right] \neq 0$$

Proof: The function $h \in C_{p,q}(\mu)$ if and only if

$$\Re\left\{\frac{D_{p,q}\left(tD_{p,q}h(t)\right)}{D_{p,q}h(t)}\right\} > \mu, \qquad for \quad all \quad t \in \mathcal{U}. \tag{2.1}$$

Since $\frac{D_{p,q}(tD_{p,q}h)}{D_{p,q}h} = 1$ at $t = 0$, so (2.4) is equivalent to

$$\frac{\frac{D_{p,q}(tD_{p,q}h)}{D_{p,q}h} - \mu}{1-\mu} \neq \frac{\lambda - 1}{\lambda + 1}, \quad (|t| < R, \ |\lambda| = 1, \ \lambda \neq -1)$$

which implies

$$(1+\lambda)D_{p,q}\left(tD_{p,q}h\right)+(1-2\mu-\lambda)D_{p,q}h\neq 0. \tag{2.2}$$

Setting $h(t)=t+\sum_{k=2}^{\infty}a_kt^k$, we have

$$D_{p,q}h=1+\sum_{k=2}^{\infty}[k]_{p,q}a_kt^{k-1}$$

$$\begin{aligned}D_{p,q}\left(tD_{p,q}h\right)&=1+\sum_{k=2}^{\infty}[k]_{p,q}^2a_kt^{k-1}\\&=\left[1+\sum_{k=2}^{\infty}[k]_{p,q}a_kt^{k-1}\right]*\left[\sum_{k=1}^{\infty}[k]_{p,q}t^{k-1}\right]\\&=D_{p,q}h*\left[\sum_{k=1}^{\infty}[k]_{p,q}t^{k-1}\right]\\&=D_{p,q}h*\frac{1}{(1-t)(p-qt)}.\end{aligned}$$

The left hand side of (2.2) is equivalent to

$$\begin{aligned}(1+\lambda)&\left[D_{p,q}h*\sum_{k=1}^{\infty}[k]_{p,q}t^{k-1}\right]+D_{p,q}h*\sum_{k=1}^{\infty}(1-2\mu-\lambda)t^{k-1}\\&=D_{p,q}h*\sum_{k=1}^{\infty}\left[(1-2\mu-\lambda)+(1+\lambda)[k]_{p,q]}\right]t^{k-1}\\&=D_{p,q}h*\left(\frac{1-2\mu-\lambda}{1-t}+\frac{1+\lambda}{(1-t)(p-qt)}\right)\\&=D_{p,q}h*\left(\frac{[(1-2\mu)p+1]+(1-p)\lambda+(\lambda+2\mu-1)qt}{(1-t)(p-qt)}\right).\end{aligned}$$

Thus

$$\frac{1}{t}\left[tD_{p,q}h*\frac{t+\frac{\lambda+2\mu-1}{[(1-2\mu)p+1]+(1-p)\lambda}qt^2}{(1-t)(p-qt)}\right]\neq 0. \tag{2.3}$$

By using (1.3), we can write (2.3) as

$$\frac{1}{t}\left[h*\frac{pt+\left(\frac{[2]_{p,q}(\lambda+2\mu-1)q}{[(1-2\mu)p+1]+(1-p)\lambda}+(p+q-1)\right)t^2+\frac{(p+q-[2]_{p,q})(\lambda+2\mu-1)}{[(1-2\mu)p+1]+(1-p)\lambda}pqt^3}{(1-pt)(1-qt)(p-q^2t)}\right]\neq 0$$

which completes the proof. □

For $p=1$, we have following result proved by Hamid and Latha [2].

Corollary 2.1 *The function $f\in C_q(\mu)$ in $|t|<R\leq 1$ if and only if*

$$\frac{1}{t}\left[h*\frac{t+\left(\frac{[2]_q(\lambda+2\mu-1)}{2-2\mu}+1\right)qt^2+\frac{(1+q-[2]_q)(\lambda+2\mu-1)}{2-2\mu}qt^3}{(1-t)(1-qt)(1-q^2t)}\right]\neq 0$$

As $q\to 1^-$ and $p=1$, we have following result proved by Silverman and et al. [6].

Corollary 2.2 *The function $h \in C(\mu)$ in $|t| < R \leq 1$ if and only if*

$$\frac{1}{t}\left[h * \frac{t + \frac{\lambda+\mu}{1-\mu}t^2}{(1-t)^3}\right] \neq 0.$$

As $q \to 1^-$, $p = 1$ and $\mu = 0$, we have following result proved by Ruschewyh [5].

Corollary 2.3 *The function $f \in C$ in $|t| < R \leq 1$ if and only if*

$$\frac{1}{t}\left[h * t\frac{1-\lambda t}{(1-t)^3}\right] \neq 0$$

Theorem 2.2 *The function $h \in S^*_{p,q}(\mu)$ in $|t| < R \leq 1$ if and only if*

$$\frac{1}{t}\left[h * \frac{t + \left(\frac{(\lambda+2\mu-1)}{[(1-2\mu)p+1]+(1-p)\lambda}\right)qt^2}{(1-t)(p-qt)}\right] \neq 0$$

Proof: Since h is (p,q)-starlike of order μ if and only if $g(t) = \int_0^t \frac{h(\zeta)}{\zeta} d_{p,q}\zeta$ is (p,q)-convex of order μ, we have

$$\frac{1}{t}\left[g * \frac{pt + \left(\frac{[2]_{p,q}(\lambda+2\mu-1)q}{[(1-2\mu)p+1]+(1-p)\lambda} + (p+q-1)\right)t^2 + \frac{(p+q-[2]_{p,q})(\lambda+2\mu-1)}{[(1-2\mu)p+1]+(1-p)\lambda}pqt^3}{(1-pt)(1-qt)(p-q^2t)}\right]$$

$$= \frac{1}{t}\left[h * \frac{t + \left(\frac{(\lambda+2\mu-1)}{[(1-2\mu)p+1]+(1-p)\lambda}\right)qt^2}{(1-t)(p-qt)}\right].$$

Thus the result follows from Theorem 2.1. □

For $p = 1$, we have following result proved by Hamid and Latha [2].

Corollary 2.4 *The function $h \in S^*_q(\mu)$ in $|t| < R \leq 1$ if and only if*

$$\frac{1}{t}\left[h * \frac{t + \frac{\lambda+2\mu-1}{2-2\mu}qt^2}{(1-t)(1-qt)}\right] \neq 0, \quad (|t| < R, \ |\lambda| = 1).$$

As $q \to 1^-$ and $p = 1$, we have following result proved by Silverman et al. [6].

Corollary 2.5 *The function $h \in S^*(\mu)$ in $|t| < R \leq 1$ if and only if*

$$\frac{1}{t}\left[g * \frac{t + \frac{\lambda+\mu}{1-\mu}t^2}{(1-t)^3}\right] = \frac{1}{t}\left[h * \frac{t + \frac{\lambda+2\mu-1}{2-2\mu}t^2}{(1-t)^2}\right].$$

As $q \to 1^-$, $p = 1$ and $\mu = \frac{1}{2}$, we have following result proved by Ruschewyh [5].

Corollary 2.6 *The function $f \in S^*(\frac{1}{2})$ in $|t| < R \leq 1$ if and only if*

$$\frac{1}{t}\left[g * \frac{t + (2\lambda+1)t^2}{(1-t)^3}\right] = \frac{1}{t}\left[h * \frac{t + \lambda t^2}{(1-t)^2}\right].$$

Now by using the concept of (p,q)-derivative we define the classes of (p,q)-spirallike and convex (p,q)-spirallike functions as the following:

Definition 2.1 A function $h \in \mathcal{A}$ is said to be (p,q)-spirallike if and only if

$$\Re\left\{e^{i\theta}\frac{tD_{p,q}h(t)}{h(t)}\right\} > 0, \quad |t| < R \leq 1, \quad \theta \in \mathbb{R} \text{ with } |\theta| < \frac{\pi}{2}.$$

Definition 2.2 A function $h \in \mathcal{A}$ is said to be convex (p,q)-spirallike if and only if

$$\Re\left\{e^{i\theta}\left(\frac{D_{p,q}\left(tD_{p,q}h(t)\right)}{D_{p,q}h(t)}\right)\right\} > 0, \quad |t| < R \le 1, \quad \theta \in \mathbb{R} \text{ with } |\theta| < \frac{\pi}{2}.$$

Theorem 2.3 *For $|t| < R \le 1, \quad \theta \in \mathbb{R}$ with $|\theta| < \frac{\pi}{2}$ and $|\lambda| = 1$, we have*

$$\Re\left\{e^{i\theta}\left(\frac{D_{p,q}\left(tD_{p,q}h(t)\right)}{D_{p,q}h(t)}\right)\right\} > 0$$

if and only if

$$\frac{1}{t}\left[h * \frac{pt + \left(\frac{[2]_{p,q}(\lambda - e^{-2i\theta})q}{[1+(1-p)\lambda + e^{-2i\theta}]} + (p+q-1)\right)t^2 + \frac{(p+q-[2]_{p,q})(\lambda - e^{-2i\theta})}{[1+(1-p)\lambda+e^{-2i\theta}]}pqt^3}{(1-pt)(1-qt)(p-q^2t)}\right] \neq 0.$$

Proof: We have, $\Re\left\{e^{i\theta}\left(\frac{D_{p,q}(tD_{p,q}h(t))}{D_{p,q}h(t)}\right)\right\} > 0$ if and only if

$$\frac{e^{i\theta}\frac{D_{p,q}(tD_{p,q}h)}{D_{p,q}h} - i\sin\theta}{\cos\theta} \neq \frac{\lambda - 1}{\lambda + 1}, \quad (|t| < R, \ |\lambda| = 1, \ \lambda \neq -1)$$

which implies

$$(1+\lambda)D_{p,q}\left(tD_{p,q}h\right) + (e^{-2i\theta} - \lambda)D_{p,q}h \neq 0. \tag{2.4}$$

Setting $h(t) = t + \sum_{k=2}^{\infty} a_k t^k$, we have

$$D_{p,q}h = 1 + \sum_{k=2}^{\infty}[k]_{p,q}a_k t^{k-1}$$

$$D_{p,q}\left(tD_{p,q}h\right) = 1 + \sum_{k=2}^{\infty}[k]_{p,q}^2 a_k t^{k-1} = D_{p,q}h * \frac{1}{(1-t)(p-qt)}.$$

The left hand side of (2.4) is equivalent to

$$\begin{aligned}(1+\lambda)&\left[D_{p,q}h * \sum_{k=1}^{\infty}[k]_{p,q}t^{k-1}\right] + D_{p,q}h * \sum_{k=1}^{\infty}(e^{-2i\theta} - \lambda)t^{k-1} \\ &= D_{p,q}h * \sum_{k=1}^{\infty}\left[(e^{-2i\theta} - \lambda) + (1+\lambda)[k]_{p,q}\right]t^{k-1} \\ &= D_{p,q}h * \left(\frac{e^{-2i\theta} - \lambda}{1-t} + \frac{1+\lambda}{(1-t)(p-qt)}\right) \\ &= D_{p,q}h * \left(\frac{(1+(1-p)\lambda + pe^{-2i\theta}) + (\lambda - e^{-2i\theta})qt}{(1-t)(p-qt)}\right).\end{aligned}$$

Thus

$$\frac{1}{t}\left[tD_{p,q}h * \frac{t + \frac{\lambda - e^{-2i\theta}}{1+(1-p)\lambda + pe^{-2i\theta}}qt^2}{(1-t)(p-qt)}\right] \neq 0. \tag{2.5}$$

By using (1.3), we can write (2.5) as

$$\frac{1}{t}\left[h * \frac{pt + \left(\frac{[2]_{p,q}(\lambda - e^{-2i\theta})q}{[1+(1-p)\lambda + e^{-2i\theta}]} + (p+q-1)\right)t^2 + \frac{(p+q-[2]_{p,q})(\lambda - e^{-2i\theta})}{[1+(1-p)\lambda+e^{-2i\theta}]}pqt^3}{(1-pt)(1-qt)(p-q^2t)}\right] \neq 0,$$

which completes the proof. □

For $p = 1$, we have following result proved by Hamid and Latha [2].

Corollary 2.7 *For* $|t| < R \leq 1$, $\theta \in \mathbb{R}$ *with* $|\theta| < \frac{\pi}{2}$ *and* $|\lambda| = 1$, *we have*

$$\Re\left\{e^{i\theta}\left(\frac{D_q\left(tD_qh(t)\right)}{D_qh(t)}\right)\right\} > 0$$

if and only if

$$\frac{1}{t}\left[h * \frac{t + \left(\frac{[2]_q(\lambda - e^{-2i\theta})}{1+e^{-2i\theta}} + 1\right)qt^2 + \frac{(1-q)(\lambda - e^{-2i\theta})}{1+e^{-2i\theta}}qt^3}{(1-pt)(1-qt)(p-q^2t)}\right] \neq 0$$

As $q \to 1^-$, $p = 1$, we have following result proved by Silverman and et al. [6].

Corollary 2.8 *For* $|t| < R \leq 1$, $\theta \in \mathbb{R}$ *with* $|\theta| < \frac{\pi}{2}$ *and* $|\lambda| = 1$, *we have*

$$\Re\left\{e^{i\theta}\left(\frac{(th'(t))'}{h'(t)}\right)\right\} > 0$$

if and only if

$$\frac{1}{t}\left[h * \frac{t + \frac{2\lambda+1-e^{-2i\theta}}{1+e^{-2i\theta}}t^2}{(1-t)^3}\right] \neq 0.$$

Theorem 2.4 *For* $|t| < R \leq 1$, $\theta \in \mathbb{R}$ *with* $|\theta| < \frac{\pi}{2}$ *and* $|\lambda| = 1$, *we have*

$$\Re\left\{e^{i\theta}\frac{tD_{p,q}h(t)}{h(t)}\right\} > 0$$

if and only if

$$\frac{1}{t}\left[h * \frac{t + \frac{\lambda - e^{-2i\theta}}{1+(1-p)\lambda + e^{-2i\theta}}qt^2}{(1-t)(p-qt)}\right] \neq 0.$$

Proof: The result follows from Theorem 2.3 in the same manner that Theorem 2.2 followed from Theorem 2.1. □

For $p = 1$, we have following result proved by Hamid and Latha [2].

Corollary 2.9 *For* $|t| < R \leq 1$, $\theta \in \mathbb{R}$ *with* $|\theta| < \frac{\pi}{2}$ *and* $|\lambda| = 1$, *we have*

$$\Re\left\{e^{i\theta}\frac{tD_qh(t))}{h(t)}\right\} > 0$$

if and only if

$$\frac{1}{t}\left[h * \frac{t + \frac{\lambda - e^{-2i\theta}}{1+e^{-2i\theta}}qt^2}{(1-t)(1-qt)}\right] \neq 0.$$

As $q \to 1^-$ and $p = 1$, we have following result proved by Silverman et al. [6].

Corollary 2.10 *For* $|t| < R \leq 1$, $\theta \in \mathbb{R}$ *with* $|\theta| < \frac{\pi}{2}$ *and* $|\lambda| = 1$, *we have*

$$\Re\left\{e^{i\theta}\frac{th'(t))}{h(t)}\right\} > 0$$

if and only if

$$\frac{1}{t}\left[h * \frac{t + \frac{\lambda - e^{-2i\theta}}{1+e^{-2i\theta}}t^2}{(1-t)^2}\right] \neq 0.$$

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J. Mamatha,
Department of Mathematics,
Government Women's College,
Hunsur-571 105, India.
E-mail address: mamatha_j910@yahoo.in

and

K. Renuka Devi (Corresponding author),
Department of Mathematics,
Government First Grade College for Women,
Krishnarajanagara-571 602, India.
E-mail address: renukk84@gmail.com; drrenukadevik3@gmail.com

and

M. Manjunatha,
Department of Mathematics,
PES College of Science, Arts and Commerce,
Mandya-571 401, India.
E-mail address: mmanjunathapes@gmail.com