Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 10** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.79883

Certain Properties on Appell-Type Changhee Polynomials Associated with Hermite Polynomials

Haitham Qawaqneh and Waseem Ahmad Khan

ABSTRACT: In this paper, we introduce a new class of Hermite-based Appell-type Changhee polynomials and generalized Gould-Hopper-Appell-type Changhee polynomials and present some properties and identities of these polynomials. The resulting formulas allow a considerable unification because the given definition uncovers and brings into focus patterns and properties shared by different classes of special functions and number theory. We obtain new representations, sums and identities which arise from various forms of generating functions.

Keywords: Hermite polynomials, Hermite-based appell type changhee polynomials, summation formulae, symmetric identities.

Contents

1. Introduction

The Hermite polynomials play an essential position within extension of the classical, unique features. Starting from the Hermite polynomials. It has already been possible to attain a few extensions of a few classically unique sets of capabilities, which include Bessel capabilities, Dickson polynomials, Laguerre polynomials, Chebyshev polynomials (see [1-3, 9-11, 25-38]).

The Hermite (or 2 variable Kampé de Fériet) polynomials is given by (see [1, 2])

$$H_j(\xi,\eta) = j! \sum_{r=0}^{[\frac{j}{2}]} \frac{\eta^r \xi^{j-2r}}{r!(j-2r)!}. \tag{1.1}$$

These polynomials are usually defined by the generating function

$$e^{\xi z+\eta z^2} = \sum_{j=0}^{\infty} H_j(\xi,\eta)\frac{z^j}{j!}, \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_j(\xi)$ (see [1]) when $\eta=-1$ and ξ is replaced by 2ξ.

Jedda *et al.* [10] introduced a class of two-index real Hermite polynomials of degree $p+j$ by

$$h_{p,j}(\xi) = (-\frac{d}{d\xi}+2\xi)^p(\xi)^j = p!j! \sum_{k=0}^{min(p,j)} \frac{(-1)^k \xi^{p-k} H_{p-k}(\xi)}{k!(j-k)!(p-k)}. \tag{1.3}$$

2020 *Mathematics Subject Classification:* 05A30, 05A40, 11B68, 11S80, 33C45.
Submitted November 03, 2025. Published April 17, 2026

Note that $h_{p,0}(\xi)=H_p\xi(\xi),\ h_{0,j\xi}(\xi)=\xi^j$ and $h_{p,j}(0)=0$, if $p<j$.

The generating function of $h_{p,j}$ is given by

$$\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}h_{p,j}(\xi)\frac{u^p v^j}{p!j!}=e^{-u^2+(2u+v)\xi-uv}. \tag{1.4}$$

Furthermore, for $\eta=u=-v$, Jedda *et al.* [9] proved that

$$e^{\xi\eta}=\sum_{p,j=0}^{\infty}(-1)^j h_{p,j}(\xi)\frac{\eta^{p+j}}{p!j!}. \tag{1.5}$$

The generating function of Gould-Hopper polynomials $G_m^{(q)}(w|\gamma)$ introduced by Dattoli *et al.* ([3, p. 72]) is given by

$$e^{wv+\gamma v^q}=\sum_{m=0}^{\infty}G_m^{(q)}(w|\gamma)\frac{v^m}{m!}, \tag{1.6}$$

so that for every complex numbers u,v,z and w, we have (see [7],pages (5) and (6)):

$$\sum_{j=0}^{\infty}H_{j,m}^{(p,q)}(z,w|\gamma)\frac{u^n}{n!}=G_m^{(q)}(w|u^p\gamma)e^{zu}, \tag{1.7}$$

and

$$\sum_{m=0}^{\infty}H_{j,m}^{(p,q)}(z,w|\gamma)\frac{v^m}{m!}=G_j^{(q)}(w|v^p\gamma)e^{zv}, \tag{1.8}$$

where the polynomials $H_{j,m}^{(p,q)}(z,w|\gamma)$ contain all the classes given above. Moreover, they give rise to new classes of polynomials of Hermite type. The concrete study of this polynomial is presented in [7] in a unified way and includes the connection to Gould-Hopper polynomials [8], operational representations and connection to hypergeometric function, generating functions, addition formulas of Runge type, multiplication formulas, Nielson formulas and higher order differential equation they obey.

The falling factorial sequence is defined by

$$(\xi)_0=1,\quad (\xi)_j=\xi(\xi-1)\cdots(\xi-j+1),(j\geq 1). \tag{1.9}$$

The first kind of Stirling numbers are defined by

$$(\xi)_j=\sum_{k=0}^{j}S_1(j,k)\xi^k,(j\geq 0),(\text{see [1-10]}). \tag{1.10}$$

and as an inversion formula of (1.10), the Stirling numbers of the second kind are given by (see [4-11, 19-24]):

$$\xi^j=\sum_{k=0}^{j}S_2(j,k)(\xi)_k. \tag{1.11}$$

From (1.9) and (1.10), we note that the generating function of Stirling numbers of the first kind and that of the second kind are respectively given by (see [1-18):

$$\frac{1}{k!}(\log(1+z))^k=\sum_{j=k}^{\infty}S_1(j,k)\frac{z^j}{j!}, \tag{1.12}$$

and

$$\frac{1}{k!}(e^z-1)^k=\sum_{j=k}^{\infty}S_2(j,k)\frac{z^j}{j!},(k\geq 0). \tag{1.13}$$

Let p be a fixed odd prime number. Throughout the article, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The normalized p-adic is given by $|\, p\,|_p=\frac{1}{p}$. For $f:\mathbb{Z}_p\to\mathbb{C}_p$ (f being a continuous function), the fermionic p-adic integral of f is defined by Kim (see [13]):

$$\int_{\mathbb{Z}_p}f(\xi)d\mu_{-1}(\xi)=\lim_{N\to\infty}\sum_{\xi=0}^{p^N-1}f(\xi)\mu_{-1}(\xi+p^N\mathbb{Z}_p)=\lim_{N\to\infty}\frac{1}{p^N}\sum_{\xi=0}^{p^N-1}f(\xi)(-1)^{\xi}. \tag{1.14}$$

It is apparent from (1.14) that

$$\int_{\mathbb{Z}_p}f(\xi+1)d\mu_{-1}(\xi)+\int_{\mathbb{Z}_p}f(\xi)d\mu_{-1}(\xi)=2f(0), \quad \text{(see [14, 15, 16])}. \tag{1.15}$$

The Changhee polynomials are defined by (see [15])

$$\int_{\mathbb{Z}_p}(1+z)^{\xi+\eta}d\mu_{-1}(\eta)=\frac{2}{2+z}(1+z)^{\xi}=\sum_{j=0}^{\infty}Ch_j(\xi)\frac{z^j}{j!}. \tag{1.16}$$

At the point $\xi=0$, $Ch_j=Ch_{j\xi}(0)$ are called the Changhee numbers.

The Euler polynomials are represented by the p-adic q-integral on $\mathbb{Z}_p$ as follows (see [13])

$$\int_{\mathbb{Z}_p}e^{(\xi+\eta)}d\mu_{-1}(\eta)=\frac{2}{e^z+1}e^{\xi z}=\sum_{j=0}^{\infty}E_j(\xi)\frac{z^j}{j!}. \tag{1.17}$$

Lim and Qi [17] introduced the Appell-type λ-Changhee polynomials which can be represented by the p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p}e^{\ln(1+\lambda z)^{\frac{\eta}{\lambda}}+\xi z}d\mu_{-1}(\eta)=\frac{2}{(1+\lambda z)^{\frac{1}{\lambda}}+1}e^{\xi z}=\sum_{j=0}^{\infty}Ch_j(\xi|\lambda)\frac{t^j}{j!}. \tag{1.18}$$

At the point $\xi=0$, $Ch_j(\lambda)=Ch_j(0|\lambda))$ are called the λ-Changhee numbers. Note that $Ch_j(1)=Ch_j$ for $j\geq 0$.

They have proved that

$$Ch_j^{(r)}(\xi|\lambda)=\sum_{l=0}^{j}\binom{j}{l}Ch_l^{(r)}(\lambda)\xi^{j-l}.$$

Recently, Lee *et al.* [18] introduced the Appell-type Changhee polynomials are defined by

$$\frac{2}{2+z}e^{\xi z}=\sum_{j=0}^{\infty}Ch_j^*(\xi)\frac{z^j}{j!}. \tag{1.19}$$

In the case $\xi=0$, $Ch_j^*=Ch_j^*(0)$ are the Appell-type Changhee numbers.

In this paper, we have presented the generalized Hermite-based Appell-type Changhee polynomials and discussed, in particular, some interesting series representations. We have deduced some relevant properties by using the structure and the relations satisfied by the recently generalized Hermite polynomials. Section 2 incorporates the definition of Hermite-based Appell-type Changhee polynomials and a preliminary study of these polynomials. Some theorems on implicit summation formulae for Hermite-based Appell-type Changhee polynomials ${}_HCh_j^{(*,r)}(\xi,\eta)$ and their special cases are given in Section 3. Section 4 is a consequence of the definition of the two-index real Hermite-Appell-type Changhee polynomials and generalized Gould-Hopper-Appell-type Changhee polynomials combined with their properties and special cases.

2. Hermite-Based Appell-Type Changhee Polynomials

This section incorporates the definition of Hermite-based Appell-type Changhee polynomials and a preliminary study of these polynomials.

Let $\xi,\eta\in\mathbb{R}$, we define Hermite-based Appell-type Changhee polynomials as

$$\frac{2}{2+z}e^{\xi z+\eta z^2}=\sum_{j=0}^{\infty}{}_HCh_j^*(\xi,\eta)\frac{z^j}{j!}. \tag{2.1}$$

When $\eta=0$ then ${}_HCh_j^*(\xi,0)=Ch_j^*(\xi)$ and Hermite-based Appell-type Changhee polynomials reduce to Appell-type Changhee polynomials. When $\xi=\eta=0$, we write $Ch_j^*={}_HCh_j^*(0,0)$, the Appell-type Changhee numbers for $j\geq 0$.

Using (1.2), we can extend (2.1) in the form

$$\left(\frac{2}{2+z}\right)^r e^{\xi z+\eta z^2}=\sum_{j=0}^{\infty}{}_HCh_j^{(*,r)}(\xi,\eta)\frac{z^j}{j!}. \tag{2.2}$$

Theorem 2.1. Let $j\geq 0$. Then

$${}_HCh_j^*(\xi,\eta|\lambda)=\sum_{l=0}^{j}\binom{j}{l}Ch_j^*H_{j-l}(\xi,\eta). \tag{2.3}$$

Proof: By using (1.2) and (2.1), we complete the proof. □

Theorem 2.2. Let $j\geq 0$. Then

$${}_HCh_j^{(*,r)}(\xi,\eta)=\sum_{l=0}^{[\frac{j}{2}]}\binom{j}{2l}Ch_l^{(*,r)}(\xi)\eta^l. \tag{2.4}$$

Proof: In (2.2), we expand $e^{\eta z^2}$ in series, use (1.2) and then compare the coefficients of t on both the sides to get the result. □

Theorem 2.3. Let $j\geq 0$. Then

$${}_HCh_j^{(*,r)}(\xi,\eta)=\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}(\xi)_lS_2(k,l){}_HCh_{j-k}^{(*,r)}(0,\eta). \tag{2.5}$$

Proof: From (2.2), we have

$$\sum_{j=0}^{\infty}{}_HCh_j^{(*,r)}(\xi,\eta)\frac{z^j}{j!}=\left(\frac{2}{2+z}\right)^r e^{\xi z+\eta z^2}$$

$$=\left(\frac{2}{2+z}\right)^r e^{\eta z^2}(e^z-1+1)^{\xi}$$

$$= \sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(0,\eta)\frac{z^j}{j!}\sum_{l=0}^{\infty}(\xi)_l\frac{1}{l!}(e^z-1)^l$$

$$= \sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(0,\eta)\frac{z^j}{j!}\sum_{l=0}^{\infty}(\xi)_l\sum_{k=l}^{\infty}S_2(k,l)\frac{z^k}{k!}$$

$$= \sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(0,\eta)\frac{z^j}{j!}\sum_{k=0}^{\infty}\sum_{l=0}^{k}(\xi)_lS_2(k,l)\frac{z^k}{k!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}(\xi)_lS_2(k,l){}_HCh_{j-k}^{(*,r)}(0,\eta)\right)\frac{z^j}{j!}. \tag{2.6}$$

In view of (2.2) and (2.6), we get (2.5). □

Theorem 2.4. Let $j \geq 0$. Then

$${}_HCh_j^{(*,r)}(\xi+\alpha,\eta) = \sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}(\xi)_lS_2(k+\alpha,l+\alpha){}_HCh_{j-k}^{(*,r)}(0,\eta). \tag{2.7}$$

Proof: On changing ξ by $\xi+\alpha$ in (2.2), we see that

$$\sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(\xi+\alpha,\eta)\frac{z^j}{j!} = \left(\frac{2}{2+z}\right)^r e^{\xi z+\eta z^2}e^{\alpha z}$$

$$= \left(\frac{2}{2+z}\right)^r e^{\eta z^2}e^{\alpha z}(e^z-1+1)^{\xi}$$

$$= \sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(0,\eta)\frac{z^j}{j!}e^{\alpha z}\sum_{l=0}^{\infty}(\xi)_l\frac{1}{l!}(e^z-1)^l$$

$$= \sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(0,\eta)\frac{z^j}{j!}e^{\alpha z}\sum_{l=0}^{\infty}(\xi)_l\sum_{k=l}^{\infty}S_2(k,l)\frac{z^k}{k!}$$

$$= \sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(0,\eta)\frac{z^j}{j!}\sum_{k=0}^{\infty}\sum_{l=0}^{k}(\xi)_lS_2(k+\alpha,l+\alpha)\frac{z^k}{k!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}(\xi)_lS_2(k+\alpha,l+\alpha){}_HCh_{j-k}^{(*,r)}(0,\eta)\right)\frac{z^j}{j!}. \tag{2.8}$$

Therefore, by (2.2) and (2.8), we obtain the result (2.7). □

Theorem 2.5. Let $j \geq 0$. Then

$${}_HCh_j^{(*,r)}(\xi,\eta) = \sum_{l=0}^{j}\binom{j}{l}{}_HCh_{j-l}^{(*,r-k)}(\xi,\eta){}_HCh_j^{(*,k)}(0,0). \tag{2.9}$$

Proof: We observe that

$$\sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(\xi,\eta)\frac{z^j}{j!} = \left(\frac{2}{2+z}\right)^r e^{\xi z+\eta z^2}$$

$$= \left(\frac{2}{2+z}\right)^{r-k}\left(\frac{2}{2+z}\right)^k e^{\xi z+\eta z^2}$$

$$=\sum_{j=0}^{\infty}{}_HCh_j^{(*,r-k)}(\xi,\eta)\frac{z^j}{j!}\sum_{l=0}^{\infty}{}_HCh_j^{(*,k)}(0,0)\frac{z^l}{l!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l}{}_HCh_{j-l}^{(*,r)}(\xi,\eta){}_HCh_j^{(*,k)}(0,0)\right)\frac{z^j}{j!}. \tag{2.10}$$

By (2.2) and (2.10), we get the required result. □

Theorem 2.6. Let $j \geq 0$. Then

$${}_HCh_j^{(*,r)}(\zeta,u)=\sum_{k=0}^{j}\binom{j}{k}H_k(\alpha-\xi+\zeta,\beta-\eta+u){}_HCh_{j-k}^{(*,r)}(\xi-\alpha,\eta-\beta). \tag{2.11}$$

Proof: By (2.2), we note that

$$\sum_{j=0}^{\infty}{}_HCh_j^{(*,r)}(\zeta,u)\frac{z^j}{j!}=\left(\frac{2}{2+z}\right)^r e^{(\xi-\alpha)z+(\eta-\beta)z^2}e^{-(\xi-\zeta-\alpha)z-(\eta-u-\beta)z^2}$$

$$=e^{-(\xi-\zeta-\alpha)z-(\eta-u-\beta)z^2}\sum_{j=0}^{\infty}{}_HCh_j^{(*,r)}(\xi-\alpha,\eta-\beta)\frac{z^j}{j!}$$

$$=\sum_{k=0}^{\infty}H_k(\alpha-\xi+\zeta,\beta-\eta+u)\frac{z^k}{k!}\sum_{j=0}^{\infty}{}_HCh_j^{(*,r)}(\xi-\alpha,\eta-\beta)\frac{z^j}{j!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}H_k(\alpha-\xi+\zeta,\beta-\eta+u){}_HCh_{j-k}^{(*,r)}(\xi-\alpha,\eta-\beta)\right)\frac{z^j}{j!}. \tag{2.12}$$

In view of (2.2) and (2.12), we arrive at the desired result. □

Remark 2.1. Letting $u=\zeta=0$ in Theorem 2.6, we get

Corollary 2.1. Let $j \geq 0$. Then

$$Ch_j^{(*,r)}=\sum_{k=0}^{j}\binom{j}{k}H_k(\alpha-\xi,\beta-\eta){}_HCh_{j-k}^{(*,r)}(\xi-\alpha,\eta-\beta). \tag{2.13}$$

Theorem 2.7. Let $j \geq 0$. Then

$$\sum_{q=0}^{j}\sum_{l=0}^{[\frac{j-q}{2}]}\left(\frac{\xi}{\eta^2}-\frac{\eta}{\xi^2}\right)^l\frac{{}_HCh_{j-2l-q}^{(*,k)}(\xi,\eta)Ch_q^{(*,k)}}{l!q!(j-q-2l)!\eta^q\xi^{j-q-2l}}$$

$$=\sum_{l=0}^{j}\frac{Ch_l^{(*,k)}{}_HCh_{j-l}^{(*,k)}(\eta,\xi)}{(j-l)!l!\xi^l\eta^{j-l}}. \tag{2.14}$$

Proof: On changing z with $\frac{z}{\xi}$ and r by k, we can write (2.2) as

$$\sum_{j=0}^{\infty}{}_HCh_j^{(*,k)}(\xi,\eta)\frac{z^j}{\xi^j j!}=\left(\frac{2}{2+\frac{z}{\xi}}\right)^k e^{z+\eta\frac{z^2}{\xi^2}}. \tag{2.15}$$

Now interchanging ξ by η, we have

$$\sum_{j=0}^{\infty} {}_{H}Ch_j^{(*,k)}(\eta,\xi)\frac{z^j}{\eta^j j!} = \left(\frac{2}{2+\frac{z}{\eta}}\right)^k e^{z+\xi\frac{z^2}{\eta^2}}. \tag{2.16}$$

Comparison of (2.15) and (2.16) yields

$$e^{\xi\frac{z^2}{\eta^2}-\eta\frac{z^2}{\xi^2}}\left(\frac{2}{2+\frac{z}{\eta}}\right)^k \sum_{j=0}^{\infty} {}_{H}Ch_j^{(*,k)}(\xi,\eta)\frac{z^j}{\xi^j j!}$$

$$= \left(\frac{2}{2+\frac{z}{\xi}}\right)^k \sum_{j=0}^{\infty} {}_{H}Ch_j^{(*,k)}(\eta,\xi)\frac{z^j}{\eta^j j!}$$

$$= \sum_{l=0}^{\infty}\frac{(\frac{\xi}{\eta^2}-\frac{\eta}{\xi^2})^l}{l!}z^{2l}\sum_{q=0}^{\infty}Ch_q^{(*,k)}\frac{z^q}{\eta^q q!}\sum_{j=0}^{\infty} {}_{H}Ch_j^{(*,k)}(\xi,\eta)\frac{z^j}{\xi^j j!}$$

$$= \sum_{l=0}^{\infty}Ch_l^{(*,k)}\frac{z^l}{\xi^l l!}\sum_{j=0}^{\infty} {}_{H}Ch_j^{(*,k)}(\eta,\xi)\frac{z^j}{\eta^j j!}$$

$$\sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\sum_{l=0}^{[\frac{j-q}{2}]}\left(\frac{\xi}{\eta^2}-\frac{\eta}{\xi^2}\right)^l\frac{{}_{H}Ch_{j-2l-q}^{(*,k)}(\xi,\eta)Ch_q^{(*,k)}}{l!q!(j-q-2l)!\eta^q\xi^{j-q-2l}}\right)z^j$$

$$= \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\frac{Ch_l^{(*,k)}\,{}_{H}Ch_{j-l}^{(*,k)}(\eta,\xi)}{(j-l)!l!\xi^l\eta^{j-l}}\right)z^j. \tag{2.17}$$

Therefore, by (2.2) and (2.17), we obtain the result (2.14). □

3. A Class of Two-Index Real Hermite-Based Appell Type Changhee Polynomials

This section is a consequence of the definition of the two-index real Hermite-based Appell-type Changhee polynomials and generalized Gould-Hopper Appell-type Changhee polynomials combined with their properties and special cases.

We define two-index real Hermite-based Appell-type Changhee polynomials by the generating function

$$\left(\frac{2}{2+z}\right)^r e^{-u^2z^2+(2u\zeta+v)\xi-uvz} = \sum_{j=0}^{\infty} {}_{h}Ch_j^{(*,r)}(\xi,u,v)\frac{z^j}{j!}. \tag{3.1}$$

For $v=0$, (3.1) reduces to

$$\left(\frac{2}{2+z}\right)^r e^{-u^2z^2+(2uz)\xi} = \sum_{j=0}^{\infty} {}_{h}Ch_j^{(*,r)}(\xi,u,0)\frac{z^j}{j!}$$

$$= \sum_{j=0}^{\infty} Ch_j^{(*,r)}(\xi)\frac{z^j}{j!}\sum_{m=0}^{\infty}H_m(\xi)\frac{u^m z^m}{m!}.$$

Replacing j by $j-m$ and comparing the coefficients of z^j, we get

$$ {}_{h}Ch_j^{(r)}(\xi,u,0) = \sum_{m=0}^{j}\binom{j}{m}u^m Ch_{j-m}^{(*,r)}(\xi)H_m(\xi),$$

where $H_m(\xi)$ is ordinary Hermite polynomials.

Note that the above result for $u=1$ is a special case of (2.2) because when ξ is replaced by 2ξ and $\eta=-1$ then we have

$$\left(\frac{2}{2+z}\right)^r e^{2\xi z-z^2}=\sum_{j=0}^{\infty}{}_HCh_j^{(*,r)}(2\xi,-1)\frac{z^j}{j!}. \tag{3.2}$$

In other words

$${}_hCh_j^{(*,r)}(\xi,1,0)={}_HCh_j^{(*,r)}(2\xi,-1).$$

Theorem 3.1. Let $j\geq 0$. Then

$${}_hCh_m^{(*,r)}(\xi,u,v)=\sum_{j=0}^{\infty}\sum_{s=0}^{m}Ch_s^{(*,r)}h_{m-s,j}(\xi)\frac{u^{m-s}v^j}{(m-s)!j!}. \tag{3.3}$$

Proof: On replacing u by uz in (1.4), we have

$$\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}h_{m,j}(\xi)\frac{(uz)^m v^j}{m!j!}=e^{-u^2z^2+(2uz+v)\xi-uvz}.$$

Then using (3.1), we can write

$$\sum_{s=0}^{\infty}{}_hCh_s^{(*,r)}(\xi,u,v)\frac{z^s}{s!}=\left(\frac{2}{2+z}\right)^r e^{-u^2z^2+(2uz+v)\xi-uvz}$$

$$=\sum_{s=0}^{\infty}Ch_s^{(*,r)}\frac{z^s}{s!}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}h_{m,j}(\xi)\frac{(uz)^m v^j}{m!j!}.$$

Now replacing m by $m-s$ and comparing the coefficients of z^s, we get the required result. □

As immediate consequence of Theorem 3.1, we assert the following.

Theorem 3.2. Let $j\geq 0$. Then

$$Ch_m^{(*,r)}(\xi\eta)=\sum_{j=0}^{\infty}\sum_{s=0}^{m}Ch_s^{(*,r)}h_{m-s,j}(\xi)\frac{y^{m-s}}{(m-s)!j!}. \tag{3.4}$$

Proof: We multiply both the sides of (1.5) by

$$\left(\frac{2}{2+z}\right)^r,$$

and replace η by ηz to get

$$\left(\frac{2}{2+z}\right)^r e^{\xi\eta z}=\left(\frac{2}{2+z}\right)^r\sum_{m,j=0}^{\infty}(-1)^j h_{m,j}(\xi)\frac{(\eta z)^{m+j}}{m!j!}$$

$$=\sum_{s=0}^{\infty}Ch_s^{(*,r)}\frac{z^s}{s!}\sum_{m,j=0}^{\infty}(-1)^j h_{m,j}(\xi)\frac{(\eta z)^{m+j}}{m!j!}.$$

Thus we have

$$\sum_{s=0}^{\infty}Ch_s^{(*,r)}(\xi\eta)\frac{z^s}{s!}=\sum_{s=0}^{\infty}Ch_s^{(*,r)}\frac{z^s}{s!}\sum_{m,j=0}^{\infty}(-1)^j h_{m,j}(\xi)\frac{(\eta z)^{m+j}}{m!j!}.$$

Now replacing m by $m-s$ and comparing the coefficients of z^s, we get the required result. □

Ghanmi and Lamsaf [7] analyzed a new class of polynomials generalizing different classes of Hermite polynomials such as the real Gould-Hopper, as well as the 1-d and 2-d holomorphic, ternary and polyanalytic complex Hermite polynomials. In the following theorem, we are concerned with a special and unified generalization. More precisely, we deal with the generalized Gould-Hopper polynomials and Appell-type Changhee polynomials.

First, we define generalized Gould-Hopper Appell-type Changhee polynomials by the generating function

$$\sum_{j=0}^{\infty} {}_GCh_j^{(*,r)}(w,\gamma,\zeta|u,v)\frac{z^j}{j!} = \left(\frac{2}{2+z}\right)^r e^{wv+\zeta uz+\gamma u^p v^q}. \tag{3.5}$$

Note that for $r=0$, (3.5) reduces to

$$\sum_{m=0}^{\infty} G_m^{(q)}(w|(uz)^p\gamma)\frac{v^m}{m!}e^{\zeta uz} = e^{wv+\zeta uz+\gamma u^p v^q},$$

where $G_m^{(q)}$ is defined by (1.6).

The next generating function is a consequence of the above one (see [7]) and gives the closed expression of $R_\gamma^{p,q}(z,w|u,v)$ in the form

$$R_\gamma^{p,q}(\zeta,w|u,v) = e^{\zeta u+wv+\gamma u^p v^q},$$

where

$$R_\gamma^{p,q}(\zeta,w|u,v) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} H_{j,m}^{(p,q)}(\zeta,w|\gamma)\frac{u^j}{j!}\frac{v^m}{m!}. \tag{3.6}$$

Furthermore, the polynomials $H_{j,m}^{(p,q)}(z,w|\gamma)$ are given by (1.7) and (1.8).

Theorem 3.3. For every $u,v,w,\zeta,\in\mathbb{C}$ and $j,r\geq 0$, we have

$${}_GCh_j^{(*,r)}(w,\gamma,\zeta|u,v) = \sum_{m=0}^{\infty}\sum_{s=0}^{j} H_{j-s,m}^{(p,q)}(\zeta,w|\gamma)Ch_s^{(*,r)}\frac{u^{j-s}v^m}{(j-s)!m!}. \tag{3.7}$$

Proof: Start with (1.10), replace γ by $u^p\gamma$ and multiply both the sides by $e^{\zeta uz}$ to get

$$e^{wv+\gamma u^p v^q}e^{\zeta uz} = \sum_{m=0}^{\infty} G_m^{(q)}(w|(uz)^p\gamma)\frac{v^m}{m!}e^{\zeta uz}.$$

Again, we multiply both the sides by

$$\left(\frac{2}{2+z}\right)^r,$$

to get

$$e^{wv+\gamma u^p v^q}e^{\zeta uz}\left(\frac{2}{2+z}\right)^r = \sum_{m=0}^{\infty} G_m^{(q)}(w|(uz)^p\gamma)\frac{v^m}{m!}e^{zut}\sum_{s=0}^{\infty} Ch_s^{(*,r)}\frac{z^s}{s!}.$$

Thus by using the definitions (3.5) and (3.6), we have

$$\sum_{s=0}^{\infty} {}_GCh_s^{(*,r)}(w,\gamma,\zeta|u,v)\frac{z^s}{s!}$$

$$= \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} H_{j,m}^{(p,q)}(\zeta, w|\gamma)\frac{(uz)^j}{j!}\frac{v^m}{m!}\sum_{s=0}^{\infty} Ch_s^{(*,r)}\frac{z^s}{s!}. \tag{3.8}$$

Now replacing j by $j-s$ and comparing the coefficients of z^j, we get the required result. □

As immediate consequence of the above theorem, we have

Corollary 3.1. For every $u, v, w, \zeta, \in \mathbb{C}$ and $j \geq 0$, we have

$${}_{G}Ch_j^{(0)}(w, \gamma, \zeta|u, v|\lambda) = R_{\gamma}^{p,q}(\zeta, w|u, v) = e^{\zeta u + wv + \gamma u^p v^q}.$$

4. Implicit Formulae Involving Hermite-Based Appell-Type Changhee Polynomials

We begin by considering the theorems on implicit summation formulae for Hermite-based Appell-type Changhee polynomials ${}_{H}Ch_j^{(*,r)}(\xi, \eta)$ and their special cases.

Theorem 4.1. Let $j \geq 0$. Then

$${}_{H}Ch_{q+l}^{(*,r)}(\zeta, \eta) \;=\; \sum_{j,p=0}^{q,l}\begin{pmatrix} q \\ j \end{pmatrix}\begin{pmatrix} l \\ p \end{pmatrix}(\zeta-\xi)^{j+p}\,{}_{H}Ch_{q+l-p-j}^{(*,r)}(\xi, \eta). \tag{4.1}$$

Proof: By changing z by $z + u$ in (2.2), we get

$$\left(\frac{2}{2+(z+u)}\right)^r e^{\eta(z+u)^2} = e^{-\xi(z+u)}\sum_{q,l=0}^{\infty} {}_{H}Ch_{q+l}^{(*,r)}(\xi, \eta|\lambda)\frac{z^q}{q!}\frac{u^l}{l!}, \quad \text{(see [11, 12])}. \tag{4.2}$$

Again changing ξ with ζ in above equation, we have

$$e^{(\zeta-\xi)(z+u)}\sum_{q,l=0}^{\infty} {}_{H}Ch_{q+l}^{(*,r)}(\xi, \eta)\frac{z^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_{H}Ch_{q+l}^{(*,r)}(\zeta, \eta)\frac{z^q}{q!}\frac{u^l}{l!}. \tag{4.3}$$

$$\sum_{N=0}^{\infty}\frac{[(\zeta-\xi)(z+u)]^N}{N!}\sum_{q,l=0}^{\infty} {}_{H}Ch_{q+l}^{(*,r)}(\xi, \eta)\frac{z^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_{H}Ch_{q+l}^{(*,r)}(\zeta, \eta)\frac{z^q}{q!}\frac{u^l}{l!}, \tag{4.4}$$

$$\sum_{N=0}^{\infty} f(N)\frac{(\xi+\eta)^N}{N!} = \sum_{j,m=0}^{\infty} f(j+m)\frac{\xi^j}{j!}\frac{\eta^m}{m!}, \tag{4.5}$$

in the left hand side becomes

$$\sum_{j,p=0}^{\infty}\frac{(\zeta-\xi)^{j+p}z^j u^p}{j!p!}\sum_{q,l=0}^{\infty} {}_{H}Ch_{q+l}^{(*,r)}(\xi, \eta)\frac{z^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_{H}Ch_{q+l}^{(*,r)}(\zeta, \eta)\frac{z^q}{q!}\frac{u^l}{l!}. \tag{4.6}$$

$$\sum_{q,l=0}^{\infty}\sum_{j,p=0}^{q,l}\frac{(\zeta-\xi)^{j+p}}{j!p!}\,{}_{H}Ch_{q+l-j-p}^{(*,r)}(\xi, \eta)\frac{z^q}{(q-j)!}\frac{u^l}{(l-p)!}$$

$$= \sum_{q,l=0}^{\infty} {}_{H}Ch_{q+l}^{(*,r)}(\zeta, \eta)\frac{z^q}{q!}\frac{u^l}{l!}. \tag{4.7}$$

The complete of the proof. □

Theorem 4.2. Let $j \geq 0$. Then

$${}_{H}Ch_j^{(*,r)}(\xi+\zeta, \eta+u) = \sum_{s=0}^{j}\begin{pmatrix} j \\ s \end{pmatrix}{}_{H}Ch_{j-s}^{(*,r)}(\xi, \eta)H_s(\zeta, u). \tag{4.8}$$

Proof: By changing ξ by $\xi+\zeta$ and η by $\eta+u$ in (2.2) and using (1.2), we have

$$\left(\frac{2}{2+z}\right)^r e^{(\xi+\zeta)z+(\eta+u)z^2} = \sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(\xi,\eta)\frac{z^j}{j!}\sum_{s=0}^{\infty} H_s(\zeta,u)\frac{z^s}{s!}$$

$$= \sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(\xi+\zeta,\eta+u)\frac{z^j}{j!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\binom{j}{s} {}_HCh_{j-s}^{(*,r)}(\xi,\eta)H_s(\zeta,u)\right)\frac{z^j}{j!}. \tag{4.9}$$

In view of (2.9), we get (4.8). □

A special case for $z=1$ and $u=0$ may be deduced from Theorem 4.2, which assumes the form

Corollary 4.1. Let $j \geq 0$. Then

$${}_HCh_j^{(*,r)}(\xi+1,\eta) = \sum_{s=0}^{j}\binom{j}{s} {}_HCh_{j-s}^{(*,r)}(\xi,\eta).$$

Theorem 4.3. Let $j \geq 0$. Then

$${}_HCh_j^{(*,r)}(\eta,\xi) = \sum_{s=0}^{[\frac{j}{2}]} Ch_{j-2s}^{(*,r)}(\eta)\frac{\xi^s}{(j-2s)!s!}. \tag{4.10}$$

Proof: By replacing ξ by η and η by ξ in equation (2.2) to get

$$\sum_{j=0}^{\infty} {}_HCh_j^{(*,r)}(\eta,\xi)\frac{z^j}{j!} = \sum_{j=0}^{\infty} Ch_j^{(*,r)}(\eta)\frac{z^j}{j!}\sum_{s=0}^{\infty}\frac{\xi^s z^{2s}}{s!}.$$

$$= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{[\frac{j}{2}]} Ch_{j-2s}^{(*,r)}(\eta)\frac{\xi^s}{(j-2s)!s!}\right)\frac{z^j}{j!},$$

which complete the proof.

□

Theorem 4.4. Let $j \geq 0$. Then

$${}_HCh_j^{(*,r)}(\xi,\eta) = \sum_{s=0}^{j}\binom{j}{s} Ch_{j-s}^{(*,r)}(\xi-\zeta)H_s(\zeta,\eta). \tag{4.11}$$

Proof: By (2.2) as

$$\left(\frac{2}{2+z}\right)^r e^{(\xi-\zeta)z}e^{\zeta z+\eta z^2} = \sum_{j=0}^{\infty} Ch_j^{(*,r)}(\xi-\zeta)\frac{z^j}{j!}\sum_{s=0}^{\infty} H_s(\zeta,\eta)\frac{z^s}{s!}. \tag{4.12}$$

$$\sum_{n=0}^{\infty} {}_HCh_j^{(*,r)}(\xi,\eta)\frac{z^j}{j!} = \sum_{j=0}^{\infty}\sum_{r=0}^{j} Ch_{j-s}^{(*,r)}(\xi-\zeta)H_r(\zeta,\eta)\frac{z^j}{(j-s)!s!}.$$

On equating the coefficients of z, we get the result (4.11). □

5. Further Remarks

In this section, by using the fermionic p-adic integral on $\mathbb{Z}_p$, we derive some identities for Appell-type Changhee polynomials, Stirling numbers of the first kind, and Euler numbers. By (1.15), we note that

$$\begin{aligned}\frac{2}{2+z}e^{\xi z+\eta z^2} &= \int_{\mathbb{Z}_p}(1+z)^{\zeta}e^{\xi z+\eta z^2}d\mu_{-1}(\zeta)\\ &= \int_{\mathbb{Z}_p}e^{\zeta\log(1+z)+\xi z+\eta z^2}d\mu_{-1}(\zeta).\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}e^{\xi z+\eta z^2}e^{\zeta\log(1+z)} &= \left(\sum_{j=0}^{\infty}H_j(\xi,\eta)\frac{z^j}{j!}\right)\left(\sum_{m=0}^{\infty}\frac{\zeta^m(\log(1+z)^m)}{m!}\right)\\ &= \left(\sum_{j=0}^{\infty}H_j(\xi,\eta)\frac{z^j}{j!}\right)\left(\sum_{m=0}^{\infty}\zeta^m\sum_{k=m}^{\infty}S_1(k,m)\frac{z^k}{k!}\right)\\ &= \left(\sum_{j=0}^{\infty}H_j(\xi,\eta)\frac{z^j}{j!}\right)\left(\sum_{k=0}^{\infty}\sum_{m=0}^{k}\zeta^m S_1(k,m)\frac{z^k}{k!}\right)\\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}H_{j-k}(\xi,\eta)\zeta^m S_1(k,m)\right)\frac{z^j}{j!}.\end{aligned} \tag{5.2}$$

Thus, by (5.1) and (5.2), we note that

$$\begin{aligned}\sum_{j=0}^{\infty}{}_HCh^*(\xi,\eta)\frac{z^j}{j!} &= \int_{\mathbb{Z}_p}e^{\zeta\log(1+z)+\xi z+\eta z^2}d\mu_{-1}(\zeta)\\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}H_{j-k}(\xi,\eta)\int_{\mathbb{Z}_p}\zeta^m S_1(k,m)d\mu_{-1}(\zeta)\right)\frac{z^j}{j!}\\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}H_{j-k}(\xi,\eta)E_m S_1(k,m)\right)\frac{z^j}{j!}.\end{aligned} \tag{5.3}$$

Therefore, by (5.3), we obtain the following theorem.

Theorem 5.1. Let $j\in\mathbb{N}$. Then

$${}_HCh_j^*(\xi,\eta) = \sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}H_{j-k}(\xi,\eta)E_m S_1(k,m). \tag{5.4}$$

References

1. L. C. Andrews, *Special functions for engineers and mathematicians*, Macmillan Co., New York, 1985.
2. E. T. Bell, Exponential polynomials, *Ann. of Math.*, 35 (1934), 258–277.
3. G. Dattoli, C. Chiccoli, S. Lorenzutta, G. Maino, A. Torre, Theory of generalized Hermite polynomials, *Comput. Math. Appl.*, 28(4) (1994), 71–83.
4. G. Dattoli, C. Cesarano, D. Sacchetti, A note on truncated polynomials, *Appl. Math. Comput.*, 134(2-3) (2003), 595–605.
5. D. V. Dolgy, W. A. Khan, A note on type two degenerate poly-Changhee polynomials of the second kind, *Symmetry*, 13 (2021), 579.
6. F. Gori, Flattened Gaussian beams, *Opt. Commun.*, 107(5-6) (1994), 335–341.

7. A. Ghanmi, K. Lamsaf, A unified generalization of real Gould-Hopper, 1-d and 2-d holomorphic and polyanalytic Hermite polynomials, *arXiv:1907.10744* (2019).

8. H. W. Gould, A. T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.*, 29 (1962), 51–63.

9. H. Haroon, W. A. Khan, Degenerate Bernoulli numbers and polynomials associated with Hermite polynomials, *Commun. Korean Math. Soc.*, 33(2) (2018), 651–669.

10. N. A. Jedda, A. Ghanmi, On a class of two-index real Hermite polynomials, *Palestine J. Math.*, 3(2) (2014), 185–190.

11. W. A. Khan, S. Araci, M. Acikgoz, H. Haroon, A new class of partially degenerate Hermite-Genocchi polynomials, *J. Nonlinear Sci. Appl.*, 10(9) (2017), 5072–5081.

12. W. A. Khan, M. A. Pathan, On generalized Lagrange-Hermite-Bernoulli and related polynomials, *Acta Comment. Univ. Tartu. Math.*, 23(2) (2019), 211–224.

13. T. Kim, A study on the q-Euler numbers and the fermionic q-integral of the product of several type q-Bernstein polynomials on $\mathbb{Z}_p$, *Adv. Stud. Contemp. Math.*, 23(1) (2013), 5–11.

14. T. Kim, Some identities on the q-Euler polynomials of higher-order and q-Stirling numbers by the fermionic p-adic invariant integral on $\mathbb{Z}_p$, *Russ. J. Math. Phys.*, 16(4) (2009), 484–491.

15. D. S. Kim, T. Kim, J. Seo, A note on Changhee polynomials and numbers, *Adv. Stud. Theor. Phys.*, 7(1) (2013), 993–1003.

16. T. Kim, D. S. Kim, A note on nonlinear Changhee differential equations, *Russ. J. Math. Phys.*, 23(1) (2016), 88–92.

17. D. Lim, F. Qi, On the Appell type λ-Changhee polynomials, *J. Nonlinear Sci. Appl.*, 9 (2016), 1872–1876.

18. J. G. Lee, L. C. Jang, J.-J. Seo, S.-K. Choi, H. I. Kwon, On Appell-type Changhee polynomials and numbers, *Adv. Differ. Equ.*, 2016:160 (2016), 1–10.

19. R. B. Corcino, C. R. Corcino, W. A. Khan, New type of degenerate Changhee numbers polynomials of the second kind, *J. Inequal. Spec. Funct.*, 15(4) (2024), 1–14.

20. W. A. Khan, New type of degenerate Changhee numbers polynomials of the second kind, *Southeast Asian J. Math. Math. Sci.*, 20(2) (2024), 31–46.

21. W. A. Khan, A. Sarosh, J. Miandad, Generalized degenerate Changhee-Genocchi numbers and polynomials, *Southeast Asian J. Math. Math. Sci.*, 20(1) (2024), 31–46.

22. M. S. Alatawi, W. A. Khan, U. Duran, Several symmetric identities of the generalized degenerate Fubini polynomials by fermionic p-adic integral on $\mathbb{Z}_p$, *Symmetry*, 16 (2024), 686.

23. M. S. Alatawi, W. A. Khan, U. Duran, Symmetric identities involving the extended degenerate central Fubini polynomials arising from fermionic p-adic integral on $\mathbb{Z}_p$, *Axioms*, 13 (2024), 421.

24. M. Nadeem, W. A. Khan, Symmetric identities for degenerate q-Genocchi numbers and polynomials, *Southeast Asian J. Math. Math. Sci.*, 19(1) (2023), 17–28.

25. W. A. Khan, V. Yadav, A study on q-analogue of degenerate Changhee numbers and polynomials, *Southeast Asian J. Math. Math. Sci.*, 19(1) (2023), 29–42.

26. W. A. Khan, M. S. Alatawi, A note on modified degenerate Changhee-Genocchi polynomials of the second kind, *Symmetry*, 15 (2023), 136.

27. H. Alsamir, M. S. M. Noorani, W. Shatanawi, H. Aydi, H. Qawaqneh, Fixed point results in metric-like spaces via σ-simulation functions, *Eur. J. Pure Appl. Math.*, 12(1) (2019), 88–100.

28. H. Alsamir, H. Qawaqneh, G. Al-Musannef, R. Khalil, Common fixed point of generalized Berinde type contraction and an application, *Eur. J. Pure Appl. Math.*, 17(4) (2024), 2492–2504.

29. H. Qawaqneh, H. A. Hammad, H. Aydi, Exploring new geometric contraction mappings and their applications in fractional metric spaces, *AIMS Math.*, 9(1) (2024), 521–541.

30. H. Qawaqneh, M. S. Alatawi, New functions for fixed point results in metric spaces with some applications, *Indian J. Math.*, 66(1) (2024), 55–84.

31. H. Qawaqneh, M. S. Noorani, H. Aydi, W. Shatanawi, On common fixed point results for new contractions with applications to graph and integral equations, *Mathematics*, 7(11) (2019), 1082.

32. M. Meneceur, H. Qawaqneh, H. Alsamir, G. Al-Musannef, Common fixed point of generalized Berinde type contraction and an application, *Eur. J. Pure Appl. Math.*, 17(4) (2024), 3093–3108.

Haitham Qawaqneh,
Department of Basic Sciences,
Al-Zaytoonah University of Jordan, Amman 11733, Jordan
E-mail address: h.alqawaqneh@zuj.edu.jo

and

Waseem Ahmad Khan,
Department of Electrical Engineering,
Prince Mohammad Bin Fahd University,
P.O Box 1664, Al Khobar 31952
Saudi Arabia.
E-mail address: wkhan1@pmu.edu.sa; khan346@gmail.com