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Remarks on $F-$Contraction in $S-$Metric Space

Süheyla Elmas and Ferdi Çınar

ABSTRACT: In this approach, we focus on fixed points of $F-$contraction on an abstract space. In particular, we propose the idea of $F-$Suzuki contraction. The practical relevance of the established results is demonstrated via suitable examples.

Keywords: Fixed point, $S-$metric space, $F-$contraction.

Contents

1. Introduction and Preliminaries

The Banach contraction principle (BCP) [1] is widely acknowledged as one of the key findings in metric fixed point theory, owing to its simplicity and broad applicability across various mathematical disciplines. It exemplifies an elegant synthesis of analysis, topology, and geometry. This foundational framework not only establishes the existence and uniqueness of fixed points for a class of self-map in metric spaces but also yields valuable methodologies for their determination. Among these methodologies is the extension of classical contraction conditions to more generalized or abstract settings. The literature encompasses a wide range of contraction-type mappings, several of which have been modified to overcome the limitations inherent in the classical contraction framework—such as the Khan type [13], Meir-Keeler Khan type [14], Hardy-Rogers type contractions [15].

BCP [1] extensively studied and generalized by numerous mathematicians, leading to substantial progress in fixed point theory. Furthermore, It serves as a valuable technique across diverse areas of mathematics, including approximation theory, game theory, quantum theory, economic analysis, and the study of differential and integral equations.

In our approach, we use $S-$metric spaces [6], which were introduced as a natural extension of standard metric spaces, to generalize classical results and investigate broader conditions under which fixed point theorems hold.

Definition 1.1 *[6] Let $\mathcal{M} \neq \varnothing$ be a set and let $\theta : \mathcal{M}^3 \to [0, \infty)$ be a map that fulfills certain conditions for any η, μ, $\zeta \in \mathcal{M}$:*
$(s1)$ $\theta(\eta, \mu, \zeta) \geq 0$
$(s2)$ $\theta(\eta, \mu, \zeta) = 0$ *iff* $\eta = \mu = \zeta = 0$;
$(s3)$ $\theta(\eta, \mu, \zeta) \leq \theta(\eta, \eta, \nu) + \theta(\mu, \mu, \nu) + \theta(\zeta, \zeta, \nu)$.
Then the map θ is said a $S-$metric on $\mathcal{M}$ and $(\mathcal{M}\ ,\theta)$ is called a $S-$metric space.

The literature provides examples of $S-$that cannot be derived from any standard metric ([7], [8]), underscoring the necessity of formulating novel fixed point theorems within the framework of $S-$metric spaces. Several such findings have been established by employing various analytical techniques aimed at extending classical fixed point theorems ([9], [10], [11], [12]).

Theorem 1.1 *[3] Let $(\mathcal{M}, d_s)$ be a complete $S-$metric space and let $\theta : \mathcal{M} \to \mathcal{M}$ be a self-map. Assume that $d_s(\theta_\eta, \theta_\eta, \theta_\mu) < d_s(\eta, \eta, \mu)$ applies for every $\eta, \mu \in \mathcal{M}$ via $\eta \neq \mu$. Hence, θ has exactly one fixed point in $\mathcal{M}$.*

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Suzuki, in 2008, [2] established extensions of Edelstein's fixed point results involving complete $S-$metric space, thereby extending the applicability of classical theorems to this broader setting.

Theorem 1.2 *[2] Let $(\mathcal{M}, d_s)$ be a complete $S-$metric space and let $\theta : \mathcal{M} \to \mathcal{M}$ be a self-map. Suppose that for every $\eta, \mu \in \mathcal{M}$ via $\eta \neq \mu$,*

$$\frac{1}{2}S(\eta, \eta, \theta\eta) < d_s(\eta, \eta, \mu) \Rightarrow d_s(\theta\eta, \theta\eta, \theta\mu) < d_s(\eta, \eta, \mu).$$

Hence, θ has exactly one fixed point in $\mathcal{M}$.

Wardowski, in 2012, [4] put forward a novel class of contractive mappings, referred to as $F-$contractions, and set forth a new fixed point theorem related to this class. Through this approach, Wardowski provided a conceptual expansion of the BCP that differs significantly from the classical extensions available in related works. The concept of $F-$contraction is defined in the following manner.

Definition 1.2 *Let $(\mathcal{M}, d_s)$ be a metric space. A map $\theta : \mathcal{M} \to \mathcal{M}$ is called to be a $F-$contraction if there exists $\lambda > 0$ such that*

$$\forall \eta, \mu \in \mathcal{M}, \left[\begin{array}{c} d_s(\theta\eta, \theta\mu) > 0 \Rightarrow \\ \lambda + F(d_s(\theta\eta, \theta\mu)) \leq F(d_s(\eta, \mu)) \end{array} \right]$$

where $F : R_+ \to R$ is a function that fulfills certain conditions:

(F1) F is strictly increasing for any $\eta, \mu \in R_+$ such that $\eta < \mu$, $F(\eta) < F(\mu)$;

(F2) Given any sequence $\{\eta_n\}_{n=1}^{\infty}$ of strictly positive terms, $\lim_{n\to\infty} \eta_n = 0$ iff $\lim_{n\to\infty} F(\eta_n) = -\infty$;

(F3) There exists $k \in (0, 1)$ such that $\lim_{\eta\to 0^+} \eta^k F(\eta) = 0$.

Definition 1.3 *Let $(\mathcal{M}, d_s)$ be a $S-$metric space. A map $\theta : \mathcal{M} \to \mathcal{M}$ is called to be a $F-$contraction if there exists $\lambda > 0$ such that*

$$\forall \eta, \mu \in \mathcal{M}, \left[\begin{array}{c} d_s(\theta\eta, \theta\eta, \theta\mu) > 0 \Rightarrow \\ \lambda + F(d_s(\theta\eta, \theta\eta, \theta\mu)) \leq F(d_s(\eta, \eta, \mu)) \end{array} \right] \tag{1}$$

where $F : R_+ \to R$ is a function that meets the assumptions (F1)-(F3) listed above.

We identified $\mathcal{F}$ as the family of functions satisfying the assumptions *(F1)-(F3)*. Illustrative examples of such functions can be found in [4] and [5].

Remark 1.1 *Based on assumption (F1) and the result in (1), one can deduce that every $F-$contraction is necessarily continuous. exhibits continuity. In this context, the author [4] proposed a revised formulation of the BCP.*

Theorem 1.3 *[4] Let $(\mathcal{M}, d_s)$ be a complete $S-$metric space and let $\theta : \mathcal{M} \to \mathcal{M}$ be a $F-$contraction. Then θ has exactly one fixed point $\eta^* \in \mathcal{M}$ and for any $\eta \in \mathcal{M}$ the sequence $\{\theta^n \eta\}_n \in N$ converges to η^*.*

The author [5] presented the following auxiliary result.

Lemma 1.1 *[5] Let $F : R_+ \to R$ be an increasing function and $\{\eta_n\}_{n=1}^{\infty}$ be a sequence of strictly positive terms. Then, the statements below are valid;*

(a) if $\lim_{n\to\infty} F(\eta_n) = -\infty$, then $\lim_{n\to\infty} \eta_n = 0$;

(b) if $\inf F = -\infty$, and $\lim_{n\to\infty} \eta_n = 0$, then $\lim_{n\to\infty} F(\eta_n) = -\infty$.

Lemma 1.1 by Secelean reveals that the original formulation of condition *(F2)* in Definition 1.3 can be substituted by a more concise yet equivalent condition;

(F2´) $\inf F = -\infty$;

in other cases, also, using

(F2´´) there exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ of strictly positive terms such that $\lim_{n\to\infty} F(\eta_n) = -\infty$.

Remark 1.2 *Let $F_\mu : R_+ \to R$ be identified by $F_\mu(\eta) = \ln \eta$. Clearly, $F_\mu \in \mathcal{F}$. Observe that when $F = F_\mu$, the notion of a $F-$contraction coincides with BCP. Hence, BCP emerge as a special instance of the more general class of $F-$contractions. Nevertheless, it should be noted that there exist $F-$contractions which do not fall under the scope of BCP ([4], [5]).*

In our approach, the condition *(F3)* in the original definition is replaced by the following alternative assumption 1.3:

(F3´) F is continuous on $(0, \infty)$.

We referred to $\Im$ as the set of all functions fulfilling the conditions *(F1)*, *(F2´)* and *(F3´)*.

Example 1.1 *Let $F_1(\eta) = \frac{-1}{\eta}$, $F_2(\eta) = \frac{-1}{\eta} + \eta$, $F_3(\eta) = \frac{1}{1-e^{\eta}}$, $F_4(\eta) = \frac{1}{e^{\eta}-e^{-\eta}}$. Then F_1, F_2, F_3, $F_4 \in \Im$.*

Remark 1.3 *It is clear that conditions (F3) and (F3´) are independent. For example, for $p \geq 1$, $F(\eta) = \frac{-1}{\eta^p}$ satisfies (F1) and (F2), but not (F3), while it satisfies (F3´); thus, $\Im \nsubseteq \mathcal{F}$. Conversely, $F(\eta) = \frac{-1}{(\eta+[\eta])^t}$, where $t \in (0, 1/\eta)$, fulfills (F1) and (F2), satisfies (F3), but not (F3´); hence, $\mathcal{F} \nsubseteq \Im$. However, $F(\eta) = \ln \eta$, belongs to both, so $\mathcal{F} \cap \Im \neq \varnothing$.*

Motivated by Remark 1.3, , it is natural to extend the author's result [4], by considering maps $F \in \Im$ instead of $F \in \mathcal{F}$. In this context, we introduce the notion of a F-Suzuki contraction and present a revised formulation of Theorem 1.3.

Definition 1.4 *Let $(\mathcal{M}, d_s)$ be a $S-$metric space. A map $\theta : \mathcal{M} \to \mathcal{M}$ is said to be a $F-$Suzuki contraction if there exists $\lambda > 0$ such that for each $\eta, \mu \in M$ via $T\eta \neq T\mu$*

$$\frac{1}{2} S(\eta, \eta, \theta\eta) < d_s(\eta, \eta, \mu) \Rightarrow \lambda + F(d_s(\theta\eta, \theta\eta, \theta\mu)) \leq F(d_s(\eta, \eta, \mu)),$$

where $F \in \mathcal{F}$.

The current work focuses on $F-$contraction endowed with $S-$metric space. Furthermore, the validity and effectiveness of the proposed results are demonstrated through illustrative examples.

2. Main Results

Theorem 2.1 *Let $\mathcal{M}$ be a complete $S-$metric space. Suppose $\theta : \mathcal{M} \to \mathcal{M}$ is a $F-$contraction assume $F \in \mathcal{F}$ and there exist $\lambda > 0$ such that for any $\eta, \mu \in \mathcal{M}$,*

$$d_s(\theta\eta, \theta\eta, \theta\mu) > 0 \Rightarrow \lambda + F(d_s(\theta\eta, \theta\eta, \theta\mu)) \leq F(d_s(\eta, \eta, \mu)) + N_\gamma(\eta, \mu),$$

where

$$N_\gamma(\eta, \mu) = \gamma \min\{d_s(\eta, \eta, \theta\mu),\ d_s(\eta, \eta, \theta\eta),\ d_s(\mu, \mu, \theta\mu),\ d_s(\mu, \mu, \theta\eta)\} \text{ such that } \gamma \geq 0.$$

Then θ has exactly one fixed point in $\eta^* \in \mathcal{M}$ and for every $\eta_0 \in \mathcal{M}$ the sequence $\{\theta^n \eta_0\}_{n=0}^{\infty}$ converges to η^*.

Proof: Let $\mathcal{M}$ be a complete $S-$metric space. Choose $\eta_0 \in \mathcal{M}$ and define a sequence $\{\eta_n\}_{n=0}^{\infty}$ as follows;

$$\begin{array}{rcl} \eta_1 & = & \theta\eta_0, \eta_2 = \theta\eta_1 = \theta^2\eta_0, ... \\ \eta_{n+1} & = & \theta\eta_n = \theta_{\eta_0}^{n+1}, \text{ for every } n \in N. \end{array}$$

In the context of the $S-$metric space when $d_s(\eta_n, \eta_n, \eta_{n+1}) > 0$ the given condition

$$\lambda + F(d_s(\eta_n, \eta_n, \eta_{n+1})) \leq F((d_s(\eta_n, \eta_n, \eta_{n+1}))$$

holds for a chosen $\lambda > 0$. This satisfies the definition of a $S-$metric space, implying that there exist a unique fixed point $\eta^* \in \mathcal{M}$ for the map θ, and for every $\eta_0 \in \mathcal{M}$ the sequence $\{\theta\eta_0\}_{n=0}^{\infty}$ converges to η^*.

If there exist $n \in N$ such that $d_s(\eta_n, \eta_n, \theta\eta_n) = 0$. Thus, having completed the proof, we proceed by assuming that $0 < d_s(\eta_n, \eta_n, \theta\eta_n) = d_s(\theta\eta_{n-1}, \theta\eta_{n-1}, \theta\eta_n)$ for each $n \in N$ we attain

$$\begin{aligned} \lambda + F(d_s(\theta\eta_{n-1}, \theta\eta_{n-1}, \theta\eta_n)) &\leq d_s(\eta_{n-1}, \eta_{n-1}, \eta_n) + N\gamma\left(\eta_{n-1}, \eta_n\right) \\ & \quad i.e \\ F(d_s(\theta\eta_{n-1}, \theta\eta_{n-1}, \theta\eta_n)) &\leq d_s(\eta_{n-1}, \eta_{n-1}, \eta_n) + N_\gamma\left(\eta_{n-1}, \eta_n\right) - \lambda, \end{aligned}$$

where

$$\begin{aligned} & N_\gamma\left(\eta_{n-1}, \eta_n\right) \\ = \; & \gamma \min\left\{d_s\left(\eta_{n-1}, \eta_{n-1}, \theta\eta_n\right),\ d_s\left(\eta_{n-1}, \eta_{n-1}, \theta\eta_{n-1}\right),\ d_s\left(\eta_n, \eta_n, \theta\eta_n\right),\ d_s\left(\eta_n, \eta_n, \theta\eta_{n-1}\right)\right\} \\ = \; & \gamma \min\left\{d_s\left(\eta_{n-1}, \eta_{n-1}, \eta_{n+1}\right),\ d_s\left(\eta_{n-1}, \eta_{n-1}, \eta_n\right),\ d_s\left(\eta_n, \eta_n, \eta_{n+1}\right),\ d_s\left(\eta_n, \eta_n, \eta_n\right)\right\} \\ = \; & 0. \end{aligned}$$

Continuing this process, the resulting expression is as follows

$$\begin{aligned} F(d_s(\theta\eta_{n-1}, \theta\eta_{n-1}, \theta\eta_n)) &\leq d_s(\eta_{n-1}, \eta_{n-1}, \eta_n) - \lambda \\ &= F(d_s(\theta\eta_{n-2}, \theta\eta_{n-2}, \theta\eta_{n-1})) - \lambda \\ &\leq F(d_s(\theta\eta_{n-2}, \theta\eta_{n-2}, \theta\eta_{n-1})) - 2\lambda \\ &= F(d_s(\theta\eta_{n-3}, \theta\eta_{n-3}, \theta\eta_{n-2})) - 2\lambda \\ &\leq F(d_s(\eta_{n-3}, \eta_{n-3}, \eta_{n-2})) - 3\lambda \\ &\;\vdots \\ &\leq d_s(\eta_0, \eta_0, \eta_1) - n\lambda, \end{aligned} \tag{2.1}$$

from 2.1 we obtain $\lim_{n\to\infty} F(d_s(\theta\eta_{n-1}, \theta\eta_{n-1}, \theta\eta_n)) = -\infty$, which combined with *(F2´)* and Lemma 1.1, leads to $F(d_s(\theta\eta_{n-1}, \theta\eta_{n-1}, \theta\eta_n)) \neq 0$, i.e

$$\lim_{n\to\infty} d_s(\eta_n, \eta_n, \theta\eta_n) = 0. \tag{5}$$

At this stage, we aim to show that the sequence $\{\eta_n\}$ is a Cauchy. Suppose, for the sake of contradiction, that there exists a real number $\varepsilon > 0$ and two strictly increasing sequences $\{\rho(n)\}$ and $\{\omega(n)\}$ in N.

$$\rho(n) > \omega(n) > n, d_s(\eta_{\rho(n)}, \eta_{\omega(n)}, \eta_{\omega(n)}) \geq \varepsilon, d_s(\eta_{\rho(n)-1}, \eta_{\rho(n)-1}, \eta_{\omega(n)}) < \varepsilon \tag{6}$$

Thus, we derive

$$\begin{aligned} \varepsilon &\leq d_s(\eta_{\rho(n)}, \eta_{\rho(n)}, \eta_{\omega(n)}) \leq d_s(\eta_{\rho(n)}, \eta_{\rho(n)}, \eta_{\rho(n-1)}) + d_s(\eta_{\rho(n)-1}, \eta_{\rho(n)-1}, \eta_{\omega(n)}) \\ &\leq d_s(\eta_{\rho(n)}, \eta_{\rho(n)}, \eta_{\rho(n-1)}) + \varepsilon = d_s(\eta_{\rho(n)-1}, \eta_{\rho(n)-1}, \theta\eta_{\rho(n)-1}) + \varepsilon. \end{aligned} \tag{2.2}$$

In light of (5) and (2.2), one deduces that

$$\lim_{n\to\infty} d_s(\eta_{\rho(n)}, \eta_{\rho(n)}, \eta_{\omega(n)}) = \varepsilon. \tag{7}$$

As implied by (5) there is $n \in N$ such that

$$d_s(\eta_{\rho(n)}, \eta_{\rho(n)}, \theta\eta_{\rho(n)}) < \frac{\varepsilon}{4} \text{ and } d_s(\eta_{\omega(n)}, \eta_{\omega(n)}, \theta\eta_{\omega(n)}) < \frac{\varepsilon}{4} \quad \forall n \geq N. \tag{8}$$

We now assert that

$$d_s(\theta\eta_{\rho(n)}, \theta\eta_{\rho(n)}, \theta\eta_{\rho(n)}) = d_s(\eta_{\rho(n)+1}, \eta_{\rho(n)+1}, \eta_{\omega(n)+1}) > 0 \quad \forall n \in N. \tag{9}$$

Reasoning by contradiction, there is $m \geq n$ such that

$$d_s(\eta_{\rho(m)+1}, \eta_{\rho(m)+1}, \eta_{\omega(m)+1}) = 0. \tag{10}$$

An application of (6), (8) and (10) yields,

$$\begin{aligned} \varepsilon &\leq d_s(\eta_{\rho(m)}, \eta_{\rho(m)}, \eta_{\omega(m)}) \\ &\leq d_s(\eta_{\rho(m)}, \eta_{\rho(m)}, \eta_{\rho(m)+1}) + d_s(\eta_{\rho(m)+1}, \eta_{\rho(m)+1}, \eta_{\omega(m)}) \\ &\leq d_s(\eta_{\rho(m)}, \eta_{\rho(m)}, \eta_{\rho(m)+1}) + d_s(\eta_{\rho(m)+1}, \eta_{\rho(m)+1}, \eta_{\omega(m)+1}) \\ &\quad + d_s(\eta_{\omega(m)+1}, \eta_{\omega(m)+1}, \eta_{\omega(m)}) \\ &= d_s(\eta_{\rho(m)}, \eta_{\rho(m)}, \theta\eta_{\rho(m)}) + d_s(\eta_{\rho(m)+1}, \eta_{\rho(m)+1}, \eta_{\omega(m)+1}) \\ &\quad + d_s(\eta_{\omega(m)}, \eta_{\omega(m)}, \theta\eta_{\omega(m)}) \\ &< \varepsilon. \end{aligned}$$

This contradiction substantiates (9). Accordingly, by applying both (9) and the initial assumption, we get

$$\lambda + F(d_s(\theta\eta_{\rho(n)}, \theta\eta_{\rho(n)}, \theta\eta_{\omega(n)})) \leq F(d_s(\eta_{\rho(n)}, \eta_{\rho(n)}, \eta_{\omega(n)})) + N_\gamma\left(\eta_{\rho(n)}, \eta_{\omega(n)}\right) \quad \text{for any } n \in N, \tag{11}$$

where

$$\begin{aligned} & N_\gamma\left(\eta_{\rho(n)}, \eta_{\omega(n)}\right) \\ =\ & \gamma \min\left\{ \begin{array}{c} d\left(\eta_{\rho(n)}, \eta_{\rho(n)}, \theta\eta_{\omega(n)}\right),\ d\left(\eta_{\rho(n)}, \eta_{\rho(n)}, \theta\eta_{\rho(n)}\right), \\ d\left(\eta_{\omega(n)}, \eta_{\omega(n)}, \theta\eta_{\omega(n)}\right),\ d\left(\eta_{\omega(n)}, \eta_{\omega(n)}, \theta\eta_{\rho(n)}\right) \end{array} \right\} \\ =\ & \gamma \min\left\{ \begin{array}{c} d\left(\eta_{\rho(n)}, \eta_{\rho(n)}, \eta_{\omega(n)+1}\right),\ d\left(\eta_{\rho(n)}, \eta_{\rho(n)}, \eta_{\rho(n)+1}\right), \\ d\left(\eta_{\omega(n)}, \eta_{\omega(n)}, \eta_{\omega(n)+1}\right),\ d\left(\eta_{\omega(n)}, \eta_{\omega(n)}, \eta_{\rho(n)+1}\right) \end{array} \right\} \end{aligned} \tag{2.3}$$

such that $\gamma \geq 0$. Letting $n \to \infty$ in (11) and (2.3), and using *(F3´)* and (7), it follows that $\lambda + F(\varepsilon) \leq F(\varepsilon)$. Hence, the contradiction confirms that $\{\eta_n\}_{n=1}^{\infty}$ forms a Cauchy sequence. Given that $(\mathcal{M}, d_s)$ is a complete, $\{\eta_n\}_{n=1}^{\infty}$ converges to a $\eta \subset \mathcal{M}$.

This continuity of θ consequently results in

$$d_s(\theta\eta, \theta\eta, \eta) = \lim_{n\to\infty} d_s(\theta\eta_n, \theta\eta_n, \eta_n) = \lim_{n\to\infty} d_s(\eta_{n+1}, \eta_{n+1}, \eta_n) = d_s(\eta^*, \eta^*, \eta^*) = 0.$$

We now proceed to demonstrate the uniqueness of the fixed point of θ. Suppose, for the sake of contradiction, that there exist two distinct fixed points $\eta, \mu \in \mathcal{M}$ such that $\theta\eta = \eta \neq \mu = \theta\mu$. Therefore,

$$d_s(\theta\eta, \theta\eta, \theta\mu)) = d_s(\eta, \eta, \mu) > 0,$$

then we attain,

$$F(d_s(\eta, \eta, \mu)) = F(d_s(\theta\eta, \theta\eta, \theta\mu)) < \lambda + F(d_s(\theta\eta, \theta\eta, \theta\mu)) \leq F(d_s(\eta, \eta, \mu)) + N_\gamma\left(\eta, \mu\right),$$

where

$$\begin{aligned} N_\gamma\left(\eta, \mu\right) &= \gamma \min\left\{d_s\left(\eta, \eta, \theta\mu\right),\ d_s\left(\eta, \eta, \theta\eta\right),\ d_s\left(\mu, \mu, \theta\mu\right),\ d_s\left(\mu, \mu, \theta\eta\right)\right\} \\ &= 0 \text{ such that } \gamma \geq 0, \end{aligned}$$

which is a contradiction. Thus, θ *has exactly one fixed point in* η^*. □

Example 2.1 *Take* $\mathcal{M} = R$. *Define the* $S-$*metric* $d_s : \mathcal{M}^3 \to [0, \infty)$ *as follows;*

$$d_s(\kappa, y, l) = |\kappa - l| + |y - l| + |\kappa - y| \textit{ for every } \kappa, y, l \in \mathcal{M}.$$

Consider the map $\theta : \mathcal{M} \to \mathcal{M}$ *given by:*

$$\theta(a) = \frac{a}{2} + c \text{ for every } a \in \mathcal{M}.$$

Next, take into account the function $F : R_+ \to R_+$ *as:*

$$F(t) = kt \text{ for any } k \in R_+.$$

We proceed to confirm the conditions of the theorem in the following way:

$$d_s(\theta a, \theta a, \theta b) > 0 \Rightarrow \lambda + F(d_s(\theta a, \theta a, \theta b)) \leq F(d_s(a, a, b)) + N_\gamma (a, b)$$

where

$$N_\gamma (a, b) = \gamma \min \left\{ d_s (a, a, \theta b),\ d_s (a, a, \theta a),\ d_s (b, b, \theta b),\ d_s (b, b, \theta a) \right\}.$$

Assume the following expressions:

$$\begin{aligned} \theta(a) &= \frac{a}{2} + c, \\ \theta(b) &= \frac{b}{2} + c, \\ d_s(\theta a, \theta a, \theta b) &= |a - b|, \\ F(d_s(\theta a, \theta a, \theta b)) &= k |a - b|, \\ d_s(a, a, b) &= |a - b| + |a - b| = 2 |a - b|, \\ F(d_s(a, a, b)) &= 2k |a - b|. \end{aligned}$$

Accordingly, the following expression is derived from the above considerations for $\lambda > 0$ *and* $\gamma \geq 0$,

$$\begin{aligned} & \lambda + k |a - b| \\ \leq\ & 2k |a - b| + \gamma \min \left\{ \begin{array}{c} d_s (a, a, \theta b),\ d_s (a, a, \theta a), \\ d_s (b, b, \theta b),\ d_s (b, b, \theta a) \end{array} \right\} \\ \leq\ & 2k |a - b|. \end{aligned}$$

Subsequently, we compute the fixed point of θ

$$\begin{aligned} \theta(a^*) &= a^* \Rightarrow \frac{a^*}{2} + c = a^* \\ a^* &= 2c. \end{aligned}$$

As a final step, we verify that the sequence converges $\theta^n(a_0)_{n=0}^{\infty}$ *for any* $a_0 \in \mathcal{M}$;

$$\begin{aligned} \theta^n(a_0) &= \frac{a_0}{2^n} + (1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}})c \\ \theta^n(a_0) &\to 2c = a^* \text{ as } n \to \infty. \end{aligned}$$

Given the $S-$*metric, The space* $\mathcal{M}$ *fulfills the hypotheses of Theorem 2.1. The map* $\theta(a) = \frac{a}{2} + c$ *is an* $F-$*contraction that satisfies* $\lambda > 0$. *The unique fixed point of* θ *is* $a^* = 2c$. *For any initial point* $a_0 \in \mathcal{M}$ *the sequence* $\theta^n(a_0)$ *converges to* a^*.

Example 2.2 *Let* $\mathcal{M} = R_+$. *Define the* $S-$*metric* $d_s : \mathcal{M}^3 \to [0, \infty)$ *as following;*

$$d_s(\kappa, y, l) = |\kappa - l| + |y - l| + |\kappa - y| \text{ for every } \kappa, y, l \in \mathcal{M}.$$

Take the maps $\theta : \mathcal{M} \to \mathcal{M}$ *and* $F : \mathcal{M} \to \mathcal{M}$ *identified by* $\theta(a) = \ln(a + 2)$ *and* $F(t) = \frac{t}{5}$, *resp.*

Assume the following expressions:

$$\begin{aligned}
d_s(\theta a, \theta a, \theta b) &= \left|\ln(\frac{a+2}{b+2})\right| + \left|\ln\left(\frac{b+2}{a+2}\right)\right|, \\
d_s(a,a,b) &= |a-b| + |b-a| = 2\,|a-b|, \\
F(d_s(a,a,b)) &= 0,4\,|a-b|, \\
F(d_s(\theta a, \theta a, \theta b)) &= 0,4\left|\ln(\frac{a+2}{b+2})\right|, \\
\theta(a) &= \ln(a+2) \textit{ and } \theta(b) = \ln(b+2) \textit{ for all } a,b \in \mathcal{M}, \\
N_\gamma(a,b) &= \gamma \min\left\{d_s(a,a,\theta b),\ d_s(a,a,\theta a),\ d_s(b,b,\theta b),\ d_s(b,b,\theta a)\right\} = 0.
\end{aligned}$$

According to the hypothesis of the Theorem 2.1, we get

$$\lambda + 0,4\left|\ln(\frac{a+2}{b+2})\right| \leq 0,4\,|a-b|.$$

Therefore, $\lambda > 0$ is found to satisfy the inequality.

Let a^* be the fixed point of the map $\theta(a) = \ln(a+2)$ and there exists $a^* \in \mathcal{M}$ such that $\ln(a^*+2) = a^*$, which implies that $(a^*+2) = e^{a^*}$. Solving this equation yields approximately $a^* \approx 1,146$.

Assume that $\{a_n\}$ is a sequence with the initial condition $a_0 > 0$, thus,

$$\begin{aligned}
a_{n+1} &= \theta(a_n) = \ln(a_n+2) \\
a_1 &= \ln(a_0+2) \\
a_2 &= \ln(a_1+2) = \ln(\ln(a_0+2)) \\
a_3 &= \ln(a_2+2) = \ln(\ln(\ln(a_0+2))) \\
&\vdots \\
\ln(a+2) &< a.
\end{aligned}$$

That is, for $a > 0$ the series converges decreasingly to a^*.

3. Conclusion

This approach explores the existence of fixed points for mappings satisfying the conditions of a $F-$contraction in the context of $S-$metric spaces. The theoretical findings have been further substantiated by carefully constructed examples, which illustrate the applicability and robustness of the proposed results. These contributions not only generalize existing fixed point theorems but also enrich the fixed point theory in non-standard metric frameworks, paving the way for potential applications in various branches of mathematical analysis and related fields.

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Süheyla Elmas,
Department of Mathematics and Sciences,
Institute of Science,
Atatürk University
Türkiye.
E-mail address: `suheylaelmas@atauni.edu.tr`

and

Ferdi Çınar,
Department of Mathematics and Sciences,
Institute of Science,
Atatürk University
Türkiye.
E-mail address: `ferdi.cinar18@ogr.atauni.edu.tr`