



## Approximate Fixed Point Computation via $F$ -Contraction in $A$ -Metric Spaces for High-Dimensional Data Clustering

Süheyla Elmas and Aleyna Kamber Özel

**ABSTRACT:** In this study, we investigate Wardowski's contraction principle for  $F$ -contraction mappings and establish the existence and uniqueness of fixed points involving  $A$ -metric spaces. A  $F$ -contraction-based iterative method in an  $A$ -metric space is proposed and shown to converge to an approximate fixed point, demonstrating its effectiveness for clustering in high-dimensional data.

**Keywords:** Data Clustering, fixed points,  $A$ -Metric Spaces,  $F$ -contraction mappings.

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### 1. Introduction and Preliminaries

The Banach contraction principle (BCP) [1] is considered a pivotal result in metric fixed point theory due to its elegant simplicity and broad applicability in numerous areas of mathematics. The theory of fixed points combines ideas from topology, analysis, and geometry to investigate the existence and uniqueness of fixed points of a map. Over time, the BCP has undergone a variety of generalizations in different directions. For example, Wardowski (2012) [2] extended the BCP and formulated the  $F$ -contraction, which inspired a range of subsequent studies on  $F$ -contractions more, [3], [4], [5] and [6].

**Definition 1.1** [2] *Let  $(\Phi, d)$  be a metric space. A map  $H : \Phi \rightarrow \Phi$  is called to be a  $F$ -contraction if there is  $\tau > 0$  such that for any  $\varsigma, \tau \in \Phi$*

$$d(H\varsigma, H\tau) > 0 \implies \tau + F(d(H\varsigma, H\tau)) \leq F(d(\varsigma, \tau)),$$

here  $F : R_+ \rightarrow R$  is a map fulfilling criterias listed below:

- (F1)  $F$  is strictly increasing, viz, for every  $\varsigma, \tau \in R_+$  such that  $\varsigma < \tau$ ,  $F(\varsigma) < F(\tau)$ ;
- (F2) For all sequence  $\{\xi_n\}_{n=1}^\infty \subseteq R_+$ ,  $\lim_{n \rightarrow \infty} \xi_n = 0$  iff  $\lim_{n \rightarrow \infty} F(\xi_n) = -\infty$ ;
- (F3) There is  $k \in (0, 1)$  such that  $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$

The set  $\mathcal{F}$  is defined as the collection of whole maps fulfilling states (F1)-(F3).

Gähler (1963) [8] formulated a 2-metric space as an enlargement of the conventional metric structure. For valuable investigations on fixed point results, Dhage (1992) [9] recommended refining the basic framework of the 2-metric space. In (1984) [10] presented the notion of a  $D$ -metric space. Naidu et al. (2004) [11] demonstrated that  $D$ -metric spaces do not offer a well-defined structure for convergent sequences and sequential continuity. In (2005) [12], they highlighted certain limitations in the notion of open balls in  $D$ -metric spaces. The authors (2006) [13] developed the concept of  $G$ -metric space and examined its topological characteristics. Unlike the theory of  $G$ -metric spaces, here states

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(D1)  $D(\varsigma, \tau, \nu) \geq 0$  for every  $\varsigma, \tau, \nu \in \Phi$  and equality is valid iff  $\varsigma = \tau = \nu$ , where  $D : \Phi^3 \rightarrow R$  is the function;

was replaced by the following three separate axioms, here  $G : \Phi^3 \rightarrow R$  is the function

(G1)  $G(\varsigma, \tau, \nu) = 0$  if  $\varsigma = \tau = \nu$  for every  $\varsigma, \tau, \nu \in \Phi$ ;

(G2)  $G(\varsigma, \tau, \tau) \geq 0$  for every  $\varsigma, \tau \in \Phi$ ;

(G3)  $G(\varsigma, \varsigma, \tau) \leq G(\varsigma, \tau, \nu)$  for every  $\varsigma, \tau, \nu \in \Phi$  via  $\tau \neq \nu$ ;

Sedghi et al. (2007) [14] pointed out that condition (D1) could be substituted by just two essential rules and thereby proposed the concept of a  $D^*$ -metric space.

**Remark 1.1** [7] *Each  $G$ -metric space can be regarded as a  $D^*$ -metric space.*

The authors (2012) [15] highlighted that condition (G3) is a distinct limitation of the  $G$ -metric space, whereas the symmetry condition was identified as a shared weakness in both  $D^*$ - and  $G$ -metric spaces. To tackle these problems, these authors developed a new generalized metric space referred to as a  $S$ -metric space.

**Remark 1.2** [7] *Each  $D^*$ -metric space is also a  $S$ -metric space.*

**Remark 1.3** [7] *The  $S$ -metric space serves as a broader concept that includes both the  $G$ -metric space and the  $D^*$ -metric space.*

Abbas, et al. (2015) [7] gave the concept of an  $A$ -metric space as outlined:

**Definition 1.2** [7] *Let  $\gamma$  be a nonvoid set. A map  $A : \gamma^n \rightarrow [0, \infty)$  is said an  $A$ -metric on  $\gamma$  if for any  $\gamma_{1(i)}, \xi \in \gamma, i = 1, 2, \dots, n$  the following terms apply:*

$$(A1) \ A\left(\gamma_{1(1)}, \gamma_{1(2)}, \dots, \gamma_{1(n-1)}, \gamma_{1(n)}\right) \geq 0;$$

$$(A2) \ A\left(\gamma_{1(1)}, \gamma_{1(2)}, \dots, \gamma_{1(n-1)}, \gamma_{1(n)}\right) = 0$$

*iff*

$$\gamma_{1(1)} = \gamma_{1(2)} = \dots = \gamma_{1(n-1)} = \gamma_{1(n)};$$

$$(A3) \ A\left(\gamma_{1(1)}, \gamma_{1(2)}, \dots, \gamma_{1(n-1)}, \gamma_{1(n)}\right) \leq A\left(\gamma_{1(1)}, \gamma_{1(1)}, \dots, \gamma_{1(n-1)}, \xi\right) \\ + A\left(\gamma_{1(2)}, \gamma_{1(2)}, \dots, \gamma_{1(2)(n-1)}, \xi\right) \\ + A\left(\gamma_{1(3)}, \gamma_{1(3)}, \dots, \gamma_{1(3)(n-1)}, \xi\right) \\ \vdots \\ + A\left(\gamma_{1(n-1)}, \gamma_{1(n-1)}, \dots, \gamma_{1(n-1)(n-1)}, \xi\right) \\ + A\left(\gamma_{1(n)}, \gamma_{1(n)}, \dots, \gamma_{1(n)(n-1)}, \xi\right).$$

*The dual  $(\gamma, A)$  is said an  $A$ -metric space. From here it can be seen that the  $A$ -metric space is actually the  $n$ -dimensional  $S$ -metric space.*

The  $A$ -metric space is a generalization of classical metric spaces, based on the concept of determining the distances between  $n$  distinct points at once. This approach provides a new perspective on spatial relationships and structure.

Encouraged by these points, we first introduce and examine  $F$ -contraction in  $A$ -metric space as indicated below;

**Definition 1.3** *Let  $(\gamma, A)$  be an  $A$ -metric space. A map  $H : \gamma \rightarrow \gamma$  is called to be  $F$ -contraction if there is  $\tau > 0$  such that for any  $\gamma_1, \gamma_2 \in \gamma$ ,*

$$A(H\gamma_1, H\gamma_1, \dots, H\gamma_1, H\gamma_2) > 0 \implies \\ \tau + F(A(H\gamma_1, H\gamma_1, \dots, H\gamma_1, H\gamma_2)) \leq F(A(\gamma_1, \gamma_1, \dots, \gamma_1, \gamma_2))$$

here  $F : R_+ \rightarrow R$  is a satisfying the following states:

(F1)  $F$  is strictly increasing, viz, for every  $\gamma_1, \gamma_2 \in R_+$  such that  $\gamma_1 < \gamma_2$ ,  $F(\gamma_1) < F(\gamma_2)$ ;

(F2) For all sequence  $\{\xi_n\}_{n=1}^\infty \subseteq R_+$ ,  $\lim_{n \rightarrow \infty} \xi_n = 0$  iff  $\lim_{n \rightarrow \infty} F(\xi_n) = -\infty$ ;

(F3) There exist  $c \in (0, 1)$  such that  $\lim_{k \rightarrow 0^+} k^c F(k) = 0$ .

The concepts necessary for developing a new complete  $A$ -metric space from a existing complete  $A$ -metric space will be outlined in the subsequent sections;

**Definition 1.4** [7] Let  $(\gamma, A)$  be an  $A$ -metric space;

(i) A sequence  $\{\gamma_n\}$  is called  $A$ -convergence to a point  $\rho$  if  $\forall \varepsilon > 0, \exists \in \mathbb{N}$ , such that  $n > N$  implies  $A(\gamma_n, \gamma_n, \dots, \rho) < \varepsilon$ . Shortly,  $\lim_{n \rightarrow \infty} A(\gamma_n, \gamma_n, \dots, \rho) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \gamma_n = \rho$ .

(ii) A sequence  $\{\gamma_n\}$  is said a  $A$ -Cauchy sequence if  $\forall \varepsilon > 0, \exists \in \mathbb{N}$ , such that for each  $k, s > N$  implies  $A(\gamma_k, \gamma_k, \dots, \gamma_s) < \varepsilon$ .

(iii) An  $A$ -metric space is called teh complete metric space in which any  $A$ -Cauchy sequence  $A$ -convergent.

**Lemma 1.1** [16] Let  $F : R^+ \rightarrow R$  be an increasing map and  $\{k_n\}_{n=1}^\infty \subseteq R_+$ . Then the following terms apply:

- (a) If  $\lim_{n \rightarrow \infty} F(\xi_n) = -\infty$ , then  $\lim_{n \rightarrow \infty} \xi_n = 0$ ;
- (b) If  $\inf F = -\infty$  and  $\lim_{n \rightarrow \infty} \xi_n = 0$ , then  $\lim_{n \rightarrow \infty} F(\xi_n) = -\infty$ .

By proving Lemma 1.1, Secelan [17] verified that the condition (F2) in Definition 1.1 can be substituted with an equivalent, simpler condition,

$$(F2') \inf F = -\infty$$

as an alternative, additionally, with

$$(F2'') \text{ there exist a sequence } \{\xi_n\}_{n=1}^\infty \subseteq R_+ \text{ such that } \lim_{n \rightarrow \infty} F(\xi_n) = -\infty.$$

In the current paper, we use  $F$ -contraction maps in  $A$ -metric spaces and aim to extend Wardowski's theorem to  $A$ -metric spaces. There has been a rising interest in the generalization of classical metric spaces in the past few years. In this framework, 2-metric,  $D$ -metric, and  $G$ -metric spaces are viewed as generalized forms of usual metric spaces. The most crucial of these generalizations is  $A$ -metric spaces, since this space was developed to measure the distance between  $n$  points at the same time.

## 2. Fixed Point Results

**Theorem 2.1** Let  $(\gamma, A)$  be a complete  $A$  metric space and let a map  $H : \gamma_1 \rightarrow \gamma_1$  be a  $F$ -contraction. Suppose  $F \in \mathcal{F}$  and  $\tau > 0$  such that for any  $\gamma_{1(1)}, \gamma_{1(2)} \in \gamma$ ,

$$A(H\gamma_1, H\gamma_1, \dots, H\gamma_1, H\gamma_2) > 0 \implies \tau + F(A(H\gamma_1, H\gamma_1, \dots, H\gamma_1, H\gamma_2)) \leq F(A(\gamma_1, \gamma_1, \dots, \gamma_1, \gamma_2))$$

Then  $H$  has a unique fixed point  $\gamma_1^* \in \gamma_1$  and for each  $\gamma_{1(0)} \in \gamma_1$  the sequence  $\{H^n \gamma_{1(0)}\}_{n=1}^\infty$  converges to  $\gamma_1^*$ .

**Proof:** Select  $\gamma_{1(0)} \in \gamma_1$  and express a sequence  $\{\gamma_{1(n)}\}_{n=1}^\infty$  by

$$\gamma_{1(1)} = H\gamma_{1(0)}, \gamma_{1(2)} = H\gamma_{1(1)} = H\gamma_{1(0)}^2, \dots, \gamma_{1(n+1)} = H\gamma_{1(n)} = H\gamma_{1(0)}^{n+1}, \quad (2.1)$$

for each  $n \in N$ . If there is  $n \in N$  such that  $A(\gamma_{1_n}, \gamma_{1_n}, \dots, \gamma_{1_n}, H\gamma_{1_n}) = 0$ , the proof is concluded. Thus, we presume that for every  $n \in N$

$$0 < A(\gamma_{1_n}, \gamma_{1_n}, \dots, \gamma_{1_n}, H\gamma_{1_n}) = A(H\gamma_{1_{n-1}}, H\gamma_{1_{n-1}}, \dots, H\gamma_{1_{n-1}}, H\gamma_{1_n}). \quad (2.2)$$

For all  $n \in N$ , we attain

$$\tau + F\left(A\left(H\gamma_{1_{n-1}}, H\gamma_{1_{n-1}}, \dots, H\gamma_{1_{n-1}}, H\gamma_{1_n}\right)\right) \leq F\left(A\left(\gamma_{1_{n-1}}, \gamma_{1_{n-1}}, \dots, \gamma_{1_{n-1}}, \gamma_{1_n}\right)\right),$$

i.e.,

$$F\left(A\left(H\gamma_{1_{n-1}}, H\gamma_{1_{n-1}}, \dots, H\gamma_{1_{n-1}}, H\gamma_{1_n}\right)\right) \leq F\left(A\left(\gamma_{1_{n-1}}, \gamma_{1_{n-1}}, \dots, \gamma_{1_{n-1}}, \gamma_{1_n}\right)\right) - \tau.$$

Reapplying this method, we arrive at

$$\begin{aligned}
& F\left(A\left(H\gamma_{1_{n-1}}, H\gamma_{1_{n-1}}, \dots, H\gamma_{1_{n-1}}, H\gamma_{1_n}\right)\right) \\
& \leq F\left(A\left(\gamma_{1_{(n-1)}}, \gamma_{1_{(n-1)}}, \dots, \gamma_{1_{(n-1)}}, \gamma_{1_{(n)}}\right)\right) - \tau \\
& = F\left(A\left(H\gamma_{1_{(n-2)}}, H\gamma_{1_{(n-2)}}, \dots, H\gamma_{1_{(n-2)}}, H\gamma_{1_{(n-1)}}\right)\right) - \tau \\
& \leq F\left(A\left(\gamma_{1_{(n-2)}}, \gamma_{1_{(n-2)}}, \dots, \gamma_{1_{(n-2)}}, \gamma_{1_{(n-1)}}\right)\right) - 2\tau \\
& = F\left(A\left(H\gamma_{1_{(n-3)}}, H\gamma_{1_{(n-3)}}, \dots, H\gamma_{1_{(n-3)}}, H\gamma_{1_{(n-2)}}\right)\right) - 2\tau \\
& \leq F\left(A\left(\gamma_{1_{(n-3)}}, \gamma_{1_{(n-3)}}, \dots, \gamma_{1_{(n-3)}}, \gamma_{1_{(n-2)}}\right)\right) - 3\tau \\
& \quad \vdots \\
& \leq F\left(A\left(\gamma_{1_{(o)}}, \gamma_{1_{(o)}}, \dots, \gamma_{1_{(o)}}, \gamma_{1_{(1)}}\right)\right) - n\tau.
\end{aligned} \tag{2.3}$$

Due to (2.3), we acquire  $\lim_{n \rightarrow \infty} F\left(A\left(\gamma_{1_{n-1}}, H\gamma_{1_{n-1}}, \dots, H\gamma_{1_{n-1}}, H\gamma_{1_n}\right)\right) = -\infty$ , which together with (F2') and Lemma 1.1 gives

$$\lim_{n \rightarrow \infty} A\left(\gamma_{1_{n-1}}, H\gamma_{1_{n-1}}, \dots, H\gamma_{1_{n-1}}, H\gamma_{1_n}\right) = 0$$

i.e.,

$$\lim_{n \rightarrow \infty} A(\gamma_{1_n}, \gamma_{1_n}, \dots, \gamma_{1_n}, H\gamma_{1_n}) = 0. \tag{2.4}$$

Next, we maintain that  $\{\gamma_{1_n}\}_{n=1}^{\infty}$  is a Cauchy sequence. Reasoning by contradiction, we presume that there exist  $\epsilon > 0$  and sequences  $\{k_{(n)}\}_{n=1}^{\infty}$ ,  $\{m_{(n)}\}_{n=1}^{\infty} \subseteq N$  such that  $k_{(n)} > m_{(n)} > n$  for every  $n \in N$

$$A\left(\gamma_{1_{k_{(n)}}}, \gamma_{1_{k_{(n)}}}, \dots, \gamma_{1_{k_{(n)}}}, \gamma_{1_{m_{(n)}}}\right) \geq \epsilon \tag{2.5}$$

$$\left(\gamma_{1_{k_{(n)}-1}}, \gamma_{1_{k_{(n)}-1}}, \dots, \gamma_{1_{k_{(n)}-1}}, \gamma_{1_{m_{(n)}}}\right) < \epsilon. \tag{2.6}$$

So, we have

$$\begin{aligned}
\epsilon &\leq A\left(\gamma_{1_{k(n)}}, \gamma_{1_{k(n)}}, \dots, \gamma_{1_{k(n)}}, \gamma_{1_{m(n)}}\right) \\
&\leq (n-1)A\left(\gamma_{1_{k(n)}}, \gamma_{1_{k(n)}}, \dots, \gamma_{1_{k(n)}}, \gamma_{1_{m(n)}}\right) \\
&\quad + A\left(\gamma_{1_{k(n)-1}}, \gamma_{1_{k(n)-1}}, \dots, \gamma_{1_{k(n)-1}}, \gamma_{1_{m(n)}}\right) \\
&\leq (n-1)A\left(\gamma_{1_{k(n)}}, \gamma_{1_{k(n)}}, \dots, \gamma_{1_{k(n)}}, \gamma_{1_{m(n)}}\right) + \epsilon \\
&\leq (n-1)A\left(\gamma_{1_{k(n)}}, \gamma_{1_{k(n)}}, \dots, \gamma_{1_{k(n)}}, H\gamma_{1_{k(n)}}\right) + \epsilon \\
&\leq (n-1)A\left(\gamma_{1_{k(n)}}, \gamma_{1_{k(n)}}, \dots, \gamma_{1_{k(n)}}, \gamma_{1_{m(n)}}\right) + \epsilon \\
&\leq (n-1)F\left[A\left(H\gamma_{1_{k(n)-1}}, H\gamma_{1_{k(n)-1}}, \dots, H\gamma_{1_{k(n)-1}}, H\gamma_{1_{k(n)}}\right)\right] \\
&\leq (n-1)\left[F\left[A\left(\gamma_{1_{k(n)-1}}, \gamma_{1_{k(n)-1}}, \dots, \gamma_{1_{k(n)-1}}, \gamma_{1_{k(n)}}\right)\right] - \tau\right] \\
&= (n-1)\left[F\left[A\left(H\gamma_{1_{k(n)-2}}, H\gamma_{1_{k(n)-2}}, \dots, H\gamma_{1_{k(n)-2}}, H\gamma_{1_{k(n)-1}}\right)\right] - \tau\right] \\
&\leq (n-1)\left[F\left[A\left(\gamma_{1_{k(n)-2}}, \gamma_{1_{k(n)-2}}, \dots, \gamma_{1_{k(n)-2}}, \gamma_{1_{k(n)-1}}\right)\right] - 2\tau\right] \\
&= (n-1)\left[F\left[A\left(H\gamma_{1_{k(n)-3}}, H\gamma_{1_{k(n)-3}}, \dots, H\gamma_{1_{k(n)-3}}, H\gamma_{1_{k(n)-2}}\right)\right] - 2\tau\right] \\
&\leq (n-1)\left[F\left[A\left(\gamma_{1_{k(n)-3}}, \gamma_{1_{k(n)-3}}, \dots, \gamma_{1_{k(n)-3}}, \gamma_{1_{k(n)-2}}\right)\right] - 3\tau\right] \\
&\quad \vdots \\
&\leq (n-1)\left[F\left[A\left(\gamma_{1_{k(0)}}, \gamma_{1_{k(0)}}, \dots, \gamma_{1_{k(0)}}, \gamma_{1_{k(1)}}\right)\right] - n\tau\right].
\end{aligned}$$

From above inequality, we obtain

$$\lim_{n \rightarrow \infty} F\left[A\left(H\gamma_{1_{k(n)-1}}, H\gamma_{1_{k(n)-1}}, \dots, H\gamma_{1_{k(n)-1}}, H\gamma_{1_{k(n)}}\right)\right] = -\infty$$

which combined with  $(F2')$  and Lemma 1.1 gives rise to

$$\lim_{n \rightarrow \infty} A\left(H\gamma_{1_{k(n)-1}}, H\gamma_{1_{k(n)-1}}, \dots, H\gamma_{1_{k(n)-1}}, H\gamma_{1_{k(n)}}\right) = \epsilon. \quad (2.7)$$

On the other hand from (2.4) there is  $m \in N$  such that

$$A\left(\gamma_{1_{k(n)}}, \gamma_{1_{k(n)}}, \dots, \gamma_{1_{k(n)}}, H\gamma_{1_{k(n)}}\right) < \frac{\epsilon}{4(n-1)} \text{ and} \quad (2.8)$$

$$A\left(\gamma_{1_{m(n)}}, \gamma_{1_{m(n)}}, \dots, \gamma_{1_{m(n)}}, H\gamma_{1_{k(n)}}\right) < \frac{\epsilon}{4}, \quad \forall n \geq m. \quad (2.9)$$

Next we affirm that for any  $n \geq m$

$$A\left(H\gamma_{1_{k(n)}}, H\gamma_{1_{k(n)}}, \dots, H\gamma_{1_{k(n)}}, H\gamma_{1_{m(n)}}\right) = A\left(\gamma_{1_{k(n)+1}}, \gamma_{1_{k(n)+1}}, \dots, \gamma_{1_{k(n)+1}}, \gamma_{1_{m(n)+1}}\right) > 0. \quad (2.10)$$

Arguing by contraction, there is  $r \geq m$  such that

$$A\left(\gamma_{1_{k(r)+1}}, \gamma_{1_{k(r)+1}}, \dots, \gamma_{1_{k(r)+1}}, \gamma_{1_{m(r)+1}}\right) = 0. \quad (2.11)$$

It follows from (2.5), (2.6), (2.8), (2.9) and (2.11) that

$$\begin{aligned}
& \leq \epsilon \\
& \leq A\left(\gamma_{1_{k(r)}}, \gamma_{1_{k(r)}}, \dots, \gamma_{1_{k(r)}}, \gamma_{1_{m(r)}}\right) \\
& \leq (n-1)A\left(\gamma_{1_{k(r)}}, \gamma_{1_{k(r)}}, \dots, \gamma_{1_{k(r)}}, \gamma_{1_{k(r)+1}}\right) \\
& \quad + A\left(\gamma_{1_{k(r)+1}}, \gamma_{1_{k(r)+1}}, \dots, \gamma_{1_{k(r)+1}}, \gamma_{1_{m(r)}}\right) \\
& \leq (n-1)A\left(\gamma_{1_{k(r)}}, \gamma_{1_{k(r)}}, \dots, \gamma_{1_{k(r)}}, \gamma_{1_{k(r)+1}}\right) \\
& \quad + (n-1)A\left(\gamma_{1_{k(r)+1}}, \gamma_{1_{k(r)+1}}, \dots, \gamma_{1_{k(r)+1}}, \gamma_{1_{m(r)+1}}\right) \\
& \quad + A\left(\gamma_{1_{m(r)+1}}, \gamma_{1_{m(r)+1}}, \dots, \gamma_{1_{m(r)+1}}, \gamma_{1_{m(r)}}\right) \\
& = (n-1)A\left(\gamma_{1_{k(r)}}, \gamma_{1_{k(r)}}, \dots, \gamma_{1_{k(r)}}, H\gamma_{1_{k(r)}}\right) \\
& \quad + (n-1)A\left(\gamma_{1_{k(r)+1}}, \gamma_{1_{k(r)+1}}, \dots, \gamma_{1_{k(r)+1}}, \gamma_{1_{m(r)+1}}\right) \\
& \quad + A\left(\gamma_{1_{m(r)}}, \gamma_{1_{m(r)}}, \dots, \gamma_{1_{m(r)}}, H\gamma_{1_{m(r)}}\right) \\
& < \epsilon.
\end{aligned}$$

This contradiction established the relation (2.10). Consequently, it can be inferred from (2.10) and the hypothesis of the theorem that

$$\begin{aligned}
& \tau + F\left(A\left(H\gamma_{1_{k(n)}}, H\gamma_{1_{k(n)}}, \dots, H\gamma_{1_{k(n)}}, H\gamma_{1_{m(n)}}\right)\right) \\
& \leq F\left(A\left(\gamma_{1_{k(n)}}, \gamma_{1_{k(n)}}, \dots, \gamma_{1_{k(n)}}, \gamma_{1_{m(n)}}\right)\right).
\end{aligned} \tag{2.12}$$

From (F3'), (2.7) and (2.12) we gather  $\tau + F(\epsilon) \leq F(\epsilon)$ . This contradiction confirms that  $\{\gamma_{1_n}\}_{n=1}^{\infty}$  is a Cauchy sequence. Owing to completeness of  $(\gamma_1, A)$ ,  $\{\gamma_{1_n}\}_{n=1}^{\infty}$  converges to some point  $\gamma_1$  in  $\gamma$ . Finally, the continuity of  $H$  yields.

$$\begin{aligned}
A(H\gamma_1, H\gamma_1, \dots, H\gamma_1, \gamma_1) &= \lim_{n \rightarrow \infty} A\left(H\gamma_{1_{k(n)}}, H\gamma_{1_{k(n)}}, \dots, H\gamma_{1_{k(n)}}, \gamma_{1_{k(n)}}\right) \\
&= A(\gamma_1^*, \gamma_1^*, \dots, \gamma_1^*) = 0.
\end{aligned}$$

At this point, let us illustrate that  $H$  possesses at most one fixed point. In fact, if  $\gamma_1, \gamma_2 \in \gamma$  be two distinctive fixed points of  $H$ , namely,  $H\gamma_1 = \gamma_1 \neq \gamma_2 = H\gamma_2$ . Therefore

$$A(H\gamma_1, H\gamma_1, \dots, H\gamma_1, H\gamma_2) = A(\gamma_1, \gamma_1, \dots, \gamma_1, \gamma_2) > 0,$$

then we get

$$\begin{aligned}
& F(A(\gamma_1, \gamma_1, \dots, \gamma_1, \gamma_2)) \\
& = F(A(H\gamma_1, H\gamma_1, \dots, H\gamma_1, H\gamma_2)) \\
& < \tau + F(A(H\gamma_1, H\gamma_1, \dots, H\gamma_1, H\gamma_2)) \\
& \leq F(A(\gamma_1, \gamma_1, \dots, \gamma_1, \gamma_2)),
\end{aligned}$$

which is a contradiction. Therefore, the fixed point is unique.  $\square$

### 3. Application: Approximate Fixed Point Computation via $A$ -Metric and $F$ -Contraction in $R^5$

In this section, we demonstrate an application of the  $F$ -contraction principle using a specifically defined  $A$ -metric in order to compute an approximate fixed point (i.e., cluster center) for a set of data points in a five-dimensional space.

We consider a randomly generated dataset  $\gamma = \{\gamma_i\}_{i=1}^{\infty} \subset R^5$ , where each  $\gamma_i \in R^5$  is produced using a uniform distribution. The generalized metric structure is introduced through the  $A$ -metric defined as follows:

$$A(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = \|\gamma_1 - \gamma_5\|^2 + \|\gamma_2 - \gamma_5\|^2 + \|\gamma_3 - \gamma_5\|^2 + \|\gamma_4 - \gamma_5\|^2.$$

This metric evaluates the sum of squared Euclidean distances from four fixed points to a fifth candidate point and serves as a measure of deviation or dispersion in the space.

To initialize the process, a point  $\gamma \in R^5$  is selected at random as the starting estimate of the fixed point. A perturbed version

$$\gamma_{\text{perturbed}} = \gamma + 0.01 \times N(0, I)$$

is used to compute the initial metric values, where  $N(0, I)$  denotes a sample from the standard multivariate normal distribution with zero mean vector and identity covariance matrix in  $R^5$ . This perturbation ensures that  $A_1 > 0$  while keeping the perturbation magnitude small.

The computed metric values at initialization are:

$$\begin{aligned} A_1 &= A(\gamma, \gamma, \gamma, \gamma, \gamma_{\text{perturbed}}) \approx 0.00186875, \\ A_2 &= A(\gamma, \gamma, \gamma, \gamma, \gamma) = 0. \end{aligned}$$

Since  $A_1 > 0$  and  $A_2 = 0$ , the  $F$ -contraction condition  $F(A_1) > F(A_2)$  is satisfied for the identity function  $F(t) = t$ , thereby justifying the applicability of the iterative process.

The fixed point iteration proceeds as follows:

(i) For each step  $j$ , the weights are computed based on the inverse Euclidean distances from the current estimate  $\gamma_j$  to all data points, stabilized by a small constant  $\epsilon$ .

(ii) A new point  $\gamma_{j+1}$  is calculated as the weighted average of all data points in  $\gamma$ .

(iii) The process repeats until convergence is achieved, i.e.,  $\|\gamma_{j+1} - \gamma_j\| < \text{tol}$ .

The algorithm is guaranteed to preserve the  $F$ -contraction condition at each step because the fixed reference points are always chosen as the current  $\gamma$ , resulting in  $A_2 = 0$  throughout the iterations.

The Matlab R2016a implementation confirms that the algorithm converges rapidly, typically within a few iterations, and yields a point that serves as the cluster center under the proposed  $A$ -metric.

The following pseudocode summarizes the iterative process:

**Algorithm 1** *Pseudocode: A-metric Based Fixed Point Algorithm*

```

% Input:
%  $\gamma$  - Dataset in  $R^5$  (size:  $n \times 5$ )
%  $\gamma$  - Initial point in  $R^5$ 
% epsilon - Small constant for numerical stability (e.g.,  $1e - 6$ )
% tau - Tolerance for convergence (e.g.,  $1e - 5$ )
% Define A-metric:
%  $A(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = \text{norm}(\gamma_1 - \gamma_5)^2 + \text{norm}(\gamma_2 - \gamma_5)^2 + \dots$ 
%  $\text{norm}(\gamma_3 - \gamma_5)^2 + \text{norm}(\gamma_4 - \gamma_5)^2$ 
% Step 1: Apply small perturbation to  $\gamma$ 
 $\gamma_{\text{perturbed}} = x + 0.01 * \text{randn}(1, 5)$ ; % random noise from  $N(0, I)$ 
% Step 2: Compute initial  $A_1$  and  $A_2$ 
 $A_1 = A(\gamma, \gamma, \gamma, \gamma, \gamma_{\text{perturbed}})$ ;
 $A_2 = A(\gamma, \gamma, \gamma, \gamma, \gamma)$ ; % This will be zero
% Step 3: Check contraction condition
if  $A_1 > 0 \&\& A_2 == 0$ 
% Proceed with iteration
else
error('F-contraction condition not satisfied at initialization.');
```

```

end
% Step 4: Iterative process
for  $j = 1 : \text{max\_iter}$ 
```

```

% 4a: Compute distances and weights
```

```

diffs = Set - $\gamma$ ; % difference vectors
dists = sqrt(sum(diffs.^2,2)) + epsilon; % Euclidean distances + $\epsilon$ 
weights = 1 ./ dists; % inverse-distance weights
% 4b: Weighted average (new  $\gamma$ )
 $\gamma$ _new = sum(Set.*weights,1)/sum(weights);
% 4c: Compute updated A1 and A2
A1 = A( $\gamma$ ,  $\gamma$ ,  $\gamma$ ,  $\gamma$ ,  $\gamma$ _new);
A2 = A( $\gamma$ ,  $\gamma$ ,  $\gamma$ ,  $\gamma$ ,  $\gamma$ _new); %A2still = 0since  $p_i = \gamma$ 
% 4d: Check convergence
if norm( $\gamma$ _new -  $\gamma$ ) < tau
disp(['Convergence achieved at iteration ', num2str(j)]);
break;
else
 $\gamma$  =  $\gamma$ _new; % update current point
end
end
% Output:
%  $\gamma$  - Approximate fixed point (cluster center)

```

To illustrate the result, the data is projected onto the first three dimensions and visualized in 3D space along with the computed fixed point.

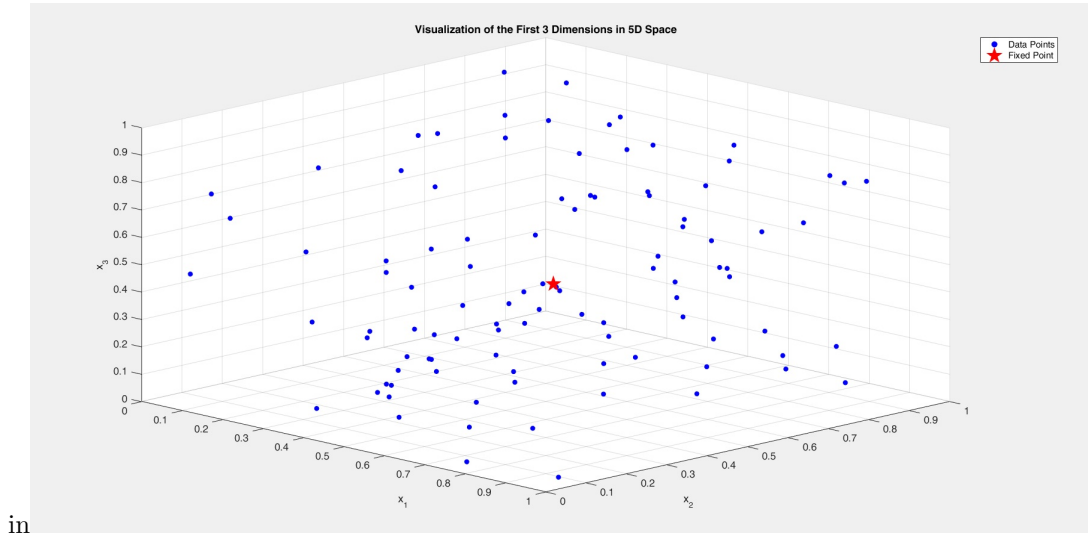


Figure 1: Projection of 5-dimensional dataset onto the first three dimensions

The red marker indicates the approximate fixed point (cluster center) computed via  $A$ -metric-based  $F$ -contraction iteration. The iterative procedure converged after 8 iterations. The approximate fixed point, representing the cluster center in the 5-dimensional space, is computed as:

$$\text{fixed\_point} = (0.5251, 0.4936, 0.4398, 0.4231, 0.5045).$$

This result confirms the existence of a fixed point under the defined  $A$ -metric and demonstrates the effectiveness of the  $F$ -contraction-based iterative scheme in high-dimensional data clustering.

#### 4. Conclusion

In this study, we investigated the existence and uniqueness of fixed points for  $F$ -contraction maps in the framework of  $A$ -metric spaces, as introduced by Wardowski. To support the theoretical findings, we implemented an iterative algorithm based on an  $A$ -metric that evaluates the deviation of a candidate point from multiple reference points. By employing a simple linear  $F$ -function and enforcing the contraction condition  $F(A_1) > F(A_2)$ , we verified the convergence behavior of the process on a randomly generated dataset in  $R^5$ . The proposed MATLAB-based procedure successfully computes an approximate fixed point, interpreted as the cluster center of the dataset. The  $A$ -metric structure and the weight-based update scheme ensure that the  $F$ -contraction inequality remains satisfied throughout the iterations. The result not only confirms the practical validity of the theoretical framework but also illustrates the potential of fixed point techniques in data-driven geometric computations, such as clustering in high-dimensional spaces. This application highlights the effectiveness of  $A$ -metric and  $F$ -contraction-based fixed point algorithms in concrete settings, offering a computational perspective for further research in nonlinear analysis and metric fixed point theory.

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*Süheyla Elmas,*  
*Department of Mathematics and Sciences,*  
*Faculty of Education,*  
*Atatürk University*

*Türkiye.*

*E-mail address:* `suheylaelmas@atauni.edu.tr`

*and*

*Aleyna Kamber Özel,*

*Department of Mathematics and Sciences,*

*Faculty of Education,*

*Atatürk University*

*Türkiye.*

*E-mail address:* `aleyna.kamber16@ogr.atauni.edu.tr`