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Somber Index of k-Splitting Graphs

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ABSTRACT: The objective of this paper is to derive explicit formulas for the Sombor index of the k-splitting of generalized transformation graphs, denoted as $spl_k\left(G^{ab}\right)$. The study systematically develops analytical expressions for the Sombor index in terms of the parameters of the original graph and the splitting factor k. Furthermore, analogous formulas are established for the complements of $spl_k\left(G^{ab}\right)$, providing a comprehensive characterization of their structural properties.

Keywords: Sombor index, generalized transformation graphs, k-Splitting of graphs.

Contents

1. Introduction

The vertices within the molecular graph are representative of the atoms constituting the molecule, while the edges signify the covalent bonds that interconnect these atoms. A topological index is defined as a graph invariant that assigns a unique real number to each molecular graph. A plethora of such descriptors has been examined in the realm of theoretical chemistry and has demonstrated practical applications, particularly within the contexts of Quantitative Structure-Activity Relationship (QSAR) and Quantitative Structure-Property Relationship (QSPR) [13,35]. Topological indices are classified into two principal categories, specifically degree-based indices and distance-based indices. Noteworthy examples of degree-based indices include the first Zagreb index, the second Zagreb index, the forgotten index, the hyper Zagreb index, the Randic index, the harmonic index, the geometric-arithmetic index, and the redefined third Zagreb index, among others. For a comprehensive exploration of degree-based topological indices, we direct the reader to reference [7,9,10,11,12].

Among the diverse array of degree-based indices, the first and second Zagreb indices have garnered substantial scholarly attention. In the year 1972, Gutman et al. [9] formally introduced the first and second Zagreb indices pertaining to a graph G. To date, numerous researchers worldwide are delving into these indices, pushing the boundaries of knowledge to advanced methodologies. Subsequently, in 2008, Došlić [6] articulated the definitions of the first and second Zagreb coindices, which pertain to all non-adjacent pairs of vertices. For further insights regarding the Zagreb indices and coindices, as well as their various applications, refer to [3,12,16,20,34,38]. In the year 2015, Basavanagoud et al. [4] introduced novel graph operations referred to as generalized transformations denoted as G^{ab}, and derived

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the formulations for both the first and second Zagreb indices as well as the coindices pertaining to these graphs and their corresponding complements. In 2017, Vaidya et al. [37] proposed a new graph operation termed k-splitting of a graph G and conducted an analysis on the energy associated with this operation.

In the current study, we concentrate on deriving explicit formulations for the forgotten index pertaining to the k-splitting of generalized transformation graphs $spl_k\left(G^{ab}\right)$. Subsequently, we also derive comparable expressions for the complements of $spl_k\left(G^{ab}\right)$.

2. Preliminaries

Throughout this manuscript, we examine undirected, simple, and finite graphs. Let G be defined as a graph characterized by the vertex set $V(G)$ and the edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$, with the elements of $V(G)$ referred to as vertices and the elements of $E(G)$ designated as edges of the graph G. Two vertices are considered adjacent in G if they are connected by a common edge, which is described as being incident to those two vertices. The degree of a vertex u in $V(G)$, represented by $d_G(u)$, quantifies the number of edges that are incident to u. The complement $\overline{G}$ of graph G constitutes a simple graph that retains the same vertex set as G, whereby two vertices u and v are adjacent in G if and only if they are non-adjacent in $\overline{G}$. For newly topological indices, we suggest the readers to refer the papers [1,2,14,15,17,18,19,21,22,23,24,25,26,27,28,29,30,32,33].

The first Zagreb index $M_1(G)$ [9] and second Zagreb index $M_2(G)$ [9] are defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 \quad and \quad M_2(G) = \sum_{uv\in E(G)} d_G(u)\ d_G(v)$$

respectively.

The first Zagreb can also be expressed as [5]

$$M_1(G) = \sum_{uv\in E(G)} \left[d_G(u) + d_G(v)\right].$$

I. Gutman [8] put forward a novel topological index namely the Sombor index which is defined as

$$SO(G) \;=\; \sum_{uv\in E(G)} [\sqrt{deg(u)^2 + deg(v)^2}]$$

3. The k-Splitting Graph

Let G denote a graph characterized by the vertex set $V(G)$ and the edge set $E(G)$, and let α and β represent two elements belonging to the union $V(G)\cup E(G)$. The associativity of the elements α and β is defined as $+$ if they exhibit adjacency or incidence within the graph G; conversely, it is denoted as $-$ in the absence of such a relationship. Let ab signify a 2-permutation of the set $\{+,-\}$. The elements α and β are said to correspond to the initial term a of ab if both entities reside within $V(G)$, whereas they correspond to the subsequent term b of ab if one of the entities is contained in $V(G)$ and the other is situated in $E(G)$. The generalized transformation graph G^{ab} is formulated on the vertex set $V(G)\cup E(G)$. A pair of vertices α and β in G^{ab} is connected by an edge if and only if their associativity within G aligns with the corresponding term of ab. In this context, there exist four distinct graphical transformations of graphs, specifically G^{++}, G^{+-}, G^{-+}, and G^{--}, corresponding to the four unique 2-permutations of the set $\{+,-\}$.

In an alternative formulation, the generalized transformation graph G^{ab} constitutes a graph characterized by the vertex set $V(G)\cup E(G)$, where $\alpha,\beta\in V\left(G^{ab}\right)$ are considered to be adjacent within G^{ab} if and only if the conditions (i) and (ii) are satisfied:
(i) $\alpha,\beta\in V\left(G^{ab}\right)$, with α and β being adjacent in G under the condition that $a=+$, whereas α and β are non-adjacent in G when $a=-$.
(ii) If $\alpha\in V(G)$ and $\beta\in E(G)$, then α and β are incident in G when $b=+$, while α and β are not incident in G if $b=-$.

The vertex u of G^{ab} that corresponds to a vertex u of G is designated as a point vertex. Conversely, the vertex e of G^{ab} that corresponds to an edge e of G is termed a line vertex.

The k-splitting of a graph G, denoted as $spl_k(G)$, is derived by augmenting each vertex u of G with k additional vertices, labeled as u', u'', ..., $u^{(k)}$, in such a manner that $u^{(i)}$, where $1 \leq i \leq k$, is adjacent to every vertex that shares adjacency with u in the original graph G.

Now we present the main results of our work through following sections.

4. Sombor Index of $spl_k(G^{++})$

Let G be a (n, m)-graph and $spl_k(G^{++})$ represents k-splitting of G^{++}. For P_4, the structure of 2-splitting graph of path P_4^{++}, is shown in the Figure 1

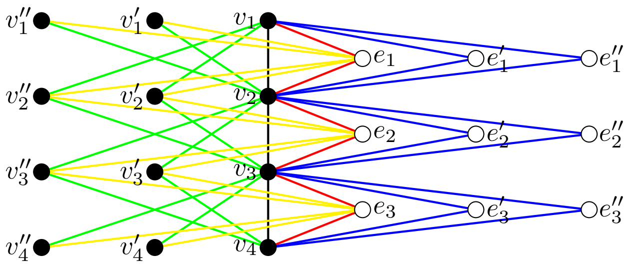

Figure 1: $spl_2\left(P_4^{++}\right)$: 2-splitting of P_4^{++}

Proposition 4.1 *Let G be a (n, m)-graph. Then*

1. $d_{spl_k(G^{++})}(v) = 2(k+1)d_G(v)$, *where* $v \in V(G)$
2. $d_{spl_k(G^{++})}(e) = 2(k+1)$, *where* $e \in E(G)$
3. $d_{spl_k(G^{++})}(v') = 2d_G(v)$, *where $v' \in G^{++}$ due to vertex v in G*
4. $d_{spl_k(G^{++})}(e') = 2$, *where e' is vertex in k-splitting of G^{++} due to edge e in G.*

Proposition 4.2 *Let G be a (n, m)-graph. Then, order and size of $spl_k(G^{++})$ are $(n+m)(k+1)$ and $3m(2k+1)$ respectively.*

Theorem 4.1 *Let G be a (n, m)-graph. Then*

$$SO\left(spl_k\left(G^{++}\right)\right) = 2(k+1)SO(G) + 2[(k+1)\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4]$$

where

$$\gamma_1 = \sum_{uv \in E_2} \sqrt{d_G(u)^2 + 1}; \qquad \gamma_2 = \sum_{uv \in E_3} \left[\sqrt{(k+1)^2 d_G(u)^2 + d_G(v)^2}\right]$$

$$\gamma_3 = \sum_{uv \in E_4} \left[\sqrt{(k+1)^2 d_G(u)^2 + 1}\right]; \qquad \gamma_4 = \sum_{ue \in E_5} \left[\sqrt{d_G(u)^2 + (k+1)^2}\right]$$

Proof: Partitioning edge set of $spl_k(G^{++})$ as follows:

$$E\left(spl_k(G^{++})\right) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$$\begin{aligned}
E_1 &= \{uv : uv \in E(G)\} \\
E_2 &= \{ue : \text{vertex } u \text{ in } G \text{ is incident to edge } e \text{ in } G\} \\
E_3 &= \left\{uv' : \text{vertex } u \text{ in } G \text{ is adjacent to vertex } v' \in G^{++} \text{ due to vertex } v \text{ in } G\right\} \\
E_4 &= \left\{ue' : \text{vertex } u \text{ in } G \text{ is incident to vertex } e' \in G^{++} \text{ due to edge } e \text{ in } G\right\} \\
E_5 &= \left\{u'e : \text{vertex } u' \in G^{++} \text{ due to vertex } u \text{ in } G \text{ incident to edge } e \text{ in } G\right\}
\end{aligned}$$

Clearly $|E_1| = m,\ |E_2| = 2m,\ |E_3| = 2mk,\ |E_4| = 2mk$ and $|E_5| = 2mk$

We have,

$$\begin{aligned}
SO\left(spl_k\left(G^{++}\right)\right) &= \sum_{uv\in\left(E\left(spl_k\left(G^{++}\right)\right)\right)} \left[\sqrt{d_{spl_k\left(G^{++}\right)}\left(u\right)^2 + d_{spl_k\left(G^{++}\right)}\left(v\right)^2}\right] \\
&= \sum_{uv\in E_1} \left[\sqrt{d_{spl_k\left(G^{++}\right)}\left(u\right)^2 + d_{spl_k\left(G^{++}\right)}\left(v\right)^2}\right] + \sum_{ue\in E_2} \left[\sqrt{d_{spl_k\left(G^{++}\right)}\left(u\right)^2 + d_{spl_k\left(G^{++}\right)}\left(e\right)^2}\right] \\
&+ \sum_{uv'\in E_3} \left[\sqrt{d_{spl_k\left(G^{++}\right)}\left(u\right)^2 + d_{spl_k\left(G^{++}\right)}\left(v'\right)^2}\right] + \sum_{ue'\in E_4} \left[\sqrt{d_{spl_k\left(G^{++}\right)}\left(u\right)^2 + d_{spl_k\left(G^{++}\right)}\left(e'\right)^2}\right] \\
&+ \sum_{u'e\in E_5} \left[\sqrt{d_{spl_k\left(G^{++}\right)}\left(u'\right)^2 + d_{spl_k\left(G^{++}\right)}\left(e\right)^2}\right] \\
&= \sum_{uv\in E_1} \left[\sqrt{4(k+1)^2 d_G\left(u\right)^2 + 4(k+1)^2 d_G\left(v\right)^2}\right] + \sum_{uv\in E_2} \left[\sqrt{4(k+1)^2 d_G\left(u\right)^2 + 4(k+1)^2}\right] \\
&+ \sum_{uv\in E_3} \left[\sqrt{4(k+1)^2 d_G\left(u\right)^2 + 4 d_G\left(v\right)^2}\right] + \sum_{uv\in E_4} \left[\sqrt{4(k+1)^2 d_G\left(u\right)^2 + 4}\right] \\
&+ \sum_{ue\in E_5} \left[\sqrt{4 d_G\left(u\right)^2 + 4(k+1)^2}\right] \\
&= 2(k+1) \sum_{uv\in E_1} \left[\sqrt{d_G\left(u\right)^2 + d_G\left(v\right)^2}\right] + 2(k+1) \sum_{uv\in E_2} \sqrt{d_G\left(u\right)^2 + 1} \\
&+ 2 \sum_{uv\in E_3} \left[\sqrt{(k+1)^2 d_G\left(u\right)^2 + d_G\left(v\right)^2}\right] + 2 \sum_{uv\in E_4} \left[\sqrt{(k+1)^2 d_G\left(u\right)^2 + 1}\right] \\
&+ 2 \sum_{ue\in E_5} \left[\sqrt{d_G\left(u\right)^2 + (k+1)^2}\right] \\
SO\left(spl_k\left(G^{++}\right)\right) &= 2(k+1)SO(G) + 2(k+1) \sum_{uv\in E_2} \sqrt{d_G\left(u\right)^2 + 1} \\
&+ 2 \sum_{uv\in E_3} \left[\sqrt{(k+1)^2 d_G\left(u\right)^2 + d_G\left(v\right)^2}\right] \\
&+ 2 \sum_{uv\in E_4} \left[\sqrt{(k+1)^2 d_G\left(u\right)^2 + 1}\right] + 2 \sum_{ue\in E_5} \left[\sqrt{d_G\left(u\right)^2 + (k+1)^2}\right]
\end{aligned}$$

Thus,

$$SO\left(spl_k\left(G^{++}\right)\right) = 2(k+1)SO(G) + 2[(k+1)\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4]$$

where,

$$\gamma_1 = \sum_{uv\in E_2} \sqrt{d_G\left(u\right)^2 + 1}; \qquad \gamma_2 = \sum_{uv\in E_3} \left[\sqrt{(k+1)^2 d_G\left(u\right)^2 + d_G\left(v\right)^2}\right]$$

$$\gamma_3 = \sum_{uv\in E_4} \left[\sqrt{(k+1)^2 d_G\left(u\right)^2 + 1}\right]; \qquad \gamma_4 = \sum_{ue\in E_5} \left[\sqrt{d_G\left(u\right)^2 + (k+1)^2}\right]$$

□

5. Sombor Index of $spl_k(G^{+-})$

Let G be a (n, m)-graph and $spl_k(G^{+-})$ represents k-splitting of G^{+-}. For path P_4, the structure of 2-splitting graph of P_4^{+-}, is shown in the Figure 2

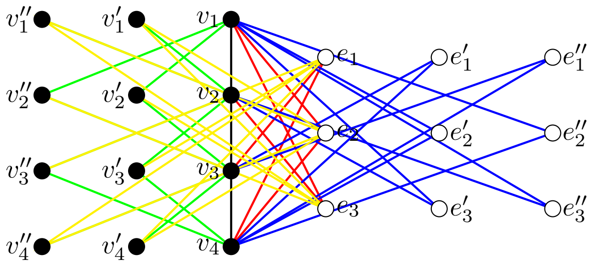

Figure 2: $spl_2\left(P_4^{+-}\right)$: 2-splitting of P_4^{+-}

Proposition 5.1 *Let G be a (n, m)-graph. Then*

1. $d_{spl_k(G^{+-})}(v) = m(k+1)$, *where* $v \in V(G)$
2. $d_{spl_k(G^{+-})}(e) = (n-2)(k+1)$, *where* $e \in E(G)$
3. $d_{spl_k(G^{+-})}(v') = m$, *where* $v' \in G^{+-}$ *due to vertex* v *in* G
4. $d_{spl_k(G^{+-})}(e') = (n-2)$, *where* $e' \in k$*-splitting of* G^{+-} *due to edge* e *in* G

Proposition 5.2 *Let G be a (n, m)-graph. Then order and size of $spl_k(G^{+-})$ are $(n+m)(k+1)$ and $m(n-1)(2k+1)$.*

Theorem 5.1 *Let G be a (n, m)-graph. Then*

$$\begin{aligned} SO\left(spl_k\left(G^{+-}\right)\right) &= m(k+1)[\sqrt{2}m + (n-2)\sqrt{m^2+(n-2)^2}] + 2m^2k\sqrt{(k+1)^2+1} \\ &\quad + mk(n-2)[\sqrt{m^2(k+1)^2+(n-2)^2} + \sqrt{m^2+(n-2)^2(k+1)^2}] \end{aligned}$$

Proof: Partitioning edge set of $spl_k(G^{+-})$ as follows:

$$E\left(spl_k(G^{+-})\right) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$$\begin{aligned} E_1 &= \{uv : uv \in E(G)\} \\ E_2 &= \{ue : \text{vertex } u \text{ in } G \text{ is not incident to edge } e \text{ in } G\} \\ E_3 &= \left\{uv' : \text{vertex } u \text{ in } G \text{ is adjacent to vertex } v' \in G^{+-} \text{ due to vertex } v \text{ in } G\right\} \\ E_4 &= \left\{ue' : \text{vertex } u \text{ in } G \text{ is incident to vertex } e' \in G^{+-} \text{ due to edge } e \text{ in } G\right\} \\ E_5 &= \left\{u'e : \text{vertex } u' \in G^{+-} \text{ due to vertex } u \in G \text{ not incident to edge } e \text{ in } G\right\} \end{aligned}$$

Clearly, $|E_1| = m$, $|E_2| = m(n-2)$, $|E_3| = 2mk$, $|E_4| = mk(n-2)$ and $|E_5| = mk(n-2)$.

Consider,

$$
\begin{aligned}
SO\left(spl_k\left(G^{+-}\right)\right) &= \sum_{uv\in(E(spl_k(G^{+-})))}\left[\sqrt{d_{spl_k(G^{+-})}(u)^2+d_{spl_k(G^{+-})}(v)^2}\right]\\
&= \sum_{uv\in E_1}\left[\sqrt{d_{spl_k(G^{+-})}(u)^2+d_{spl_k(G^{+-})}(v)^2}\right]+\sum_{ue\in E_2}\left[\sqrt{d_{spl_k(G^{+-})}(u)^2+d_{spl_k(G^{+-})}(e)^2}\right]\\
&+ \sum_{uv'\in E_3}\left[\sqrt{d_{spl_k(G^{+-})}(u)^2+d_{spl_k(G^{+-})}(v')^2}\right]+\sum_{ue'\in E_4}\left[\sqrt{d_{spl_k(G^{+-})}(u)^2+d_{spl_k(G^{+-})}(e')^2}\right]\\
&+ \sum_{u'e\in E_5}\left[\sqrt{d_{spl_k(G^{+-})}(u')^2+d_{spl_k(G^{+-})}(e)^2}\right]\\
&= \sum_{uv\in E_1}\left[\sqrt{m^2(k+1)^2+m^2(k+1)^2}\right]+\sum_{ue\in E_2}\left[\sqrt{m^2(k+1)^2+(n-2)^2(k+1)^2}\right]\\
&+ \sum_{uv'\in E_3}\left[\sqrt{m^2(k+1)^2+m^2}\right]+\sum_{ue'\in E_4}\left[\sqrt{m^2(k+1)^2+(n-2)^2}\right]\\
&+ \sum_{u'e\in E_5}\left[\sqrt{m^2+(n-2)^2(k+1)^2}\right]\\
&= \sqrt{2}m(k+1)\sum_{uv\in E_1}1+(k+1)\sqrt{m^2+(n-2)^2}\sum_{ue\in E_2}1+m\sqrt{(k+1)^2+1}\sum_{uv'\in E_3}1\\
&+ \sqrt{m^2(k+1)^2+(n-2)^2}\sum_{ue'\in E_4}1+\sqrt{m^2+(n-2)^2(k+1)^2}\sum_{u'e\in E_5}1\\
SO\left(spl_k\left(G^{+-}\right)\right) &= \sqrt{2}m^2(k+1)+m(n-2)(k+1)\sqrt{(m^2+(n-2)^2)}\\
&+ 2m^2k\sqrt{[(k+1)^2+1]}+mk(n-2)\sqrt{m^2(k+1)^2+(n-2)^2}\\
&+ mk(n-2)\sqrt{m^2+(n-2)^2(k+1)^2}\\
&= m(k+1)[\sqrt{2}m+(n-2)\sqrt{m^2+(n-2)^2}]+2m^2k\sqrt{(k+1)^2+1}\\
&+ mk(n-2)[\sqrt{m^2(k+1)^2+(n-2)^2}+\sqrt{m^2+(n-2)^2(k+1)^2}
\end{aligned}
$$

□

6. Sombor Index of $spl_k(G^{-+})$

Let G be a (n,m)-graph and $spl_k(G^{-+})$ represents k-splitting of G^{-+}. For P_4, the structure of 2-splitting graph of path P_4^{-+}, is shown in the Figure 3

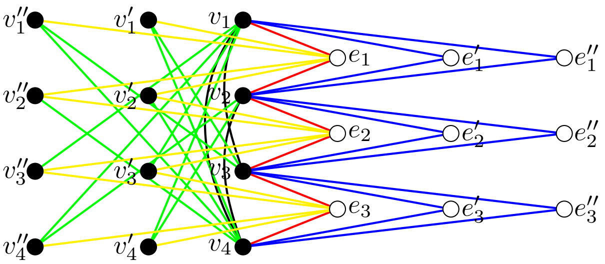

Figure 3: $spl_2\left(P_4^{-+}\right)$: 2-splitting of P_4^{-+}

Proposition 6.1 *Let G be a (n, m)-graph. Then*

1. $d_{spl_k(G^{-+})}(v) = (n-1)(k+1)$, *where* $v \in V(G)$

2. $d_{spl_k(G^{-+})}(e) = 2(k+1)$, *where* $e \in E(G)$

3. $d_{spl_k(G^{-+})}(v') = (n-1)$, *where* $v' \in G^{-+}$ *due to vertex* v *in* G

4. $d_{spl_k(G^{-+})}(e') = 2$, , *where* $e' \in G^{-+}$ *due to edge* e *in* G

Proposition 6.2 *Let G be a (n, m)-graph. Then order and size of $spl_k(G^{-+})$ are $(n+m)(k+1)$ and $\frac{1}{2}[n(n-1)+2m](2k+1)$ respectively.*

Theorem 6.1 *Let G be a (n, m)-graph. Then*

$$\begin{aligned} SO\left(spl_k\left(G^{-+}\right)\right) &= \left(\frac{n(n-1)}{2} - m\right)[\sqrt{2}(n-1)(k+1) + 2k(n-1)\sqrt{(k+1)^2+1}] \\ &+ 2m(k+1)\sqrt{(n-1)^2+4} \\ &+ 2mk[\sqrt{(n-1)^2(k+1)^2+4} + \sqrt{(n-1)^2+4(k+1)^2}] \end{aligned}$$

Proof: Partitioning edge set of $spl_k(G^{-+})$ as follows:

$$E\left(spl_k(G^{-+})\right) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$$\begin{aligned} E_1 &= \{uv : uv \notin E(G)\} \\ E_2 &= \{ue : \text{vertex } u \text{ in } G \text{ is incident to edge } e \text{ in } G\} \\ E_3 &= \left\{uv' : \text{vertex } u \in G \text{ is not adjacent to vertex } v' \in G^{-+} \text{ due to vertex } v \text{ in } G\right\} \\ E_4 &= \left\{ue' : \text{vertex } u \in G \text{ is incident to vertex } e' \in G^{-+} \text{ due to edge } e \text{ in } G\right\} \\ E_5 &= \left\{u'e : \text{vertex } u' \in G^{-+} \text{ due to vertex } u \in G \text{ is not incident to edge } e \text{ in } G\right\} \end{aligned}$$

Clearly, $|E_1| = \frac{n(n-1)}{2} - m$, $|E_2| = 2m$, $|E_3| = 2k\left(\frac{n(n-1)}{2} - m\right)$,
$|E_4| = 2mk$ and $|E_5| = 2mk$.

Consider,

$$
\begin{aligned}
SO\left(spl_k\left(G^{-+}\right)\right) &= \sum_{uv\in(E(spl_k(G^{-+})))}\left[\sqrt{d_{spl_k(G^{-+})}(u)^2+d_{spl_k(G^{-+})}(v)^2}\right]\\
&= \sum_{uv\in E_1}\left[\sqrt{d_{spl_k(G^{-+})}(u)^2+d_{spl_k(G^{-+})}(v)^2}\right]+\sum_{ue\in E_2}\left[\sqrt{d_{spl_k(G^{-+})}(u)^2+d_{spl_k(G^{-+})}(e)^2}\right]\\
&+ \sum_{uv'\in E_3}\left[\sqrt{d_{spl_k(G^{-+})}(u)^2+d_{spl_k(G^{-+})}(v')^2}\right]+\sum_{ue'\in E_4}\left[\sqrt{d_{spl_k(G^{-+})}(u)^2+d_{spl_k(G^{-+})}(e')^2}\right]\\
&+ \sum_{u'e\in E_5}\left[\sqrt{d_{spl_k(G^{-+})}(u')^2+d_{spl_k(G^{-+})}(e)^2}\right]\\
SO\left(spl_k\left(G^{-+}\right)\right) &= \sum_{uv\in E_1}\left[\sqrt{(n-1)^2(k+1)^2+(n-1)^2(k+1)^2}\right]+\sum_{ue\in E_2}\left[\sqrt{(n-1)^2(k+1)^2+4(k+1)^2}\right]\\
&+ \sum_{uv'\in E_3}\left[\sqrt{(n-1)^2(k+1)^2+(n-1)^2}\right]+\sum_{ue'\in E_4}\left[\sqrt{(n-1)^2(k+1)^2+4}\right]\\
&+ \sum_{u'e\in E_5}\left[\sqrt{(n-1)^2+4(k+1)^2}\right]\\
&= \sqrt{2}(n-1)(k+1)\sum_{uv\in E_1}1+(k+1)\sqrt{(n-1)^2+4}\sum_{ue\in E_2}1\\
&+ (n-1)\sqrt{(k+1)^2+1}\sum_{uv'\in E_3}1+\sqrt{(n-1)^2(k+1)^2+4}\sum_{ue'\in E_4}1\\
&+ \sqrt{(n-1)^2+4(k+1)^2}\sum_{u'e\in E_5}1\\
SO\left(spl_k\left(G^{-+}\right)\right) &= \sqrt{2}(n-1)(k+1)\left(\frac{n(n-1)}{2}-m\right)+2m(k+1)\sqrt{(n-1)^2+4}\\
&+ 2k(n-1)\sqrt{(k+1)^2+1}\left(\frac{n(n-1)}{2}-m\right)+2mk\sqrt{(n-1)^2(k+1)^2+4}\\
&+ 2mk\sqrt{(n-1)^2+4(k+1)^2}\\
&= \left(\frac{n(n-1)}{2}-m\right)[\sqrt{2}(n-1)(k+1)+2k(n-1)\sqrt{(k+1)^2+1}]\\
&+ 2m[(k+1)\sqrt{(n-1)^2+4}+k[\sqrt{(n-1)^2(k+1)^2+4}\\
&+ \sqrt{(n-1)^2+4(k+1)^2}]
\end{aligned}
$$

□

7. Sombor Index of $spl_k(G^{--})$

Let G be a (n,m)-graph and $spl_k(G^{--})$ represents k-splitting of G^{--}. For P_4, the structure of 2-splitting graph of P_4^{--}, is shown in the Figure 4

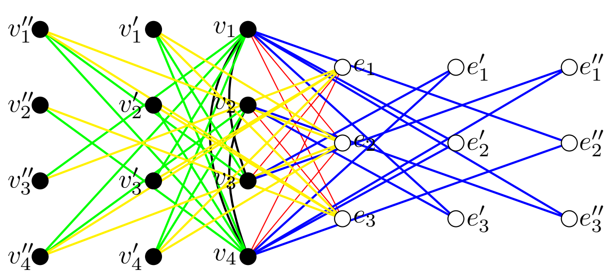

Figure 4: $spl_2\left(P_4^{--}\right)$: 2-splitting of P_4^{--}

Proposition 7.1 *Let G be a (n,m)-graph. Then*

1. $d_{spl_k(G^{--})}(v) = (n+m-1-2d_G(v))(k+1)$, *where* $v \in V(G)$

2. $d_{spl_k(G^{--})}(e) = (n-2)(k+1)$, *where* $e \in E(G)$

3. $d_{spl_k(G^{--})}(v') = (n+m-1-2d_G(v))$, *where* $v' \in G^{--}$ *due to vertex* v *in* G

4. $d_{spl_k(G^{--})}(e') = (n-2)$, *where* $e' \in G^{--}$ *due to edge* e *in* G

Proposition 7.2 *Let* G *be a* (n,m)*-graph. Then order and size of* $spl_k(G^{--})$ *are* $(n+m)(k+1)$ *and* $\frac{1}{2}[n(n-1)+2m(n-3)](2k+1)$ *respectively.*

Theorem 7.1 *Let* G *be a* (n,m)*-graph. Then*

$$SO\left(spl_k\left(G^{--}\right)\right) = (k+1)\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$$

Proof: Partitioning edge set of $spl_k(G^{--})$ as follows:

$$E\left(spl_k(G^{--})\right) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$$\begin{aligned}
E_1 &= \{uv : uv \notin E(G)\} \\
E_2 &= \{ue : \text{vertex } u \text{ in } G \text{ is not incident to edge } e \text{ in } G\} \\
E_3 &= \left\{uv' : \text{vertex } u \in G \text{ is not adjacent to vertex } v' \in G^{--} \text{ due to vertex } v \text{ in } G\right\} \\
E_4 &= \left\{ue' : \text{vertex } u \in G \text{ is not incident to vertex } e' \in G^{--} \text{ due to edge } e \text{ in } G\right\} \\
E_5 &= \left\{u'e : \text{vertex } u' \in G^{--} \text{ due to vertex } u \in G \text{ is not incident to edge } e \text{ in } G\right\}
\end{aligned}$$

Clearly, $|E_1| = \dfrac{n(n-1)}{2} - m$, $\quad |E_2| = m(n-2)$, $\quad |E_3| = 2k\left(\dfrac{n(n-1)}{2} - m\right)$
$|E_4| = mk(n-2)$ and $|E_5| = mk(n-2)$.

Consider,

$$
\begin{aligned}
SO\left(spl_k\left(G^{--}\right)\right) &= \sum_{uv\in\left(E\left(spl_k\left(G^{--}\right)\right)\right)} \left[\sqrt{d_{spl_k\left(G^{--}\right)}\left(u\right)^2 + d_{spl_k\left(G^{--}\right)}\left(v\right)^2}\right] \\
&= \sum_{uv\in E_1} \left[\sqrt{d_{spl_k\left(G^{--}\right)}\left(u\right)^2 + d_{spl_k\left(G^{--}\right)}\left(v\right)^2}\right] + \sum_{ue\in E_2} \left[\sqrt{d_{spl_k\left(G^{--}\right)}\left(u\right)^2 + d_{spl_k\left(G^{--}\right)}\left(e\right)^2}\right] \\
&\quad + \sum_{uv'\in E_3} \left[\sqrt{d_{spl_k\left(G^{--}\right)}\left(u\right)^2 + d_{spl_k\left(G^{--}\right)}\left(v'\right)^2}\right] + \sum_{ue'\in E_4} \left[\sqrt{d_{spl_k\left(G^{--}\right)}\left(u\right)^2 + d_{spl_k\left(G^{--}\right)}\left(e'\right)^2}\right] \\
&\quad + \sum_{u'e\in E_5} \left[\sqrt{d_{spl_k\left(G^{--}\right)}\left(u'\right)^2 + d_{spl_k\left(G^{--}\right)}\left(e\right)^2}\right] \\
SO\left(spl_k\left(G^{--}\right)\right) &= \sum_{uv\in E_1} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2(k+1)^2 + \left(n+m-1-2d_G(v)\right)^2(k+1)^2}\right] \\
&\quad + \sum_{ue\in E_2} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2 + (n-2)^2(k+1)^2}\right] \\
&\quad + \sum_{uv'\in E_3} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2(k+1)^2 + \left(n+m-1-2d_G(v)\right)^2}\right] \\
&\quad + \sum_{ue'\in E_4} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2(k+1)^2 + (n-2)^2}\right] \\
&\quad + \sum_{u'e\in E_5} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2 + (n-2)^2}\right] \\
SO\left(spl_k\left(G^{--}\right)\right) &= (k+1)\sum_{uv\in E_1} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2 + \left(n+m-1-2d_G(v)\right)^2}\right] \\
&\quad + \sum_{ue\in E_2} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2 + (n-2)^2(k+1)^2}\right] \\
&\quad + \sum_{uv'\in E_3} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2(k+1)^2 + \left(n+m-1-2d_G(v)\right)^2}\right] \\
&\quad + \sum_{ue'\in E_4} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2(k+1)^2 + (n-2)^2}\right] \\
&\quad + \sum_{u'e\in E_5} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2 + (n-2)^2}\right] \\
&= (k+1)\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5
\end{aligned}
$$

Where,

$$
\begin{aligned}
\gamma_1 &= \sum_{uv\in E_1} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2 + \left(n+m-1-2d_G(v)\right)^2}\right] \\
\gamma_2 &= \sum_{ue\in E_2} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2 + (n-2)^2(k+1)^2}\right] \\
\gamma_3 &= \sum_{uv'\in E_3} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2(k+1)^2 + \left(n+m-1-2d_G(v)\right)^2}\right] \\
\gamma_4 &= \sum_{ue'\in E_4} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2(k+1)^2 + (n-2)^2}\right] \\
\gamma_5 &= \sum_{u'e\in E_5} \left[\sqrt{\left(n+m-1-2d_G(u)\right)^2 + (n-2)^2}\right]
\end{aligned}
$$

□

8. Conclusion

This study derives explicit formulas for the Sombor index of the k-splitting of generalized transformation graphs, represented as $spl_k\left(G^{ab}\right)$. Furthermore, analogous expressions have been established for the complements of, extending the analytical understanding of these graph structures $spl_k\left(G^{ab}\right)$.

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