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On the Transcendence of Sum and Product of Certain Infinite Series

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ABSTRACT: In the present paper and as an application of Roth's theorem concerning the rational approximation of algebraic numbers, we give a sufficient condition that will assure us that a sum, product and quotient of some series of positive rational terms are transcendental numbers. We recall that all the infinite series that we are going to treat are Liouville numbers. At the end this article, we establish an approximation measure of these numbers.

Key Words: Infinite series, transcendental number, approximation measure.

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1. Introduction

The theory of transcendental numbers has a long history. We know since J. Liouville in 1844 that the very rapidly converging sequences of rational numbers provide examples of transcendental numbers. So, in his famous paper [8], Liouville showed that a real number admitting very good rational approximation can not be algebraic, then he explicitly constructed the first examples of transcendental numbers. In [1], the authors give a family of real numbers that are transcendental.

There are a number of sufficient conditions known within the literature for an infinite series, $\sum_{n=1}^{\infty} 1/a_n$, of positive rational numbers to converge to an irrational number, see [3, 10].These conditions, which are quite varied in form, share one common feature, namely, they all require rapid growth of the sequence a_n to deduce irrationality of the series. As an illustration consider the following results of J. Sandor which have been taken from [12] and [13].

From this direction, the transcendence of some infinite series has been studied by several authors such as M.A. Nyblom [9], J. Hančl and J. Štěpnička [5]. we also note that the transcendence of some power series with rational coefficients is given by some authors such as J. P. Allouche [3] and G. K. Gözer [4].

We recall that in [5], we have proved the transcendence of some series of positive rational terms.
In the present paper, the first aim is to give a sufficient condition that will assure us that a sum, product and quotient of some series of positive rational terms are transcendental numbers.
Let g_1, g_2 be two distinct integers ≥ 2, such that $g_1 > g_2$ and θ_1, θ_2 two infinite series which are defined by

$$\theta_1 = \sum_{n=1}^{+\infty} g_1^{-a_n},\ \theta_2 = \sum_{n=1}^{+\infty} g_2^{-a_n}$$

where $a_n \geq 1$, $b_n \geq 1$ are integers for all $n \geq 1$.
The second main result of this article is to establish an approximation measure of a real number $\theta_1 + \theta_2$.
In order to prove the transcendence of the infinite series, we will use the Roth's Theorem.

Roth's Theorem. *Let α be a real number, δ a real number > 2, if there exists an infinity rational numbers $\frac{p}{q}$ with $gcd(p, q) = 1$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\delta}},$$

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then α is a transcendental number.

2. Main Results

2.1. Transcendence of some series

Theorem 2.1 *Let g_1, g_2 be two distinct integers ≥ 2, such that $g_1 > g_2$, β a real number > 0. With the same notations as above, let*

$$\theta_1 = \sum_{n=1}^{+\infty} \frac{1}{g_1^{a_n}}, \;\; \theta_2 = \sum_{n=1}^{+\infty} \frac{1}{g_2^{a_n}},$$

where $a_{n+1} = a_n^{1+\beta}$ for all $n \geq 1$ and $a_1 \geq 2$.
Then, the real numbers $\theta_1 + \theta_2$, $\theta_1 - \theta_2$, $\theta_1 \times \theta_2$ and θ_1/θ_2 are transcendental numbers.

Remark 1. From the definition of θ_j, it is clear that these series are convergent.

Proof: Let

$$\theta_{n,1} = \sum_{k=1}^{n} \frac{1}{g_1^{a_k}} = \frac{p_{n,1}}{q_{n,1}}, \theta_{n,2} = \sum_{k=1}^{n} \frac{1}{g_2^{a_k}} = \frac{p_{n,2}}{q_{n,2}},$$

where $\gcd(p_{n,1}, q_{n,1}) = \gcd(p_{n,2}, q_{n,2}) = 1$. So, we can see that $q_{n,1} = g_1^{a_n}$ and $q_{n,2} = g_2^{a_n}$. □

(i) Transcendence of $(\theta_1 + \theta_2)$.

We have

$$|\theta_1 + \theta_2 - \theta_{n,1} - \theta_{n,2}| \leq |\theta_1 - \theta_{n,1}| + |\theta_2 - \theta_{n,2}|.$$

Therefore, we get

$$|\theta_1 - \theta_{n,1}| = \frac{1}{g_1^{a_{n+1}}} \left(1 + \sum_{k=n+1}^{+\infty} \frac{1}{g_1^{a_{k+1} - a_{n+1}}} \right).$$

which gives

$$\frac{1}{g_1^{a_{n+1}}} < |\theta_1 - \theta_{n,1}| < \frac{2}{g_1^{a_{n+1}}},$$

and

$$\frac{1}{g_2^{a_{n+1}}} < |\theta_2 - \theta_{n,2}| < \frac{2}{g_2^{a_{n+1}}}.$$

From the previous inequalities and the hypothesis $g_1 > g_2$, we deduce

$$|\theta_1 + \theta_2 - \theta_{n,1} - \theta_{n,2}| < \frac{2}{g_1^{a_{n+1}}} + \frac{2}{g_2^{a_{n+1}}} < \frac{4}{g_2^{a_{n+1}}}. \tag{2.1}$$

We can see that

$$\lim_{n \to +\infty} \frac{g_2^{a_{n+1}}}{(g_1 g_2)^{a_n}} = +\infty.$$

The above inequality is true since one has

$$\frac{a_{n+1} \ln g_2}{a_n \ln(g_1 g_2)} = a_n^{\beta} \frac{\ln g_2}{\ln(g_1 g_2)} \geq 2^{\beta(1+\delta)^{n-1}} \frac{\ln g_2}{\ln(g_1 g_2)}.$$

and $\lim_{n\to+\infty} 2^{\beta(1+\delta)^{n-1}} \frac{\ln g_2}{\ln(g_1 g_2)} = +\infty$.

Hence, for all positive integer $d > 2$, $\exists n_0 = n_0(d)$ such that for all $n \geq n_0$, we have

$$\ln(g_2^{a_{n+1}}) > d\ln(g_1 g_2)^{a_n} = \ln\left((g_1 g_2)^{a_n}\right)^d.$$

Which yields,

$$g_2^{a_{n+1}} > (g_1^{a_n} g_2^{a_n})^d \tag{2.2}$$

and the inequality (1) becomes

$$|\theta_1 + \theta_2 - (\frac{p_{n,1}}{q_{n,1}} + \frac{p_{n,2}}{q_{n,2}})| < \frac{4}{(q_{n,1}q_{n,2})^d}.$$

Since, $d > 2$, by Roth's Theorem, we deduce that $\theta_1 + \theta_2$ is a transcendental number. Similarely, it can easily be proven that $\theta_1 - \theta_2$ is also a transcendental number.

(ii) Transcendence of $\theta_1 \times \theta_2$.
One has

$$\begin{aligned} |\theta_1 \times \theta_2 - \theta_{n,1} \times \theta_{n,2}| &= |(\theta_1 - \theta_{n,1})\theta_{n,2} + (\theta_{n,2} - \theta_2)\theta_{n,1}| \\ &\leq |\theta_1 - \theta_{n,1}|\theta_2 + (1+\theta_1)|\theta_2 - \theta_{n,2}| \\ &< \frac{2\theta_2}{g_1^{a_{n+1}}} + \frac{2(1+\theta_1)}{g_2^{a_{n+1}}} \end{aligned}$$

for all n sufficiently large. Since $g_1 > g_2$ and using the relationship (2), we obtain

$$|\theta_1 \times \theta_2 - \theta_{n,1} \times \theta_{n,2}| < \frac{2(1+\theta_1+\theta_2)}{g_2^{a_{n+1}}} < \frac{1}{(q_{n,1}q_{n,2})^d},$$

for all n sufficiently large. Therefore, $\theta_1 \times \theta_2$ is transcendental by Roth's Theorem.

(iii) Transcendence of (θ_1/θ_2).

$$\begin{aligned} \left|\frac{\theta_1}{\theta_2} - \frac{\theta_{n,1}}{\theta_{n,2}}\right| &= |(\theta_1 - \theta_{n,1})\theta_2 + (\theta_{n,2} - \theta_2)\theta_{n,1}| \\ &= \frac{|\theta_{n,2}(\theta_1 - \theta_{n,1}) + \theta_{n,1}(\theta_{n,2} - \theta_2)|}{\theta_2\theta_{n,2}} \end{aligned}$$

As $\theta_{n,1} = p_{n,1}/q_{n,1} \leq 1 + \theta_1 < 2$ and $\theta_{n,2} = p_{n,2}/q_{n,2} > 1/g_2^{a_1}$, for $n \geq 2$, we obtain

$$\begin{aligned} \left|\frac{\theta_1}{\theta_2} - \frac{\theta_{n,1}}{\theta_{n,2}}\right| &\leq \frac{|\theta_1 - \theta_{n,1}|}{\theta_2} + \frac{2g_2^{a_1}}{\theta_2}|\theta_{n,2} - \theta_2| \\ &< \frac{2}{\theta_2 g_1^{a_{n+1}}} + \frac{4g_2^{a_1}}{\theta_2 g_2^{a_{n+1}}} \\ &< \frac{2+4g_2^{a_1}}{\theta_2}\frac{1}{g_2^{a_{n+1}}}. \end{aligned}$$

Since $p_{n,2} < (1+\theta_2)q_{n,2} < 2q_{n,2}$ and using (2), for all n sufficiently large, we find

$$\left|\frac{\theta_1}{\theta_2} - \frac{\theta_{n,1}}{\theta_{n,2}}\right| \leq \frac{4}{(q_{n,1}q_{n,2})^d} < \frac{4(1+\theta_2)}{(q_{n,1}p_{n,2})^d}.$$

Therefore θ_1/θ_2 is a transcendental number.

2.2. Approximation measure of a power series $(\theta_1 + \theta_2)$

In this subsection, we give the second main result of this article.
Throughout this section, we adopt the following notation $\theta = \theta_1 + \theta_2$ and $\theta_n = \theta_{1,n} + \theta_{2,n} = p_n/q_n$ where $\gcd(p_n, q_n) = 1$.

Theorem 2.2 *Let ξ be an algebraic number of degree $d \geq 2$ and height H. Let $\alpha > H$ and $k > 1$ be two real numbers such that for all $n \geq 1$, we have*

$$a_n^{\alpha} \leq a_{n+1} < a_n^{k\alpha}, \quad \textit{for all} \quad n \geq 1.$$

Then we get,

$$|\theta - \xi| > \frac{1}{(2Hd^2)^{1+4d}}.$$

Remark 2.1 *This approximation measure obtained of $\theta_1 + \theta_2$ is the same of $\theta_1 - \theta_2$, $\theta_1 \times \theta_2$ and θ_1/θ_2.*

Proof: Let ξ be an algebraic number of degree $d \geq 2$ and height $H > \alpha$. From Theorem 1, part i) we deduce that θ is transcendental.
To obtain an approximation measure of $\theta_1 + \theta_2$, it is sufficient to minimize $|\theta - \xi|$ by function of H and d.

We have

$$|\theta - \xi| = |\theta - \theta_n + \theta_n - \xi| \leq |\theta - \theta_n| + |\xi - \theta_n|.$$

Which becomes,

$$|\theta - \xi| \geq |\theta - \xi| - |\theta - \theta_n|.$$

According to Liouville's Theorem, it is not hard to see that

$$|\xi - \theta_n| \geq \frac{1}{Hd^2 q_n^d}. \tag{2.3}$$

It follows from section 1 that

$$|\theta - \theta_n| < \frac{1}{g_2^{a_{n+1}}}.$$

so, we obtain

$$|\xi - \theta| > \frac{1}{Hd^2 q_n^d} - \frac{1}{g_2^{a_{n+1}}}.$$

In order to have

$$|\xi - \theta| > \frac{1}{2Hd^2 q_n^d},$$

it suffices to have

$$\frac{1}{2Hd^2 q_n^d} > \frac{1}{g_2{}^{a}_{n+1}}.$$

Since $q_n = (g_1 g_2)^{a_n}$, the relationship (3) gives

$$\frac{1}{2Hd^2 q_n^d} > \frac{1}{(g_1 g_2)^{a_{n+1}}}. \tag{2.4}$$

Therefore (4) is equivalent to

$$(g_1 g_2)^{a_{n+1}} > 2Hd^2 (g_1 g_2)^{da_n}. \tag{2.5}$$

To realize the inequality (5), it suffices that the integer n satisfies

$$\begin{cases} (g_1g_2)^{a_{n+1}/2} > (g_1g_2)^{da_n} \\ \\ (g_1g_2)^{a_{n+1}/2} > 2Hd^2. \end{cases} \tag{2.6}$$

The first inequality of (6) is easily obtained because $a_{n+1} > a_n^\alpha > 2da_n$.
For the second one, let n_1 be the smallest integer n such that

$$(g_1g_2)^{a_n/2} < 2Hd^2 < (g_1g_2)^{a_{n+1}/2}. \tag{2.7}$$

This yields that

$$|A - \xi| > \frac{1}{(2Hd^2)^{1+4d}}.$$

Which proves the result of Theorem 2. □

Example. Let

$$\begin{cases} g_1 = 2, g_2 = 3, a_1 = 2 \\ a_{n+1} = a_n^2, n \geq 1, \\ \alpha = 4, k = 2. \end{cases}$$

Let ξ be an algebraic number of degree $d = 3$ and height H. By applying Theorem 2, an approximation measure of $(\theta_1 + \theta_2)$ is given by

$$|\theta_1 + \theta_2 - \xi| > \frac{1}{(18H)^{13}}.$$

References

1. S. Ahallal, A. Kacha, *Transcendental Continued Fractions*, Communications in Mathematics 30 (2022), No. 1, 251-259.
2. J.P. Allouche, *Transcendence of formal power series with rational coefficients*, Theoretical Computer Science 218 (1999), 143-160.
3. P. Erdos, *Some problems and result on the irrationality of the sum of infinite series*, J. Math. Sci. 10 (1975), 1-7.
4. G. K. Gözer, A. Pekin and A. Kiliçman, *On the transcendence of some power series*, Advances in Difference Equations 17 (2013), 1-8.
5. F. Sgiouer, K. Belhroukia, A. Kacha, *Transcendence of some infinite series.* Le Matematiche, vol. 78, Issue 1 (2023), 201-211.
6. J. Hančl and J. Štěpnička, *On the transcendence of some infinite series*, Glasgow Math. J. 50 (2008), 33–37.
7. J. Hančl, S. Dodulíková and R. Nair, *On the irrationality of infinite series of reciprocals of square roots*, Rocky Mountain Journal of Mathematics (October 2017), 1-11.
8. J. Liouville, *sur des très etendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationnelles algèbrique*, C. R. Mat. Acad. Sci. Paris 18 (1844), 883-885, 910-911.
9. M. A. Nyblom, *A theorem on transcendence of infinite series I* I, J. Number Theory 91 (2001), 71-80.
10. A.Oppenheim, *The irrationality or rationality of certain infinite series*, in " Studies in Pure Math. (presented to Richard Rado)", Academic Press, London (1971), 195-201.
11. K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika 2 (1955), 1-20.
12. J.Sandor, *Some classes of irrational numbers*, Studia Univ. Babes-Bolyai Math. 29 (1984), 3-12.
13. J.Sandor, *On some irrationality factorial series*, Studia Univ.Babes-Bolyai Math. 32 (1987), 13-17.

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