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Douglas Space with Special $(\alpha,\ \beta)$-Metric

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ABSTRACT: We deal with the conditions for a Finsler space with the special $(\alpha,\ \beta)$-metric to be a Douglas space of second kind, where α is a Riemannian metric and β is a differential 1-form.

Keywords: Finsler space, Berwald space, Douglas space of second kind, $(\alpha,\ \beta)$-metric.

Contents

1. Introduction

The Douglas spaces and weakly Berwald space are considered as generalizations of Berwald spaces. S. Bacso and M. Matsumoto [2] introduced the concept Douglas space. Finsler space is a Douglas space if and only if the Douglas tensor vanishes identically. I. Y. Lee [4] found the conditions for a Finsler space with Matsumoto metric to be a Douglas space of second kind.

In this paper, we find the condition for a Finsler space with an $(\alpha,\ \beta)$-metric $L(\alpha,\ \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$, where $c_1 \neq 0$, to be Douglas space of second kind.

2. Preliminaries

Let $F^n = (M^n, L)$ be the Finsler space with an $(\alpha,\ \beta)$-metric $L(\alpha,\ \beta)$, where $\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$ and $\beta = b_i(x)y^i$. The fundamental function G^i of a Finsler space is given by, $2G^i = \gamma_0{}^i{}_0 + 2B^i$, where $(\gamma_j{}^i{}_k)$ is the Christoffel symbol, then we have $G^i{}_j = \gamma_0{}^i{}_j + B^i{}_j$ and $G_j{}^i{}_k = \gamma_j{}^i{}_k + B_j{}^i{}_k$, where $\dot{\partial}_j B^i = B^i{}_j$, $\dot{\partial}_k B^i{}_j = B_j{}^i{}_k$ and the Berwald connection is given by $B\Gamma = (G_j{}^i{}_k, G^i{}_j)$. Denote $B_j{}^i{}_k$ the difference tensor [3] of $G_j{}^i{}_k$ from $\gamma_j{}^i{}_k$.

$$G_j{}^i{}_k(x,y) = \gamma_j{}^i{}_k + B_j{}^i{}_k(x,y).$$

Transvecting this equation by y^k, we get

$$G^i{}_j = \gamma_0{}^i{}_j + B^i{}_j \quad and \quad 2G^i = \gamma_0{}^i{}_0 + 2B^i,$$

where, $B_j{}^i{}_k = \dot{\partial}_k B^i{}_j$ and $B^i{}_j = \dot{\partial}_j B^i$.
The differential equations of geodesic of a Finsler space F^n in the parameter t is,

$$\ddot{x}^i\dot{x}^j - \ddot{x}^j\dot{x}^i + 2(G^i x^j - G^j x^i) = 0,\ \ y^i = \dot{x}^i,$$

where the functions $G^i(x,y)$ is given by

$$2G^i(x,y) = \left\{ \begin{matrix} & i & \\ j & & k \end{matrix} \right\} y^j y^k,$$

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where $\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\}$ are the Christoffel symbol constructed from $g_{ij}(x,y)$ with respect to $\dot{x}^i$. The Finsler space F^n is a Douglas space if and only if the Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n+1}\left(G_{ijk}y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j\right),$$

vanishes identically [2], where $G^h_{ijk} = \dot{\partial}_k G^h_{ij}$ is the hv-curvature tensor of the Berwald connection. F^n is said to be Douglas space if,

$$D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i, \tag{2.1}$$

is homogeneous polynomial in (y^i) of degree three. On differentiating (2.1) with respect to y^h, y^k, y^p and y^q, we have $D^{ij}_{hkpq} = 0$, which is equivalent to $D^{im}_{hkpm} = (n+1)D_h{}^i{}_{kp} = 0$. Thus, the Finsler space F^n which satisfies the condition $D^{ij}_{hkpq} = 0$, which is equivalent to $D^{im}_{hkpm} = (n+1)D_h{}^i{}_{kp} = 0$, is called Douglas space. Again differentiating (2.1) by y^m and contracting by m and j in the obtained equation, we have $D^{im}_m = (n+1)G^i - G^m_m y^i$. Thus, F^n is said to be Douglas space of the second kind if and only if

$$D^{im}_m = (n+1)G^i - G^m_m y^i, \tag{2.2}$$

is homogeneous polynomial in (y^i) of degree two. Again differentiating (2.2) with respect to y^h, y^j and y^k, we get, $D^{im}_{hjkm} = (n+1)D^i_{hjk} = 0$.

Definition 2.1 *The Finsler space F^n is said to be Douglas space of the second kind, if it satisfies the condition $D^{im}_m = (n+1)G^i - G^m_m y^i$ be homogeneous polynomials in (y^i) of degree two.*

Definition 2.2 *The Finsler space F^n with $(\alpha,\ \beta)$-metric is said to be Douglas space of the second kind, if and only if*

$$B^{im}_m = (n+1)B^i - B^m_m y^i,$$

is homogeneous polynomials in (y^i) of degree two, where B^m_m is given [4].

Further differentiating above equation with respect to y^h, y^j, and y^k, we get $B^{im}_{hkpm} = B^i_{hjk} = 0$, is called Douglas space of second kind.

We use the following [1]:

Lemma 2.1 *If $\alpha^2 \equiv 0 (mod \beta)$, that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfies $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

Now define the function $G^i(x,y)$ [5] of Finsler space F^n with $(\alpha,\ \beta)$-metric as,

$$2G^m = \gamma_0{}^i{}_0 + 2B^i,$$

where

$$B^i = \left(\frac{E}{\alpha}\right)y^i + \left(\frac{\alpha L_\beta}{L_\alpha}\right)s^i{}_0 - \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)C^*\left\{\left(\frac{y^i}{\alpha}\right) - \left(\frac{\alpha}{\beta}\right)b^i\right\}, \tag{2.3}$$

where, we put

$$\begin{aligned} E &= \left(\frac{\beta L_\beta}{L}\right)C^*, \\ C^* &= \frac{\{\alpha\beta\left(r_{00}L_\alpha - 2\alpha s_0 L_\beta\right)\}}{\{2\left(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}\right)\}}, \\ \gamma^2 &= b^2\alpha^2 - \beta^2. \end{aligned}$$

Since $\gamma_0{}^i{}_0 = \gamma_j{}^i{}_k(x)y^iy^j$ is $hp(2)$, by means of (2.1) and (2.3) we have the following [6]:

$$B^{im}_m = \frac{(n+1)\alpha L_\beta}{L_\alpha} s^i_0 + \frac{\alpha\{(n+1)\alpha^2\Omega L_{\alpha\alpha}b^i + \beta\gamma^2 A y^i\}}{2\Omega^2} r_{00} - \frac{\alpha^2\{(n+1)\alpha^2\Omega L_\beta L_{\alpha\alpha}b^i + By^i\}}{L_\alpha\Omega^2} s_0 - \frac{\alpha^3 L_{\alpha\alpha}y^i}{\Omega} r_0, \tag{2.4}$$

where

$$A = \alpha L_\alpha L_{\alpha\alpha\alpha} + 3L_\alpha L_{\alpha\alpha} - 3\alpha(L_{\alpha\alpha})^2,$$
$$B = \alpha\beta\gamma^2 L_\alpha L_\beta L_{\alpha\alpha\alpha} + \beta\{(3\gamma^2 - \beta^2)L_\alpha - 4\alpha\gamma^2 L_{\alpha\alpha}\}L_\beta L_{\alpha\alpha} + \Omega L L_{\alpha\alpha}. \tag{2.5}$$

Thus from the above, we have

Theorem 2.1 *The necessary and sufficient condition for a Finsler space F^n with an $(\alpha,\ \beta)$-metric to be Douglas space of second kind is that B^{im}_m are homogeneous polynomials in (y^m) of degree two, where B^{im}_m is given by (2.4) and (2.5), provided that $\Omega \neq 0$.*

3. Douglas Space of Second Kind with $L(\alpha,\ \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$

Theorem 3.1 *The Finsler space F^n with an $(\alpha,\ \beta)$-metric $L(\alpha,\ \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$ is a Douglas space of second kind if and only if*
(i) $\alpha^2 \not\equiv 0 (mod\ \beta)$ then (3.5) and (3.6) are satisfied and $s_0 = 0$.
(ii) $\alpha^2 \equiv 0 (mod\ \beta)$ then (3.12), (3.13) and (3.14) are satisfied and $V^i_2 = 0$.

Proof: Consider the Finsler space F^n with an $(\alpha,\ \beta)$-metric $L(\alpha,\ \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$. The partial derivatives for this metric are as follows,

$$L_\alpha = c_1 + \frac{2\alpha}{\beta},$$
$$L_\beta = c_2 - \frac{\alpha^2}{\beta^2},$$
$$L_{\alpha\alpha} = \frac{2}{\beta},$$
$$L_{\alpha\alpha\alpha} = 0,$$
$$\Omega = c_1\beta^2 + \frac{2b^2\alpha^3}{\beta}. \tag{3.1}$$

Substituting (3.1) in (2.5), we get

$$A = \frac{6c_1}{\beta},$$
$$B = \frac{1}{\beta^3}\{6c_1c_2\alpha^2\beta^3b^2 - 6c_1c_2\beta^5 - 2c_1b^2\alpha^4\beta + 8\alpha^5b^2 + 10c_1\alpha^2\beta^3 + 2c_1^2\alpha\beta^4\}. \tag{3.2}$$

Now, from (3.1) and (3.2), equation (2.4) reduces to

$$\begin{aligned}
&B^{im}_m\{c_1^3\beta^9+4c_1b^4\alpha^6\beta^3+4c_1^2b^2\alpha^3\beta^6+2c_1^2\alpha\beta^8+8b^4\alpha^7\beta^2+8c_1b^2\alpha^4\beta^5\}\\
&-s^i_0\{(n+1)c_1^3c_2\alpha\beta^9+(n+1)c_1c_2b^4\alpha^7\beta^3+4(n+1)c_1^2c_2b^2\alpha^4\beta^6+2(n+1)c_1^2c_2\alpha^2\beta^8\\
&+8(n+1)c_2b^4\alpha^8\beta^2+8(n+1)c_1c_2b^2\alpha^5\beta^5-(n+1)c_1^3\alpha^3\beta^7-4(n+1)c_1b^4\alpha^9\beta\\
&-4(n+1)c_1^2b^2\alpha^6\beta^4-2(n+1)c_1^2\alpha^3\beta^6-8(n+1)b^4\alpha^9-8(n+1)c_1b^2\alpha^6\beta^3\}\\
&-r_{00}\{b^i[(n+1)c_1^2\alpha^3\beta^6+2(n+1)c_1\alpha^4\beta^5+2(n+1)c_1b^2\alpha^6\beta^3+4(n+1)b^2\alpha^7\beta^2]\\
&+y^i[3c_1^2b^2\alpha^3\beta^5+6c_1b^2\alpha^4\beta^4-3c_1^2\alpha\beta^7-6c_1\alpha^2\beta^6]\}\\
&+s_0\{b^i[2(n+1)c_1c_2\alpha^4\beta^6-2(n+1)c_1\alpha^6\beta^4+4(n+1)c_2b^2\alpha^7\beta^3-4(n+1)b^2\alpha^9\beta]\\
&+y^i[6c_1c_2b^2\alpha^4\beta^5-6c_1c_2\alpha^2\beta^7-2c_1b^2\alpha^6\beta^3+2b^2\alpha^7\beta^2+10c_1\alpha^4\beta^5+2c_1^2\alpha^3\beta^6]\}\\
&+r_0y^i\{2c_1^2\alpha^3\beta^6+4c_1\alpha^4\beta^5+4c_1b^2\alpha^6\beta^3+8b^2\alpha^7\beta^2\}=0.
\end{aligned}$$

Assume that F^n is a Douglas space of second kind, that is, B^{im}_m be $hp(2)$. Since α is irrational in (y^i), the above equation is divided into two parts as follows:

$$\begin{aligned}
&B^{im}_m\{c_1^3\beta^9+4c_1b^4\alpha^6\beta^3+8c_1b^2\alpha^4\beta^5\}\\
&-s^i_0\{4(n+1)c_1^2c_2b^2\alpha^4\beta^6+2(n+1)c_1^2c_2\alpha^2\beta^8+8(n+1)c_2b^4\alpha^8\beta^2\\
&-4(n+1)c_1^2b^2\alpha^6\beta^4-8(n+1)c_1\alpha^6\beta^3\}\\
&-r_{00}\{b^i[2(n+1)c_1\alpha^4\beta^5+2(n+1)c_1b^2\alpha^6\beta^3]+y^i[6c_1b^2\alpha^4\beta^4-6c_1\alpha^2\beta^6]\}\\
&+s_0\{b^i[2(n+1)c_1c_2\alpha^4\beta^6-2(n+1)c_1\alpha^6\beta^4]+y^i[6c_1c_2b^2\alpha^4\beta^5-6c_1c_2\alpha^2\beta^7-2c_1b^2\alpha^6\beta^3\\
&+10c_1\alpha^4\beta^5]\}+r_0y^i\{4c_1\alpha^4\beta^5+4c_1b^2\alpha^6\beta^3\}=0,
\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}
&B^{im}_m\{4c_1^2b^2\alpha^2\beta^6+2c_1^2\beta^8+8b^4\alpha^6\beta^2\}\\
&-s^i_0\{(n+1)c_1^3c_2\beta^9+(n+1)c_1c_2b^4\alpha^6\beta^3+8(n+1)c_1c_2b^2\alpha^4\beta^5\\
&-(n+1)c_1^3\alpha^2\beta^7-4(n+1)c_1b^4\alpha^8\beta-2(n+1)c_1^2\alpha^2\beta^6-8(n+1)b^4\alpha^8\}\\
&-r_{00}\{b^i[(n+1)c_1^2\alpha^2\beta^6+4(n+1)b^2\alpha^6\beta^2]+y^i[3c_1^2b^2\alpha^2\beta^5-3c_1^2\beta^7]\}\\
&+s_0\{b^i[4(n+1)c_2b^2\alpha^6\beta^3-4(n+1)b^2\alpha^8\beta]+y^i[2b^2\alpha^6\beta^2+2c_1^2\alpha^2\beta^6]\}\\
&+r_0y^i\{2c_1^2\alpha^2\beta^6+8b^2\alpha^6\beta^2\}=0.
\end{aligned}\tag{3.4}$$

Since the term which is does not contain α^2 in (3.3) is $c_1^3\beta^9B^{im}_m$, therefore we must have $hp(9)V^i_9$ such that $c_1^3\beta^9B^{im}_m=\alpha^2V^i_9$.

We consider two different cases:

Case(i): when $\alpha^2\not\equiv 0(mod\ \beta)$, that is, $n>2$, then there exits a function $g^i(x)$ such that $V^i_9=g^i(x)c_1^3\beta^9$, then

$$B^{im}_m=\alpha^2g^i(x).\tag{3.5}$$

Similarly, the terms which does not contain α^2 in (3.4) is $B^{im}_m2c_1^2\beta^8-s^i_0(n+1)c_1^3c_2\beta^9+r_{00}3c_1^3\beta^7y^i$, hence we must have $hp(7)U^i_7$ such that

$$\beta^6\{B^{im}_m2c_1^2\beta-s^i_0(n+1)c_1^3c_2\beta^2+r_{00}3c_1^2y^i\}=\alpha^2U^i_7.$$

Above equation shows that there exits $hp(8)U^i_8$ such that $U^i_7=\beta^7U^i_8$, which gives

$$B^{im}_m2c_1^2\beta-s^i_0(n+1)c_1^3c_2\beta^2+r_{00}3c_1^2y^i=\alpha^2U^i_8,$$

substituting (3.5) in this equation, we get

$$-s_0^i(n+1)c_1^3c_2\beta^2 + r_{00}3c_1^2y^i = \alpha^2\{U_8^i - g^i(x)2c_1^2\beta\},$$

since $\alpha^2 \not\equiv 0(mod\ \beta)$, $U_8^i - g^i(x)2c_1^2\beta$ must vanish and hence

$$3y^ir_{00} = (n+1)c_1c_2\beta^2 s_0^i. \tag{3.6}$$

Now substituting (3.5) and (3.6) in (3.3), we get

$$\begin{aligned}
&g^i(x)\{c_1^3\beta^9 + 4c_1b^4\alpha^6\beta^3 + 8c_1b^2\alpha^4\beta^5\}\\
&- s_0^i\{4(n+1)c_1^2c_2b^2\alpha^2\beta^6 + 2(n+1)c_1^2c_2\beta^8 + 8(n+1)c_2b^4\alpha^6\beta^2\\
&- 4(n+1)c_1^2b^2\alpha^4\beta^4 - 8(n+1)c_1\alpha^4\beta^3b^2\}\\
&- r_{00}b^i[2(n+1)c_1\alpha^2\beta^5 + 2(n+1)c_1b^2\alpha^4\beta^3] - (n+1)c_1c_2\beta^2s_0^i[2c_1b^2\alpha^2\beta^4 - 2c_1\beta^6]\\
&+ s_0\{b^i[2(n+1)c_1c_2\alpha^2\beta^6 - 2(n+1)c_1\alpha^4\beta^4] + y^i[6c_1c_2b^2\alpha^2\beta^5 - 6c_1c_2\beta^7 - 2c_1b^2\alpha^4\beta^3\\
&+ 10c_1\alpha^2\beta^5]\} + r_0y^i\{4c_1\alpha^2\beta^5 + 4c_1b^2\alpha^4\beta^3\} = 0.
\end{aligned} \tag{3.7}$$

In the equation (3.7), the term which is free from α^2 is $c_1^3\beta^9g^i(x) - s_0^i\left(4(n+1)c_1^2c_2\beta^8\right) - 6c_1c_2y^i\beta^7s_0$, hence we must have $hp(7)V_7^i$, such that $c_1^3\beta^2g^i(x) - s_0^i\left(4(n+1)c_1^2c_2\beta\right) - 6c_1c_2y^is_0 = \alpha^2V_7^i$. If $V_7^i = h^i(x)\beta^7$, then

$$c_1^3\beta^2g^i(x) - s_0^i\left(4(n+1)c_1^2c_2\beta\right) - 6c_1c_2y^is_0 = \alpha^2h^i(x).$$

Transvecting the above equation by b_i, gives

$$c_1^3\beta^2g_b - s_0\left(4(n+1)c_1^2c_2\beta + 6c_1c_2\right) = \alpha^2h_b, \quad where \quad b_ih^i = h_b, \quad b_ig^i = g_b.$$

Again since $\alpha^2 \not\equiv 0(mod\ \beta)$, we get $h_b = 0$, which gives

$$s_0 = \frac{c_1^2\beta^2g_b}{2\left[2(n+1)c_1c_2\beta + 3c_2\right]}. \tag{3.8}$$

Substituting (3.5), (3.6) and (3.8) in (3.4), we get

$$\begin{aligned}
&\alpha^2g^i(x)\{4c_1^2b^2\alpha^2\beta^6 + 2c_1^2\beta^8 + 8b^4\alpha^6\beta^2\}\\
&- s_0^i\{(n+1)c_1^3c_2\beta^9 + 4(n+1)c_1c_2b^4\alpha^6\beta^3 + 8(n+1)c_1c_2b^2\alpha^4\beta^5\\
&- (n+1)c_1^3\alpha^2\beta^7 - 4(n+1)c_1b^4\alpha^8\beta\}\\
&- r_{00}b^i[(n+1)c_1^2\alpha^2\beta^6 + 4(n+1)b^2\alpha^6\beta^2] - (n+1)c_1c_2\beta^2s_0^i[c_1^2b^2\alpha^2\beta^5 - c_1^2\beta^7]\\
&+ \left(\frac{c_1^2\beta^2g_b}{2\left[2(n+1)c_1c_2\beta + 3c_2\right]}\right)\{b^i[4(n+1)c_2b^2\alpha^6\beta^3 - 4(n+1)b^2\alpha^8\beta] + y^i[2b^2\alpha^6\beta^2\\
&+ 2c_1^2\alpha^2\beta^6]\} + r_0y^i\{2c_1^2\alpha^2\beta^6 + 8b^2\alpha^6\beta^2\} = 0.
\end{aligned} \tag{3.9}$$

Since only the term $-2(n+1)c_1^3c_2\beta^9s_0^i$ does not contain δ, hence there exists $hp(8)V_8^i$ such that $-2(n+1)c_1^3c_2\beta^9s_0^i = \alpha^2V_8^i$, where $V_8^i = \beta^9h^i(x)$, then $-2(n+1)c_1^3c_2s_0^i = \alpha^2h^i(x)$. Since $\alpha^2 \not\equiv 0(mod\ \beta)$, therefore $h^i(x) = 0$ which gives $s_0^i = 0 \;\;\Rightarrow\;\; s_0 = 0$.

Case(ii): when $\alpha^2 \equiv 0(mod\ \beta)$, Lemma(2.1) shows that $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$. Using these conditions, equations (3.3) and (3.4), can be rewritten as

$$\begin{aligned}
&B_m^{im}c_1^3\beta^2 - s_0^i\{6c_1^2c_2\beta^2\delta\} - r_{00}\{b^i6c_1\delta^2 - y^i6c_1\delta\}\\
&+ s_0\{b^i[6c_1c_2\delta^2\beta - 6c_1\delta^3] + y^i[-6c_1c_2\beta\delta + 10c_1\delta^2\beta]\} + r_0y^i4c_1\delta^2 = 0
\end{aligned} \tag{3.10}$$

and

$$B^{im}_m 2c_1^2\beta - s^i_0\{3c_1^3c_2\beta^2 - 3c_1^3\beta\delta - 6c_1^2\delta\} - r_{00}\{b^i 3c_1^2\delta - y^i 3c_1^2\} + s_0 y^i 2c_1^2\delta + r_0 y^i 2c_1^2\delta = 0. \tag{3.11}$$

In (3.10), the only term which is free from δ is $B^{im}_m c_1^3\beta^2$, hence there exits $hp(1)U^i$ such that

$$B^{im}_m = \delta U^i. \tag{3.12}$$

Similarly, in (3.11) the term which does not contain δ is

$$B^{im}_m 2c_1^2\beta - s^i_0 3c_1^3c_2\beta^2 + r_{00}y^i 3c_1^2 = \delta V^i_2.$$

Substituting (3.12) in this equation, we get

$$r_{00}y^i 3c_1^2 = s^i_0 3c_1^3c_2\beta^2 + \delta\{V^i_2 - 2c_1^2\beta U^i\}. \tag{3.13}$$

Again substitute (3.12) and (3.13) in (3.10), we get

$$\begin{aligned} &U^i c_1^3\beta^2 - s^i_0\{6c_1^2c_2\beta^2\} - r_{00}b^i 6c_1\delta + r_{00}y^i 6c_1\beta \\ &+ s_0\{b^i[6c_1c_2\delta\beta - 6c_1\delta^2] + y^i[-6c_1c_2\beta + 10c_1\delta]\} + r_0 y^i 4c_1\delta = 0. \end{aligned}$$

Since the term which is free from δ is $U^i c_1^3\beta^2 - s^i_0 6c_1^2c_2\beta^2 + r_{00}y^i 6c_1\beta - y^i 6c_1c_2\beta s_0$, thus we must have $hp(2)U^i_2$ such that

$$U^i c_1^3\beta^2 - s^i_0 6c_1^2c_2\beta^2 + r_{00}y^i 6c_1 - y^i 6c_1c_2\beta s_0 = \delta U^i_2,$$

which gives

$$c_1^2c_2\beta^2 s^i_0 = \delta U^i_2 + y^i 6c_1c_2\beta s_0 - U^i c_1^3\beta^2. \tag{3.14}$$

Substituting (3.12), (3.13) and (3.14) in (3.11), we get

$$\begin{aligned} &2c_1^2\beta\delta U^i - \{3c_1^3c_2\beta^2 - 3c_1^3\beta\delta - 6c_1^2\delta\}\left(\frac{\delta U^i_2 + y^i 6c_1c_2\beta s_0 - U^i c_1^3\beta^2}{c_1^2c_2\beta^2}\right) \\ &- \{b^i 3c_1^2\delta - y^i 3c_1^2\}\left(\frac{s^i_0 3c_1^3c_2\beta^2 + \delta\{V^i_2 - 2c_1^2\beta U^i\}}{y^i 3c_1^2}\right) + s_0 y^i 2c_1^2\delta + r_0 y^i 2c_1^2\delta = 0. \end{aligned}$$

The term which is free from δ is $-3c_1\{y^i 6c_1c_2\beta s_0 - U^i c_1^3\beta^2\} + s^i_0 3c_1^3c_2\beta^2$, hence there exits $hp(2)V^i_2$ such that $-3c_1\{y^i 6c_1c_2\beta s_0 - U^i c_1^3\beta^2\} + 3c_1\{y^i 6c_1c_2\beta s_0 - U^i c_1^3\beta^2\} = \delta V^i_2 \ \Rightarrow\ V^i_2 = 0.$ □

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