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A Note on Non-Inclusion Principal Ideal Graph of Completely Simple Semigroups

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ABSTRACT: The non-inclusion principal left ideal graph of a semigroup, denoted by $\mathbf{nPiG}_l(S)$ is a simple, undirected graph with the nonzero elements of S as vertices and two distinct elements $a, b \in S$ are adjacent if and only if $a \notin S^1b$ and $b \notin S^1a$, where S^1a and S^1b are principal left ideals generated by a and b respectively. The non-inclusion principal right ideal graph, $\mathbf{nPiG}_r(S)$ is defined similarly. Here, we identify the structure of $\mathbf{nPiG}_l(S)$ and $\mathbf{nPiG}_r(S)$ when S is a completely simple semigroup in terms of Green's equivalences and we establish the correspondence between these graphs and the complete k-partite graphs . Furthermore, we analyze the automorphism groups and discuss some energies associated with these graph structures.

Key Words: Completely simple semigroups, non-inclusion principal ideal graph, automorphism group, energies.

Contents

Introduction

Algebraic graph theory primarily focuses on the dynamic relationship between graph structures and algebraic structures. The characterization and determination of properties of algebraic structures using their graphs have been a subject of increasing interest in the past few decades and have been actively investigated in the literature. In the modern era, it has evolved significantly due to advances in computational power, interdisciplinary applications, and new theoretical developments [1,18,15]. Graph energy is a well-defined concept in algebraic graph theory, computed using the eigenvalues of the adjacency matrix of the graph. The concept of graph energy was initially developed by Ivan Gutman [7] in the context of chemical graph theory and has evolved into a versatile tool with applications across mathematics, chemistry, physics, and computer science.

For a commutative ring R with a nonzero identity, the concept of cozero-divisor graphs was introduced in 2011 [14]. The cozero-divisor graph $\Gamma'(R)$, viewed as a dual counterpart to the zero-divisor graph $\Gamma(R)$, is an undirected graph whose vertices belong to all non-unit elements of R, and two distinct vertices a and b are adjacent exactly when $a \notin bR$ and $b \notin aR$. In the same year, principal ideal graphs of semigroups were studied by R. S. Indu and L. John as the graph with vertex set as a semigroup and two vertices are connected by an edge in the graph if their principal ideals intersect [9,10,11,12]. Recent studies on the different energies of these graphs can be found in [4,5]. Inspired by this, we introduce the notion of the non-inclusion principal ideal graph for semigroups. Specifically, the non-inclusion principal left ideal graph $\mathbf{nPiG}_l(S)$ is defined with vertex set $S \setminus \{0\}$, where two distinct vertices a and b are adjacent precisely when $a \notin S^1b$ and $b \notin S^1a$; here, $S^1a = \{a\} \cup \{sa : s \in S\}$ denotes the principal left ideal generated by a. Similarly, the non-inclusion principal right ideal graph $\mathbf{nPiG}_r(S)$ has vertex set $S \setminus \{0\}$,

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with vertices a and b adjacent if and only if $a \notin bS^1$ and $b \notin aS^1$. This study extends existing work on cozero-divisor graphs and principal ideal graphs by introducing new graph structures for semigroups that parallel the concepts used in commutative ring theory but specifically tailored to the algebraic structure of semigroups.

This article focuses on investigating the structural characteristics of the non-inclusion principal ideal graphs associated with completely simple semigroups. We explore the automorphism groups and various energy measures of these graphs.

Section 1 presents relevant preliminary notions. In Section 2, we establish necessary and sufficient conditions characterizing adjacency and non-adjacency in $\mathbf{nPiG}_l(S)$ and $\mathbf{nPiG}_r(S)$, and provide descriptions of these graphs in terms of $\mathscr{L}$-classes and $\mathscr{R}$-classes, respectively. We demonstrate that both $\mathbf{nPiG}_l(S)$ and $\mathbf{nPiG}_r(S)$ are complete k-partite graphs, and show that for each complete k-partite graph G, there exists a completely simple semigroup S whose corresponding non-inclusion principal left (or right) ideal graph is isomorphic to G. Moreover, we analyze the conditions under which the graphs for completely simple semigroups coincide with those of rectangular bands.

The automorphism group $Aut(G)$ of a graph G is the set of bijections $\psi : \mathcal{V}(G) \to \mathcal{V}(G)$ preserving adjacency and non-adjacency [6]. Section 3 characterizes the automorphism groups $Aut(\mathbf{nPiG}_l(S))$ and $Aut(\mathbf{nPiG}_r(S))$ when S is completely simple.

Finally, Section 4 investigates the characteristic polynomials of certain matrices associated with the non-inclusion principal ideal graphs of completely simple semigroups and computes the related graph energies.

1. Preliminaries

The notations and terminologies used throughout this paper are introduced in this section. By a semigroup S, we mean a nonempty set equipped with an associative binary operation. Green's equivalence relation $\mathscr{L}$ on S is such that for $a, b \in S$, we have $a \mathscr{L} b$ if and only if the principal left ideals generated by a and b coincide, i.e., $S^1a = S^1b$ [8]. The relation $\mathscr{R}$ is defined analogously but with respect to principal right ideals. An element $a \in S$ is called regular if there exists some $a' \in S$ such that $aa'a = a$. A semigroup in which every element is regular is a regular semigroup. The join of the relations $\mathscr{L}$ and $\mathscr{R}$ is denoted by $\mathcal{D}$ [8]. In this work, our attention is focused on completely simple semigroups, which are regular semigroups with exactly one $\mathcal{D}$-class. According to Rees [17], completely simple semigroups admit a structural description as a semigroup constructed from a group, two nonempty index sets, and a matrix with entries from the group.

Theorem 1.1 ([8])**.** *Let $\mathbb{G}$ be a group, and let I and Λ be nonempty sets. Consider a matrix $P = (p_{\lambda i})$ of size $\Lambda \times I$ with entries in $\mathbb{G}$. Define the set $S = \mathbb{G} \times I \times \Lambda$ with multiplication given by*

$$(h_1, i, \lambda)(h_2, j, \tau) = (h_1 p_{\lambda j} h_2, i, \tau).$$

Then S is a completely simple semigroup. Moreover, every completely simple semigroup is isomorphic to one constructed via this method.

The semigroup $S = \mathcal{M}(\mathbb{G}; I, \Lambda; P)$ is used to denote this construction. In this special case where $\mathbb{G} = \{e\}$ is the trivial group with a single element, S is called a rectangular band [8]. Here, the group component can be suppressed, and the multiplication reduces to $(i, \lambda)(j, \tau) = (i, \tau)$.

Proposition 1.2. *[8] For $(h_1, i, \lambda), (h_2, j, \tau) \in \mathcal{M}(\mathbb{G}; I, \Lambda; P)$,*

- *$(h_1, i, \lambda) \mathscr{L} (h_2, j, \tau)$ if and only if $\lambda = \tau$,*
- *$(h_1, i, \lambda) \mathscr{R} (h_2, j, \tau)$ if and only if $i = j$.*

The graphs here considered are simple and undirected. A complete k-partite graph is one whose vertex set can be partitioned into k disjoint subsets such that each vertex is connected to every vertex not in its own subset. The complete k-partite graph with each partition having size p is denoted by $T_{k,p}$ [3].

The $\mathcal{A}$-energy of a graph G, denoted $\epsilon_{\mathcal{A}}(G)$, is the sum of absolute values of the eigenvalues of the adjacency matrix $\mathcal{A}(G)$ [7]. Define $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ where $d_i = \deg(v_i)$ for each vertex v_i. The Laplacian matrix $\mathcal{L}(G)$ and the signless Laplacian matrix $\mathcal{Q}(G)$ of G are given by

$$\mathcal{L}(G) = D(G) - \mathcal{A}(G), \quad \mathcal{Q}(G) = D(G) + \mathcal{A}(G).$$

Their respective energies are denoted by $\epsilon_{\mathcal{L}}(G)$ and $\epsilon_{\mathcal{Q}}(G)$ [16]. The distance matrix, denoted $\mathbb{D}$, is defined by $\mathbb{D} = [d_{ij}]$, where d_{ij} is the distance between vertices v_i and v_j; its energy is denoted by $\epsilon_{\mathbb{D}}(G)$ [13].

For any undefined terminology related to semigroup theory, graph theory, or algebraic graph theory, the reader is referred to [8], [3], and [2].

2. Non-Inclusion Principal Ideal Graphs of Completely Simple Semigroups

Here, we outline the distinctive features of the non-inclusion principal ideal graphs associated with completely simple semigroups. Throughout this paper, unless specified otherwise, let $S = \mathcal{M}(\mathbb{G}; I, \Lambda; P)$ represent a completely simple semigroup, where $|\mathbb{G}| = g$, $|I| = m$, and $|\Lambda| = n$. Since S is a regular semigroup, it holds that $S^1a = Sa$ and $aS^1 = aS$ for any $a \in S$ [8].

We commence by establishing the necessary and sufficient conditions under which two elements of S become adjacent in the non-inclusion principal left ideal graph $\mathbf{nPiG}_l(S)$.

Proposition 2.1. *Let $(h_1, i, \lambda), (h_2, j, \tau) \in S$. Then they are adjacent in $\mathbf{nPiG}_l(S)$ if and only if $\lambda \neq \tau$.*

Proof. Let $m_1 = (h_1, i, \lambda), m_2 = (h_2, j, \tau) \in S$. If there is an edge between m_1 and m_2, then $m_1 \notin Sm_2$ and $m_2 \notin Sm_1$. Thus, $m_1 \notin Sm_2 \subseteq \{(h_2, j, \tau) : h_2 \in \mathbb{G}, j \in I\}$. That is $(h_1, i, \lambda) \notin \{(h_2, j, \tau) : h_2 \in \mathbb{G}, j \in I\}$. Hence $\lambda \neq \tau$.

Conversely, assume that $\lambda \neq \tau$. We will show that $m_1 \notin Sm_2$ and $m_2 \notin Sm_1$. If not, let $m_1 \in Sm_2$. Then there exists $(g_1, i_1, \lambda_1) \in S$ such that

$$(g_1, i_1, \lambda_1)(h_2, j, \tau) = (h_1, i, \lambda)$$

That means $(g_1 p_{\lambda_1 j} h_2, i_1, \tau) = (h_1, i, \lambda)$. This leads to a contradiction that $\lambda = \tau$. Hence, when $\lambda \neq \tau$, (h_1, i, λ) and (h_2, j, τ) are adjacent in $\mathbf{nPiG}_l(S)$. □

Now, we state the analogous result for $\mathbf{nPiG}_r(S)$.

Proposition 2.2. *Let $(h_1, i, \lambda), (h_2, j, \tau) \in S$. Then (h_1, i, λ) and (h_2, j, τ) are adjacent in $\mathbf{nPiG}_r(S)$, if and only if $i \neq j$.*

Proposition 1.2 in conjunction with Proposition 2.1 enables us to deduce that $\mathscr{L}$-related elements in S are non-adjacent in $\mathbf{nPiG}_l(S)$.

Proposition 2.3. *For $(h_1, i, \lambda), (h_2, j, \tau) \in S$ are adjacent in $\mathbf{nPiG}_l(S)$, if and only if (h_1, i, λ) and (h_2, j, τ) are not $\mathscr{L}$-related elements.*

Proof. By Proposition 1.2, $(h_1, i, \lambda) \mathscr{L} (h_2, j, \tau)$ if and only if $\lambda = \tau$. Now, by Proposition 2.1, (h_1, i, λ) and $(h_2, j, \tau) \in S$ are adjacent if and only if $\lambda \neq \tau$. Thus, it is necessary and sufficient that (h_1, i, λ) and (h_2, j, τ) are non-$\mathscr{L}$-related, for them to be adjacent in $\mathbf{nPiG}_l(S)$. □

By combining Proposition 1.2 and Proposition 2.2, we see that $\mathscr{R}$-related elements in S are non-adjacent in $\mathbf{nPiG}_r(S)$.

Proposition 2.4. *Two elements $(h_1, i, \lambda), (h_2, j, \tau) \in S$ are adjacent in $\mathbf{nPiG}_r(S)$, if and only if (h_1, i, λ) and (h_2, j, τ) are not $\mathscr{R}$- related.*

Now we characterize $\mathbf{nPiG}_l(\mathscr{L}_a)$, the induced subgraph of $\mathbf{nPiG}_l(S)$.

Proposition 2.5. *For $a \in S$, let $\mathscr{L}_a$ represent the $\mathscr{L}$-class that includes the element a. Then,*

1. *The induced subgraph* $\mathbf{nPiG}_l(\mathscr{L}_a)$ *of* $\mathbf{nPiG}_l(S)$ *with vertex set* $\mathscr{L}_a$ *is a null graph consisting of* gm *vertices.*

2. *For elements* $a, b \in S$ *with* $b \notin \mathscr{L}_a$*, each vertex in* $\mathbf{nPiG}_l(\mathscr{L}_a)$ *is connected to every vertex in* $\mathbf{nPiG}_l(\mathscr{L}_b)$*.*

Proof. 1. Let $a \in S$ and $x, y \in \mathscr{L}_a$ so that $x \mathscr{L} a$, $y \mathscr{L} a$ and hence $x \mathscr{L} y$. Then, by Proposition 2.3, there does not exist an edge between x and y. This leads to the conclusion that $\mathbf{nPiG}_l(\mathscr{L}_a)$ is a null graph. Also, $(h_x, i_x, \lambda_x) \mathscr{L} (h_1, i, \lambda)$ if and only if $\lambda_x = \lambda$. Thus, $\mathscr{L}_a = \{(h, i, \lambda) : h \in \mathbb{G}, i \in I\}$ and $\mathbf{nPiG}_l(\mathscr{L}_a)$ has $|\mathscr{L}_a| = gm$ vertices.

2. Let $x \in \mathbf{nPiG}_l(\mathscr{L}_a)$ and $y \in \mathbf{nPiG}_l(\mathscr{L}_b)$. Then $x \mathscr{L} a$ and $y \mathscr{L} b$. That is $(h_x, i_x, \lambda_x) \mathscr{L} (h_1, i, \lambda)$ and $(h_y, i_y, \lambda_y) \mathscr{L} (h_2, j, \tau)$, then $\lambda_x = \lambda$ and $\lambda_y = \tau$. Since $b \notin \mathscr{L}_a$, $\lambda \neq \tau$. Therefore, $\lambda_x \neq \lambda_y$. Hence, by Proposition 2.1, x and y are adjacent in $\mathbf{nPiG}_l(S)$.

□

Recall that the *join* of two graphs G_1 and G_2, denoted by $G_1 \bigvee G_2$, is constructed by taking disjoint copies of G_1 and G_2 and adding edges between every vertex of G_1 and every vertex of G_2. In other words, it forms the union of these two graphs along with all possible edges connecting the vertices across the separate components [2]. Utilizing this definition together with the conclusions from Proposition 2.5, we obtain a significant description of the graph $\mathbf{nPiG}_l(S)$ in terms of the $\mathscr{L}$-classes of elements $a \in S$.

Theorem 2.6. *Let* $\mathscr{L}_a$ *denote the* $\mathscr{L}$*-class containing* $a \in S$*. Then* $\mathbf{nPiG}_l(S) = \bigvee_{a \in S} \mathbf{nPiG}_l(\mathscr{L}_a)$*.*

Proof. For each $a \in S$, $\{\mathscr{L}_a : a \in S\}$ forms a partition of the vertex set of $\mathbf{nPiG}_l(S)$. By Proposition 2.5, $\mathbf{nPiG}_l(\mathscr{L}_a)$ with vertex set $\mathscr{L}_a$ is a null graph of gm vertices and for $a, b \in S$ and $b \notin \mathscr{L}_a$, every vertex of $\mathbf{nPiG}_l(\mathscr{L}_a)$ is adjacent to every vertex of $\mathbf{nPiG}_l(\mathscr{L}_b)$. Therefore, $\mathbf{nPiG}_l(S)$ is obtained as the join of the induced subgraphs $\mathbf{nPiG}_l(\mathscr{L}_a)$. □

Dually, for $\mathbf{nPiG}_r(S)$, we have the following theorem.

Proposition 2.7. *Let* $\mathscr{R}_a$ *denote the* $\mathscr{R}$*-class of an element* $a \in S$*. Then,*

1. *The induced subgraph* $\mathbf{nPiG}_r(\mathscr{R}_a)$ *of* $\mathbf{nPiG}_r(S)$ *with vertex set* $\mathscr{R}_a$ *is a null graph containing* gn *vertices.*

2. *For any* $a, b \in S$ *with* $b \notin \mathscr{R}_a$*, each vertex in* $\mathbf{nPiG}_r(\mathscr{R}_a)$ *is connected to every vertex in* $\mathbf{nPiG}_r(\mathscr{R}_b)$*.*

Theorem 2.8. *Let* $\mathscr{R}_a$ *denote the* $\mathscr{R}$*-class containing* $a \in S$*. Then* $\mathbf{nPiG}_r(S) = \bigvee_{a \in S} \mathbf{nPiG}_r(\mathscr{R}_a)$*.*

Theorem 2.6 establishes that for every element $a \in S$, the subset $\mathscr{L}_a$ induces a null subgraph, while for distinct elements a and b, every vertex in $\mathscr{L}_a$ is connected to all vertices in $\mathscr{L}_b$. This observation enables us to examine the structure of the graph $\mathbf{nPiG}_l(S)$ as a complete n-partite graph, where each part corresponds to a distinct $\mathscr{L}$-class.

Theorem 2.9. $\mathbf{nPiG}_l(S) \cong T_{n,gm}$

Proof. By Theorem 2.6, $\mathbf{nPiG}_l(S) = \bigvee_{a \in S} \mathbf{nPiG}_l(\mathscr{L}_a)$ where, each $\mathbf{nPiG}_l(\mathscr{L}_a)$ is a null graph with gm vertices. Also, for $a, b \in S$ and $b \notin \mathscr{L}_a$, every vertex of $\mathbf{nPiG}_l(\mathscr{L}_a)$ is adjacent to every other vertex of $\mathbf{nPiG}_l(\mathscr{L}_b)$. Now $b \notin \mathscr{L}_a$, gives $\lambda \neq \tau$. Therefore, there are n distinct $\mathscr{L}$ classes and it follows that $\mathbf{nPiG}_l(S) \cong T_{n,gm}$. □

Corollary 2.10. *In* $\mathbf{nPiG}_l(S)$*, the degree of each vertex is* $(n-1)gm$ *and the total degree of* $\mathbf{nPiG}_l(S)$ *is* $n(n-1)g^2m^2$*.*

Proof. Theorem 2.9 yields that the degree of each vertex is $gmn - gm = gm(n-1)$. As there are gmn vertices of degree $gm(n-1)$, the total degree of the graph is $gm(n-1)gmn = n(n-1)g^2m^2$. □

Corollary 2.11. $\mathbf{nPiG}_l(S)$ *has* $\frac{n(n-1)}{2}m^2g^2$ *edges.*

Corollary 2.12. *Consider* S *with* $|\mathbb{G}| = g$, $|I| = m$ *and* $|\Lambda| = 1$, $\mathbf{nPiG}_l(S)$ *is a null graph with* gm *vertices.*

Corollary 2.13. $\mathbf{nPiG}_l(S)$ *is a connected graph unless* $|\Lambda| = 1$.

The results corresponding to $\mathbf{nPiG}_r(S)$ are as follows.

Theorem 2.14. *The non-inclusion principal right ideal graph* $\mathbf{nPiG}_r(S)$ *forms a complete* m*-partite graph, where each partition contains exactly* gn *vertices.*

Corollary 2.15. *In* $\mathbf{nPiG}_r(S)$*, the degree of each vertex is* $(m-1)gn$ *and the total degree of* $\mathbf{nPiG}_r(S)$ *is* $m(m-1)g^2n^2$.

Corollary 2.16. *The number of edges in* $\mathbf{nPiG}_r(S)$ *is* $\frac{m(m-1)}{2}n^2g^2$.

Corollary 2.17. *Let* S *be a completely simple semigroup with* $|\mathbb{G}| = g$, $|I| = 1$ *and* $|\Lambda| = n$*, then* $\mathbf{nPiG}_r(S)$ *is a null graph with* gn *vertices.*

Corollary 2.18. $\mathbf{nPiG}_r(S)$ *is a connected graph unless* $|I| = 1$.

In the following theorem, we construct a completely simple semigroup from a complete n-partite graph, which is isomorphic to the non-inclusion left ideal graphs.

Theorem 2.19. *For a graph* G *with* $G \cong T_{n,m}$*, there corresponds a completely simple semigroup* S *such that* $\mathbf{nPiG}_l(S) \cong G$.

Proof. Suppose G is a complete n-partite graph with all parts of size m. Then $G = \bigvee_{\lambda\in\Lambda} H_\lambda$, where each subgraph H_λ is an independent set with $|H_\lambda| = m$ for all distinct indices $\lambda \in \Lambda$, and the number of parts satisfies $|\Lambda| = n$. For any group $\mathbb{G}$ of order m, choose the completely simple semigroup $S = \mathcal{M}(\mathbb{G}; I, \Lambda; P)$ with $I = \{1\}$. By Theorem 2.6, the graph $\mathbf{nPiG}_l(S)$ decomposes as a join of null subgraphs: $\bigvee_{x\in S} \mathbf{nPiG}_l(\mathscr{L}_x)$, where each $\mathbf{nPiG}_l(\mathscr{L}_x)$ is an edgeless graph on $|\mathbb{G}| \times |I| = m$ vertices. Since there are exactly n distinct $\mathscr{L}$-classes in S, it follows that $\mathbf{nPiG}_l(S)$ is a complete n-partite graph with parts of size m. Consequently, $\mathbf{nPiG}_l(S) \cong G$. □

Similarly, one can realize $\mathbf{nPiG}_r(S)$ as a complete m-partite graph following the analogous construction.

Theorem 2.20. *For a graph* G *with* $G \cong T_{m,n}$*, there corresponds a completely simple semigroup* S *such that* $\mathbf{nPiG}_r(S) \cong G$.

Now, we establish the existence of non-inclusion principal ideal graphs associated with a finite join of null graphs possessing some properties.

Theorem 2.21. *Let* $G = \{G_\lambda : \lambda \in \Lambda\}$ *and* $H = \{H_i : i \in I\}$ *be finite join of null graphs such that for some integer* g, $|G_\lambda| = g|I|$ *for every* $\lambda \in \Lambda$ *and* $|H_i| = g|\Lambda|$ *for every* $i \in I$*. Then there corresponds a completely simple semigroup* S *with* $\mathbf{nPiG}_l(S) \cong G$ *and* $\mathbf{nPiG}_r(S) \cong H$.

Proof. Consider $S = \mathcal{M}(\mathbb{G}; I, \Lambda; P)$, where $\mathbb{G}$ is a group of order g. According to Theorem 2.6, the graph $\mathbf{nPiG}_l(S)$ can be expressed as a join, $\bigvee_{a\in S} \mathbf{nPiG}_l(\mathscr{L}_a)$, with each $\mathbf{nPiG}_l(\mathscr{L}_a)$ being an edgeless graph whose vertex set is $\mathscr{L}_a$. For an element $a = (h_1, i, \lambda) \in S$, the vertex set of $\mathbf{nPiG}_l(\mathscr{L}_a)$ is $\mathscr{L}_a = \{(h_1', i', \lambda) : h_1' \in \mathbb{G}, i' \in I\}$, which contains $|\mathbb{G}| \times |I| = g|I|$ vertices. The number of distinct $\mathscr{L}$-classes in $\mathbf{nPiG}_l(S)$ equals $|\Lambda|$, and each class $\mathscr{L}_a$ is isomorphic to some unique subgraph G_λ for $\lambda \in \Lambda$. Hence, $\mathbf{nPiG}_l(S) \cong G$.

Similarly, applying Theorem 2.8, it follows that $\mathbf{nPiG}_r(S) = \bigvee_{a\in S} \mathbf{nPiG}_r(\mathscr{R}_a)$, where each $\mathbf{nPiG}_r(\mathscr{R}_a)$ is a null graph with vertex set $\mathscr{R}_a = \{(h_1', i, \lambda') : h_1' \in \mathbb{G}, \lambda' \in \Lambda\}$, which also has size $|\mathbb{G}| \times |\Lambda| = g|\Lambda|$. The total number of distinct $\mathscr{R}$-classes in $\mathbf{nPiG}_r(S)$ matches $|I|$, and each $\mathscr{R}_a$ is isomorphic to a unique H_i for some $i \in I$. From this, we conclude that $\mathbf{nPiG}_r(S) \cong H$. □

It is well-known that a completely simple semigroup becomes a rectangular band when its group component $\mathbb{G}$ is trivial. The subsequent propositions extend this idea, asserting that the non-inclusion principal ideal graphs of a completely simple semigroup and a rectangular band are, in fact, isomorphic, under appropriate conditions.

Proposition 2.22. *Let $\mathbb{G}$ be a trivial group $\{e\}$ and S^* be the rectangular band $(I \times \Lambda)$. Then* $\mathbf{nPiG}_l(S) \cong \mathbf{nPiG}_l(S^*)$.

Proposition 2.23. *Let $S = \mathbb{G} \times I \times \Lambda$ with $\mathbb{G} = \{e\}$ and S^* be the rectangular band $(I \times \Lambda)$. Then* $\mathbf{nPiG}_r(S) \cong \mathbf{nPiG}_r(S^*)$.

We conclude this section with an example illustrating $\mathbf{nPiG}_l(S)$ and $\mathbf{nPiG}_r(S)$.

Example 2.24. Let $S = \mathcal{M}(\mathbb{G}; I, \Lambda; P)$ with $\mathbb{G} = Z_2 = \{0, 1\}$, $I = \{1, 2, 3\}$, $\Lambda = \{1, 2\}$ and $P = (p_{\lambda i})$ be any $\Lambda \times I$ matrix with entries in the group $\mathbb{G}$. Then $\mathbf{nPiG}_l(S)$ is a 2-partite graph with 6 vertices in each partition as shown in Figure 1. While $\mathbf{nPiG}_r(S)$ is a 3- partite graph with 4 vertices in each partition, as shown in Figure 2.

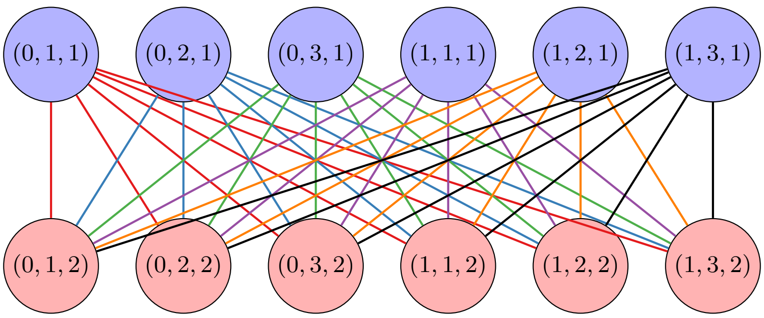

Figure 1: $\mathbf{nPiG}_l(\mathbb{Z}_2; \{1, 2, 3\}, \{1, 2\}; P)$

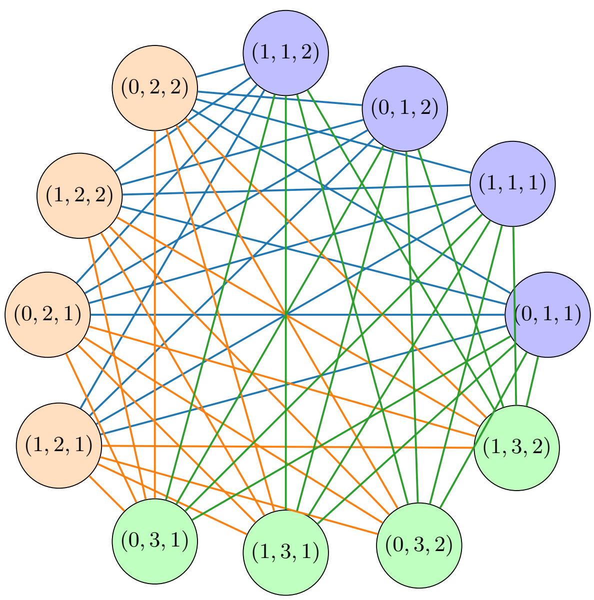

Figure 2: $\mathbf{nPiG}_r(\mathbb{Z}_2; \{1, 2, 3\}, \{1, 2\}; P)$

3. $Aut(\mathbf{nPiG}_l(S))$ and $Aut(\mathbf{nPiG}_r(S))$

This section focuses on the group of automorphisms of $\mathbf{nPiG}_l(S)$ and $\mathbf{nPiG}_r(S)$. First, we determine the automorphism group of $\mathbf{nPiG}_l(S)$.

Theorem 3.1. *$Aut(\mathbf{nPiG}_l(S)) \cong S_n \times \Pi_{k=1}^{n} S_{gm}$, where S_n denotes the symmetric group on n letters.*

Proof. By Theorem 2.9, $\mathbf{nPiG}_l(S)$ is a complete n-partite graph with each partition having gm vertices. Automorphism of these n partitions is isomorphic to $\Pi_{k=1}^{n} S_{gm}$. Since there are n partitions, by permuting them, the group of automorphisms of $\mathbf{nPiG}_l(S)$ is isomorphic to

$$S_n \times \Pi_{k=1}^{n} S_{gm}$$

□

Theorem 3.2. *$Aut(\mathbf{nPiG}_r(S)) \cong S_m \times \Pi_{k=1}^{n} S_{gn}$.*

4. Energies of $\mathbf{nPiG}_l(S)$ and $\mathbf{nPiG}_r(S)$

In this section, we determine the characteristic polynomials of $\mathbf{nPiG}_l(S)$ and $\mathbf{nPiG}_r(S)$, which are associated with various energies of the graphs. First, we describe $\epsilon_{\mathcal{A}}(\mathbf{nPiG}_l(S))$.

Theorem 4.1. *The $\mathcal{A}$-energy, $\epsilon_{\mathcal{A}}(\mathbf{nPiG}_l(S)$ is $2gm(n-1)$.*

Proof. Theorem 2.9 depicts that the adjacency matrix of $\mathbf{nPiG}_l(S)$ has a block structure, with zero matrices for vertices within the same partition and all-one matrices for vertices between different partitions. The structure of the adjacency matrix is given by,

$$A(\mathbf{nPiG}_l(S)) = \begin{bmatrix} O_{gm} & U_{gm} & \dots & U_{gm} \\ U_{gm} & O_{gm} & \dots & U_{gm} \\ . & . & \dots & . \\ . & . & \dots & . \\ U_{gm} & U_{gm} & \dots & O_{gm} \end{bmatrix}_{gmn \times gmn}$$

where,

$$O_{gm} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & 0 \end{bmatrix}_{gm \times gm}$$

and

$$U_{gm} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ . & . & \dots & . \\ . & . & \dots & . \\ 1 & 1 & \dots & 1 \end{bmatrix}_{gm \times gm}$$

The characteristic polynomial of $\mathcal{A}(\mathbf{nPiG}_l(S))$ is obtained as

$$x^{n(gm-1)}[x - gm(n-1)](x + gm)^{n-1}.$$

The A-eigenvalues obtained are 0 of multiplicity $n(gm-1)$, $gm(n-1)$ of multiplicity 1, and $-gm$ of multiplicity $(n-1)$ which yields that $\epsilon_{\mathcal{A}}(\mathbf{nPiG}_l(S)) = 2gm(n-1)$. □

Corollary 4.2. *If $\rho(\mathbf{nPiG}_l(S))$ denotes the largest A-eigenvalue of $\mathbf{nPiG}_l(S)$, then we have $\rho(\mathbf{nPiG}_l(S)) \geq 0$ and $\rho(\mathbf{nPiG}_l(S)) = gm(n-1)$ with multiplicity 1.*

Proof. The $\mathcal{A}$- eigenvalues of $\mathbf{nPiG}_l(S)$ are $0, gm(n-1)$ and $-gm$, of which the largest value is $gm(n-1)$ with multiplicity 1. Since $n \geq 1$ and hence $\rho(\mathbf{nPiG}_l(S)) \geq 0$. □

Next, we give a characterization for $\epsilon_{\mathcal{L}}(\mathbf{nPiG}_l(S))$

Theorem 4.3. *The L-energy, $\epsilon_{\mathcal{L}}(\mathbf{nPiG}_l(S))$ is $g^2m^2n(n-1)$.*

Proof. Theorem 2.9 and Corollary 2.10 yield that the Laplacian matrix has a block structure with $(n-1)gmI_{gm}$ vertices within the same partition and all -1 entry matrices for vertices between different partitions. We obtain the structure of the Laplacian matrix $\mathcal{L}(\mathbf{nPiG}_l(S))$ as

$$\mathcal{L}(\mathbf{nPiG}_l(S)) = \begin{bmatrix} (n-1)gmI_{gm} & -U_{gm} & \dots & -U_{gm} \\ -U_{gm} & (n-1)gmI_{gm} & \dots & -U_{gm} \\ . & . & \dots & . \\ . & . & \dots & . \\ -U_{gm} & -U_{gm} & \dots & (n-1)gmI_{gm} \end{bmatrix}_{gmn \times gmn}$$

where,

$$(n-1)gmI_{gm} = \begin{bmatrix} (n-1)gm & 0 & \dots & 0 \\ 0 & (n-1)gm & \dots & 0 \\ . & . & \dots & . \\ . & . & \dots & . \\ 0 & 0 & \dots & (n-1)gm \end{bmatrix}_{gm \times gm}$$

and

$$-U_{gm} = \begin{bmatrix} -1 & -1 & \dots & -1 \\ -1 & -1 & \dots & -1 \\ . & . & \dots & . \\ . & . & \dots & . \\ -1 & -1 & \dots & -1 \end{bmatrix}_{gm \times gm}$$

.
The characteristic polynomial of $\mathcal{L}(\mathbf{nPiG}_l(S))$ is obtained as

$$x[x - gm(n-1)]^{(gm-1)n}(x - gmn)^{n-1}.$$

Thus the L-eigenvalues are 0 of multiplicity 1, $gm(n-1)$ of multiplicity $(gm-1)n$, and gmn of multiplicity $(n-1)$. Thus, $\epsilon_{\mathcal{L}}(\mathbf{nPiG}_l(S)) = g^2m^2n(n-1)$. □

Corollary 4.4. *If $\mu(\mathbf{nPiG}_l(S))$ denotes the largest L-eigenvalue of $\mathbf{nPiG}_l(S)$, then $\mu(\mathbf{nPiG}_l(S)) \geq 1$ and $\mu(\mathbf{nPiG}_l(S)) = gmn$ with multiplicity $n-1$.*

Proof. By Theorem 4.3, the $\mathcal{L}$- eigenvalues of $\mathbf{nPiG}_l(S)$ are $0, gm(n-1)$ and gmn of which gmn is the largest with multiplicity $n-1$. Since G, I, Λ are nonempty, $\mu(\mathbf{nPiG}_l(S)) \geq 1$. □

Now we characterize the signless Laplacian energy of $\mathbf{nPiG}_l(S)$.

Theorem 4.5. *The $\mathcal{Q}$-energy $\epsilon_{\mathcal{Q}}(\mathbf{nPiG}_l(S))$ of the non-inclusion left ideal graph is $g^2m^2n(n-1)$.*

Proof. Theorem 2.9 and Corollary 2.10 express the signless Laplacian matrix has a block structure, with $(n-1)gmI_{gm}$ for vertices within the same partition and all-one matrix for vertices between different partitions. The structure of the signless Laplacian matrix $\mathcal{Q}(\mathbf{nPiG}_l(S))$ is obtained as

$$Q(\mathbf{nPiG}_l(S)) = \begin{bmatrix} (n-1)gmI_{gm} & U_{gm} & \dots & U_{gm} \\ U_{gm} & (n-1)gmI_{gm} & \dots & U_{gm} \\ . & . & \dots & . \\ . & . & \dots & . \\ U_{gm} & U_{gm} & \dots & (n-1)gmI_{gm} \end{bmatrix}_{gmn \times gmn}$$

where,

$$(n-1)gmI_{gm} = \begin{bmatrix} (n-1)gm & 0 & ... & 0 \\ 0 & (n-1)gm & ... & 0 \\ . & . & ... & . \\ . & . & ... & . \\ 0 & 0 & ... & (n-1)gm \end{bmatrix}_{gm\times gm}$$

and

$$U_{gm} = \begin{bmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ . & . & ... & . \\ . & . & ... & . \\ 1 & 1 & ... & 1 \end{bmatrix}_{gm\times gm}$$

The characteristic polynomial of $Q(\mathbf{nPiG}_l(S))$ is obtained as

$$[x-2gm(n-1)][x-gm(n-2)]^{(n-1)}[x-gm(n-1)]^{(m-1)n}.$$

Hence the Q-eigenvalues of $\mathcal{L}(\mathbf{nPiG}_l(S))$ are $2gm(n-1)$ of multiplicity 1, $gm(n-2)$ of multiplicity $n-1$, and $gm(n-1)$ of multiplicity $(m-1)n$. Thus the Q-energy is $\epsilon_Q(\mathbf{nPiG}_l(S)) = g^2m^2n(n-1)$. □

Corollary 4.6. *If $q(\mathbf{nPiG}_l(S))$ denotes the largest Q-eigenvalue of $\mathbf{nPiG}_l(S)$, then $q(\mathbf{nPiG}_l(S)) \geq 0$ and $q(\mathbf{nPiG}_l(S)) = 2gm(n-1)$ with multiplicity* 1.

Proof. By Theorem 4.5, the eigenvalues of $Q(\mathbf{nPiG}_l(S))$ are $2gm(n-1)$, $gm(n-2)$ and $gm(n-1)$ of which the largest value is $2gm(n-1)$ with multiplicity 1. Since Λ is nonempty $n \geq 1$ and $q(\mathbf{nPiG}_l(S)) \geq 0$. □

From Theorem 4.3 and Theorem 4.5, we can conclude that $\epsilon_{\mathcal{L}}(\mathbf{nPiG}_l(S)) = \epsilon_Q(\mathbf{nPiG}_l(S))$
Since $\mathbf{nPiG}_l(S)$ is connected, we compute its distance energy in the following theorem.

Theorem 4.7. *The $\mathbb{D}$-energy, $\epsilon_{\mathbb{D}}(\mathbf{nPiG}_l(S))$ is $3gmn + 2gm - 2n - 4$*

Proof. By Theorem 2.9, in $\mathbf{nPiG}_l(S)$, the distance between two vertices belonging to the same partition is 2 and that between those vertices belonging to different partitions is 1. Hence, the distance matrix has a block structure with $2U_{gm} - 2I_{gm}$ for vertices within the same partition and all-one matrix for vertices between different partitions. The distance matrix $\mathbb{D}(\mathbf{nPiG}_l(S)$ is given by

$$\mathbb{D}(\mathbf{nPiG}_l(S)) = \begin{bmatrix} 2U_{gm}-2I_{gm} & U_{gm} & ... & U_{gm} \\ U_{gm} & 2U_{gm}-2I_{gm} & ... & U_{gm} \\ . & . & ... & . \\ . & . & ... & . \\ U_{gm} & U_{gm} & ... & 2U_{gm}-2I_{gm} \end{bmatrix}_{gmn\times gmn}$$

where,

$$2U_{gm} - 2I_{gm} = \begin{bmatrix} 0 & 2 & ... & 2 \\ 2 & 0 & ... & 2 \\ . & . & ... & . \\ . & . & ... & . \\ 2 & 2 & ... & 0 \end{bmatrix}_{gm\times gm}$$

and

$$U_{gm} = \begin{bmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ . & . & ... & . \\ . & . & ... & . \\ 1 & 1 & ... & 1 \end{bmatrix}_{gm\times gm}$$

The characteristic polynomial of $\mathbb{D}(\mathbf{nPiG}_l(S))$ is obtained as

$$[x-(gm(n+1)-2)][x-(gm-2)](x+2)^{n(gm-1)}$$

Hence, the $\mathbb{D}$-eigenvalues of $\mathbb{D}(\mathbf{nPiG}_l(S))$ are $gm(n+1)-2$ of multiplicity 1, $gm-2$ of multiplicity 1 , and -2 of multiplicity $n(gm-1)$, and the $\mathbb{D}$-energy is
$\epsilon_{\mathbb{D}}(\mathbf{nPiG}_l(S)) = 3gmn + 2gm - 2n - 4$. □

Corollary 4.8. *If $d(\mathbf{nPiG}_l(S))$ denotes the largest $\mathbb{D}-$eigenvalue of $\mathbf{nPiG}_l(S)$, then $d(\mathbf{nPiG}_l(S)) \geq 0$ and $d(\mathbf{nPiG}_l(S)) = gm(n+1)-2$ with multiplicity 1.*

Proof. By Theorem 4.7, $\mathbb{D}$-eigenvalues of $\mathbf{nPiG}_l(S)$ are $gm(n+1)-2$, $gm-2$ and -2 of which the largest value is $gm(n+1)-2$ with multiplicity 1. Since Λ is nonempty $n \geq 1$ and $d(\mathbf{nPiG}_l(S)) \geq 0$. □

We conclude this section by stating the relevant results related to the energies of $\mathbf{nPiG}_r(S)$ without delving into the proofs.

Theorem 4.9. *The $\mathcal{A}$-energy, $\epsilon_{\mathcal{A}}(\mathbf{nPiG}_r(S))$ is $2gn(m-1)$ and the largest $\mathcal{A}$-eigenvalue of $\mathbf{nPiG}_r(S)$, $\rho(\mathbf{nPiG}_r(S)) \geq 0$ and $\rho(\mathbf{nPiG}_r(S))$ is $gn(m-1)$ with multiplicity 1.*

Theorem 4.10. *The $\mathcal{L}$-energy, $\epsilon_{\mathcal{L}}(\mathbf{nPiG}_r(S))$ is $g^2n^2m(m-1)$ the largest $\mathcal{L}$-eigenvalue of $\mathbf{nPiG}_r(S)$, $\mu(\mathbf{nPiG}_r(S)) \geq 1$ and $\mu(\mathbf{nPiG}_r(S))$ is gmn with multiplicity $m-1$.*

Theorem 4.11. *The $\mathcal{Q}$-energy, $\epsilon_{\mathcal{L}}(\mathbf{nPiG}_r(S))$ is $g^2n^2m(m-1)$ and the largest $\mathcal{Q}$-eigenvalue of $\mathbf{nPiG}_r(S)$, $q(\mathbf{nPiG}_r(S)) \geq 0$ and $q(\mathbf{nPiG}_r(S))$ is $2gn(m-1)$ with multiplicity 1.*

Theorem 4.12. *The $\mathbb{D}-$ energy, $\epsilon_D(\mathbf{nPiG}_l(S))$ is $3gmn+2gn-2m-4$ and the largest $\mathbb{D}$-eigenvalue of $\mathbf{nPiG}_r(S)$, $d(\mathbf{nPiG}_r(S)) \geq 0$ and $d(\mathbf{nPiG}_r(S))$ is $gn(m+1)-2$ with multiplicity 1.*

5. Conclusion

In this article, we introduced the notion of non-inclusion principal ideal graphs of semigroups. We studied the structure, automorphism group, and some energies of the non-inclusion principal ideal graphs when the semigroup is completely simple. We obtained that the graph structure is isomorphic to that of a complete k-partite graph G, and for a given complete k-partite graph G, we can find a completely simple semigroup S, whose non-inclusion principal ideal graphs are isomorphic to G. We summarize the main results in the following table:

	$\mathbf{nPiG}_l$	$\mathbf{nPiG}_r$
Graph structure	$T_{n,gm}$	$T_{m,gn}$
Degree of each vertex	$(n-1)gm$	$(m-1)gn$
Total degree	$n(n-1)g^2m^2$	$m(m-1)g^2n^2$
Number of edges	$\frac{n(n-1)}{2}g^2m^2$	$\frac{m(m-1)}{2}g^2n^2$
$Aut(G)$	$S_n \times \prod_{k=1}^{n} S_{gm}$	$S_m \times \prod_{k=1}^{m} S_{gn}$
$\mathcal{A}$-energy	$2gm(n-1)$	$2gn(m-1)$
largest A-eigenvalue	$gm(n-1)$	$gn(m-1)$
L-Energy	$g^2m^2n(n-1)$	$g^2n^2m(m-1)$
largest L-eigenvalue	gmn	gmn
Q-Energy	$g^2m^2n(n-1)$	$g^2n^2m(m-1)$
largest Q-eigenvalue	$2gm(n-1)$	$2gn(m-1)$
$\mathbb{D}$-Energy	$3gmn+2gm-2n-4$	$3gmn+2gn-2m-4$
largest $\mathbb{D}$-eigenvalue	$gm(n+1)-2$	$gn(m+1)-2$

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