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On the Kernel Eigenspace of Coalescence of Singular Graphs

Reeja S. B.*, John K. Rajan, Sreekumar K. G.

ABSTRACT: A finite simple undirected graph is said to be singular if its adjacency matrix has eigenvalue 0. If a vertex u in a graph G_1 is identified with a vertex v in a graph G_2, then the resulting graph $G_1 \circ G_2$, of order $|G_1| + |G_2| - 1$, is called the coalescence of G_1 and G_2 with respect to u and v. Singular graphs consist of core and noncore vertices. In this paper, we coalesce two singular graphs and study the kernel eigenspace of $G_1 \circ G_2$, and based on this analysis, determine the core and noncore vertices of the coalesced graph.

Key Words: Singular graph, adjacency matrix, core vertex, Noncore vertex, coalescence, Kernel eigenspace.

Contents

1. Introduction

A simple graph G is a graph with no multiple edges and loops. The total number of vertices in G, denoted by $o(G)$, is called the order of G . In this paper, we consider finite simple undirected graph. The adjacency matrix A of a graph G with n vertices $v_1, v_2, \cdots, v_n$ is an $n \times n$ matrix with ij^{th} entry as 1 if v_i and v_j are adjacent, and 0 otherwise. The nullity of the adjacency matrix of a graph G is called nullity of G. If the nullity η of G is greater than or equal to 1 then G is called singular graph. In this case, 0 is an eigenvalue of the adjacency matrix. The eigenvectors corresponding to the eigenvalue 0 are obtained by solving the system of linear equation ($AX = 0$). If v is a vertex in G, then the null spread of v is defined as $\eta(G) - \eta(G - v)$. Null spread of a vertex lies between -1 and 1. Null spread of a noncore vertex is either 0 or -1. Also, null spread of a core vertex is always 1. Noncore vertex is called Fiedler vertex or F-vertex or Core forbidden vertex. In [1,3], noncore vertex with null spread -1 is called upper core-forbidden vertex and noncore vertex with null spread 0 is called middle core-forbidden vertex.

The concepts of nullity, kernel eigenspaces, and core vertices have been extensively studied in spectral graph theory. Sciriha laid much of the foundation by characterizing singular graphs [8], analyzing graphs with nullity one [7], and investigating maximal core sizes [9]. Her work on extremal non-bonding orbitals [6] further emphasized the role of graph nullity in mathematical chemistry.

Building on these foundations, Kim and Shader [4] analyzed Fiedler and Parter vertices in acyclic matrices, while Edholm et al. [2] explored vertex and edge spread in relation to zero forcing number, maximum nullity, and minimum rank, highlighting the interplay between structural and spectral properties of graphs.

Several studies have focused on the spectral effects of graph operations. Ali et al. [1] examined coalescence with respect to Fiedler and core vertices, establishing important connections between coalescence and spectral characteristics. Varkey and Rajan contributed further by investigating the spectrum and energy of both coalesced singular graphs [11] and singular graphs more broadly [10]. Applications to

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fullerene structures were also demonstrated by Fowler et al. [3], where the spectral properties of singular graphs were linked to conduction phenomena.

Together, these works underscore the importance of singular graphs in both theoretical and applied contexts. However, the specific effect of coalescence on the kernel eigenspace and the classification of vertices into core and noncore categories has not been fully resolved. The present paper addresses this gap by establishing detailed conditions under which the kernel eigenspace of the coalesced graph decomposes as a direct sum of the eigenspaces of the components, or properly contains them, while confirming the preservation of core and noncore vertices across the operation.

2. Preliminaries

In this section, we start with the formal definition of core and non-core vertices of a singular graph, as introduced in [1], [5], [6], [7].

Definition 2.1 *Let G be a singular graph of n vertices $v_1, v_2, \cdots, v_n$ with nullity $\eta \geq 1$ and $\mathbb{B}$ be a basis of its kernel eigenspace. If the vertices in G are relabeled in such a way that all the vectors in $\mathbb{B}$ are of the form $X = (x_{v_1}, x_{v_2}, \cdots, x_{v_m}, 0, 0, \cdots, 0)$, where $x_{v_1}, x_{v_2}, \cdots, x_{v_m}$ are nonzero in at least one kernel eigenvector in $\mathbb{B}$, then the vertices $v_1, v_2, \cdots, v_m$ are called core vertices, and $v_{m+1}, v_{m+2}, \cdots, v_n$ are called noncore vertices.*

The distinction between core and noncore vertices plays a fundamental role in determining how the nullity of a graph changes under vertex deletion and, more generally, under graph operations such as coalescence. In particular, the concept of null spread provides a measure of this change and allows us to classify vertices according to their contribution to the kernel eigenspace. Using this framework, the following result describes how the nullity behaves when two singular graphs are coalesced at vertices with different null spreads.

Theorem 2.2 *[1] If G_1, G_2 are two singular graphs with nullity η_1, η_2, respectively, then*

1. *the nullity of the coalescence $G_1 \circ G_2$ of G_1 and G_2 with respect to noncore vertices with null spread zero is $\eta_1 + \eta_2$.*
2. *the nullity of the coalescence $G_1 \circ G_2$ of G_1 and G_2 with respect to noncore vertices with null spread zero and null spread -1 (or vice versa) is $\eta_1 + \eta_2$.*
3. *the nullity of the coalescence $G_1 \circ G_2$ of G_1 and G_2 with respect to noncore vertices with null spread -1 is $\eta_1 + \eta_2 + 1$.*
4. *the nullity of the coalescence $G_1 \circ G_2$ of G_1 and G_2 with respect to a core vertex in G_1 and a noncore vertex (null spread 0 or -1) in G_2 is $\eta_1 + \eta_2 - 1$*

3. Kernel Eigenspace of Coalescence of Singular Graphs

Let G_1, G_2 be two singular graphs with vertex sets $\{v_1, v_2, \cdots, v_n\}$ and $\{u_1, u_2, \cdots, u_m\}$, having nullities η_1, η_2, respectively. The vertex labels of $G_1 \circ G_2$ are the same as those of G_1 and G_2, except for the coalesced vertex. If the vertices $v_i \in G_1$ and $u_j \in G_2$ are used for coalescing, then the coalesced vertex in $G_1 \circ G_2$ is labeled as v_i. Thus, the vertex set of $G_1 \circ G_2$ are labeled as $v_1, v_2, \cdots, v_i, v_{i+1}, \cdots, v_n, u_1, u_2, \cdots, u_{j-1}, u_{j+1}, \cdots, u_m$. Let A_1, A_2, and A be the adjacency matrices of $G_1, G_2, G_1 \circ G_2$, respectively. Furthermore, let E_1, E_2, and E represent their respective kernel eigenspaces. Suppose G_1 and G_2 each have at least one noncore vertex. Our aim is to determine the kernel eigenspace, E of $G_1 \circ G_2$ with respect to noncore vertices using E_1 and E_2. For this purpose, we consider the subspaces $E_1^{'}, E_2^{'}$ of $\mathbb{R}^{(n+m-1)}$:

- $E_1^{'}$ is an η_1 dimensional subspace of $\mathbb{R}^{(n+m-1)}$ in which the last $(m-1)$ entries of each vector are zero and the first n entries coincide with those of the corresponding vector in E_1.

- $E_2^{'}$ is an η_2 dimensional subspace of $\mathbb{R}^{n+m-1}$ in which the first n entries of each vector are zero and the remaining $(m-1)$ entries coincide with the first $(m-1)$ entries of the corresponding vector in E_2.

Throughout the paper we use the notation O_k to denote a column vector of length k whose entries are all zero.
The following example illustrates this relation clearly and also leads to the next theorem.

Example 3.1 *In Figure 1, G_1 and G_2 are singular graphs and $G_1 \circ G_2$ is the coalescence of G_1 and G_2 with respect to noncore vertices v_3 and u_3 with null spread* 0.

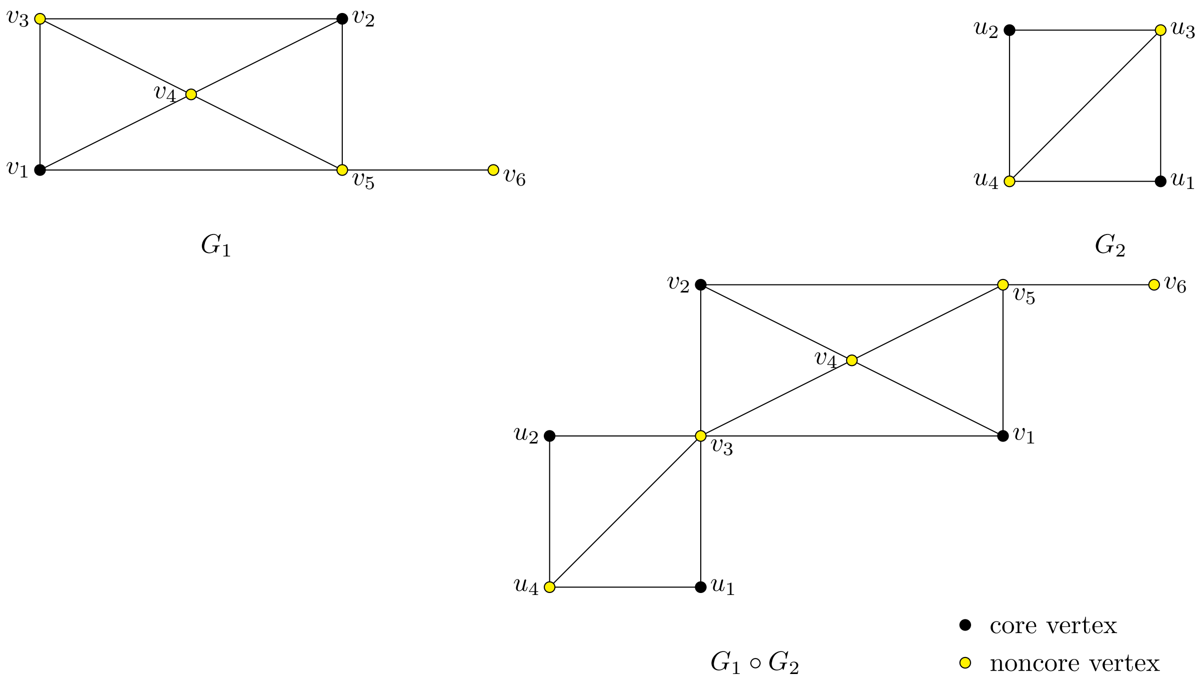

Figure 1: $G_1 \circ G_2$ with respect to the noncore vertices having null spread 0

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Using the direct method for finding kernel eigenvector, we obtain
$E_1 = span\{(-1, 1, 0, 0, 0, 0)^T\}$, $E_2 = span\{(-1, 1, 0, 0)^T\}$, *and*
$E = span\{(-1, 1, 0, 0, 0, 0, 0, 0, 0)^T, (0, 0, 0, 0, 0, 0, -1, 1, 0)^T\}$.
As noted earlier in this section, construct the subspace $E_1^{'}$ and $E_2^{'}$ of $\mathbb{R}^9$ using E_1 and E_2, respectively. Thus, $E_1^{'} = span\{(-1, 1, 0, 0, 0, 0, 0, 0, 0)^T\}$, *and* $E_2^{'} = span\{(0, 0, 0, 0, 0, 0, -1, 1, 0)^T\}$,
Hence, we conclude that $E = E_1^{'} \oplus E_2^{'}$.

Theorem 3.2 *If $G_1 \circ G_2$ is the coalescence of two singular graphs G_1 and G_2, with nullities η_1 and η_2, respectively, with respect to noncore vertices having null spread 0, then the kernel eigenspace E of $G_1 \circ G_2$ satisfies* $E = E_1^{'} \oplus E_2^{'}$.

Proof: Without loss of generality, assume that $v_1, v_2, \cdots, v_N$ and $u_1, u_2, \cdots u_M$ are core vertices of G_1 and G_2, respectively. All other vertices of G_1 and G_2 are noncore vertices. We take the noncore vertices v_n with null spread zero from G_1 and u_m with null spread zero from G_2 to construct $G_1 \circ G_2$. There is no loss of generality in making this choice. For convenience, the adjacency matrices A_1 and A_2 are represented as block matrices, each consisting of 9 blocks. Thus we have

$$A_1 = \begin{bmatrix} P_1 & P_2 & S_1 \\ P_2^T & P_3 & S_2 \\ S_1^T & S_2^T & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} Q_1 & Q_2 & T_1 \\ Q_2^T & Q_3 & T_2 \\ T_1^T & T_2^T & 0 \end{bmatrix},$$

where P_1 is an $N \times N$ matrix, P_2 is an $N \times (n-N-1)$ matrix, P_3 is an $(n-N-1) \times (n-N-1)$ matrix, S_1 is an $N \times 1$ matrix , S_2 is an $(n-N-1) \times 1$ matrix. Q_1 is an $M \times M$ matrix, Q_2 is an $M \times (m-M-1)$ matrix, Q_3 is an $(m-M-1) \times (m-M-1)$ matrix, T_1 is an $M \times 1$ matrix, T_2 is an $(m-M-1) \times 1$ matrix.

The adjacency matrix A can be represented as a block matrix such that $A = \begin{bmatrix} P_1 & P_2 & S_1 & 0 & 0 \\ P_2^T & P_3 & S_2 & 0 & 0 \\ S_1^T & S_2^T & 0 & T_1^T & T_2^T \\ 0 & 0 & T_1 & Q_1 & Q_2 \\ 0 & 0 & T_2 & Q_2^T & Q_3 \end{bmatrix}$.

Since G_1 has N core vertices, the last $n-N$ components of every vector in E_1 are zero. Let $\begin{bmatrix} X \\ O_{n-N} \end{bmatrix}$ be an arbitrary vector in E_1, where X is a column vector with N entries.

Then, $\begin{bmatrix} X \\ O_{n-N-1} \\ 0 \\ O_M \\ O_{m-M-1} \end{bmatrix} = \begin{bmatrix} X \\ O_{n-N} \\ O_{m-1} \end{bmatrix} \in E$, since $A \begin{bmatrix} X \\ O_{n-N-1} \\ 0 \\ O_M \\ O_{m-M-1} \end{bmatrix} = 0$. Thus, we get

$$E_1^{'} = \left\{ \begin{bmatrix} X \\ O_{n-N} \\ O_{m-1} \end{bmatrix} \middle| \begin{bmatrix} X \\ O_{n-N} \end{bmatrix} \in E_1 \right\} \subset E. \tag{3.1}$$

Since G_2 has M core vertices, last $(m-M)$ components of every vector in E_2 are zero. Let $\begin{bmatrix} Y \\ O_{m-M} \end{bmatrix}$ be an arbitrary vector in E_2, where Y is a column vector with M entries.

Then, $\begin{bmatrix} O_N \\ O_{n-N-1} \\ 0 \\ Y \\ O_{m-M-1} \end{bmatrix} = \begin{bmatrix} O_n \\ Y \\ O_{m-M-1} \end{bmatrix} \in E$, since $A \begin{bmatrix} O_N \\ O_{n-N-1} \\ 0 \\ Y \\ O_{m-M-1} \end{bmatrix} = 0$. Thus, we get

$$E_2^{'} = \left\{ \begin{bmatrix} O_n \\ Y \\ O_{m-M-1} \end{bmatrix} \middle| \begin{bmatrix} Y \\ O_{m-M} \end{bmatrix} \in E_2 \right\} \subset E \tag{3.2}$$

.

From (3.1) and (3.2), we get $E_1^{'}$, $E_2^{'} \subset E$. Also, $E_1^{'} + E_2^{'} \subseteq E$. By Theorem 1.1, $dimE = \eta_1 + \eta_2 = dimE_1^{'} + dimE_2^{'}$. Hence $E = E_1^{'} \oplus E_2^{'}$. □

Theorem 3.3 *If $G_1 \circ G_2$ is the coalescence of two singular graphs G_1 and G_2, with nullities η_1 and η_2, respectively, with respect to noncore vertices having null spread 0, then all core vertices of G_1 and G_2 remain core vertices of $G_1 \circ G_2$ and all noncore vertices of G_1 and G_2 remain noncore vertices of $G_1 \circ G_2$.*

Proof: Using Definition 2.1 and Theorem 3.2, we can directly prove this theorem. □

We now explore how E, E_1, and E_2 are related when coalescing noncore vertices of G_1 and G_2 having null spread 0 and -1. The following example illustrates the connection and provides insight for the subsequent theorem.

Example 3.4 *In Figure 2, G_1 and G_2 are singular graphs and $G_1 \circ G_2$ is the coalescence of G_1 and G_2 with respect to noncore vertex v_3 with null spread 0 and noncore vertex u_4 with null spread -1.*

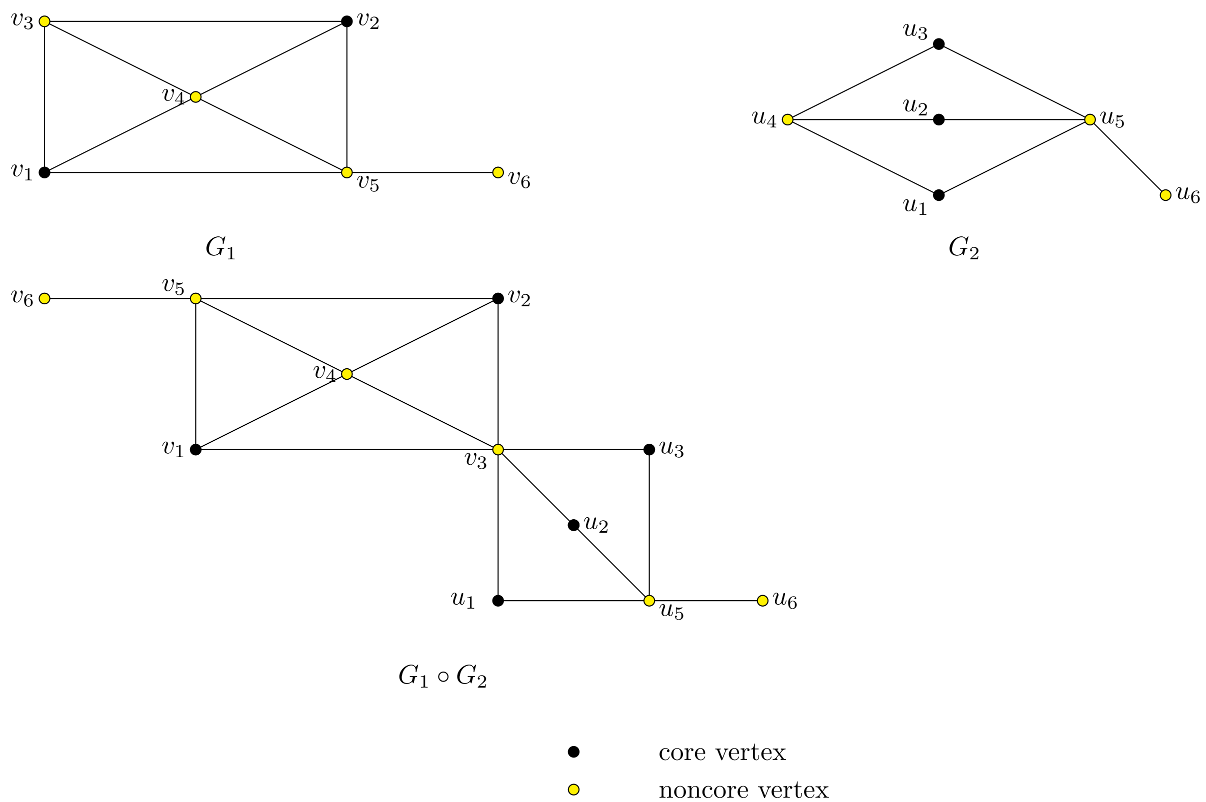

Figure 2: $G_1 \circ G_2$ with respect to noncore vertices having null spread 0 and null spread -1

Here, $E_1 = span\{(-1,1,0,0,0,0)^T\}$,
$E_2 = span\{(-1,1,0,0,0,0)^T, (-1,0,1,0,0,0)^T\}$ *and*
$E = span\{(-1,1,0,0,0,0,0,0,0,0,0)^T, (0,0,0,0,0,0,-1,1,0,0,0)^T, (0,0,0,0,0,0,-1,0,1,0,0)^T\}$.
Construct the subspaces $E_1^{'}$, $E_2^{'}$ *of* $\mathbb{R}^{11}$ *using* E_1, E_2, *respectively. Thus*
$E_1^{'} = span\{(-1,1,0,0,0,0,0,0,0,0,0)^T\}$
$E_2^{'} = span\{(0,0,0,0,0,0,-1,1,0,0,0)^T, (0,0,0,0,0,0,-1,0,-1,0,0)^T\}$,
$dimE_1^{'} + dimE_2^{'} = dimE$.
Thus, we get $E_1^{'} \oplus E_2^{'} = E$.

The above example leads to the following theorem.

Theorem 3.5 *If $G_1 \circ G_2$ is the coalescence of two singular graphs G_1 and G_2 with nullities η_1 and η_2 respectively, with respect to noncore vertices having null spreads 0 and -1 (or vice versa), then the kernel eigenspace E of $G_1 \circ G_2$ satisfies $E = E_1^{'} \oplus E_2^{'}$.*

Proof: The proof follows the same steps as those in Theorem 3.2. □

Theorem 3.6 *If $G_1 \circ G_2$ is the coalescence of two singular graphs G_1 and G_2 with nullities η_1 and η_2 respectively, with respect to non-core vertices having null spreads 0 and -1 (or vice versa), then all core verices of G_1 and G_2 remain core vertices in $G_1 \circ G_2$, and all noncore vertices of G_1 and G_2 remain noncore vertices in $G_1 \circ G_2$.*

Proof: Using Definition 2.1 and Theorem 3.5, we can directly prove this theorem. □

Example 3.7 *Consider the singular graphs G_1 and G_2 in the Example 3.4. $G_1 \circ G_2$ is the coalescence of G_1 and G_2 with respect to noncore vertices v_6 and u_6 with null spread -1.*

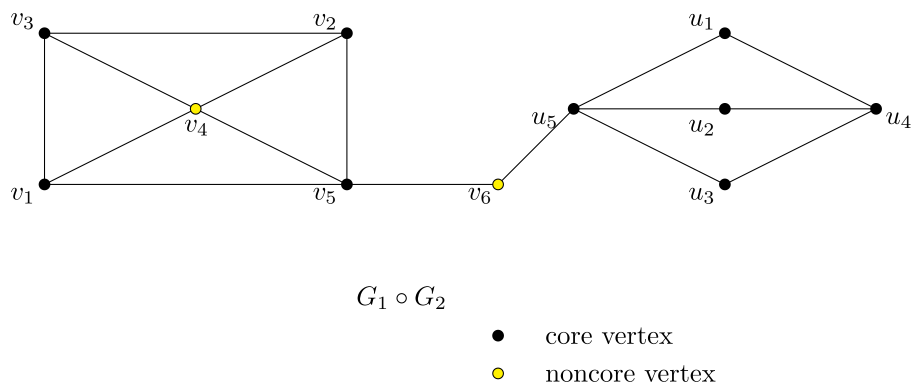

Figure 3: $G_1 \circ G_2$ with respect to noncore vertices having null spread -1

By applying the direct method to find the kernel eigenvectors, we obtain $E_1 = span\{(-1,1,0,0,0,0)^T\}$
$E_2 = span\{(-1,1,0,0,0,0)^T, (-1,0,1,0,0,0)^T\}$
$E = span\{(-1,1,0,0,0,0,0,0,0,0,0)^T, (0,0,0,0,0,0,-1,1,0,0,0)^T, (0,0,0,0,0,0,-1,0,1,0,0)^T, (0,0,1,0,-1,0,0,0,0,-1,1)^T\}$.
Construct the subspace $E_1^{'}$, $E_2^{'}$ *of* $\mathbb{R}^{11}$ *using* E_1, E_2, *respectively. Thus,*
$E_1^{'} = span\{(-1,1,0,0,0,0,0,0,0,0,0)^T\}$,
$E_2^{'} = span\{(0,0,0,0,0,0,-1,1,0,0,0)^T, (0,0,0,0,0,0,-1,0,-1,0,0)^T\}$
This implies $E_1^{'} + E_2^{'} \subset E$
Since $E_1^{'} \cap E_2^{'} = \{0\}$, *we get* $E \supset E_1^{'} \oplus E_2^{'}$.

Theorem 3.8 *If $G_1 \circ G_2$ is the coalescence of two singular graphs G_1 and G_2, with nullities η_1 and η_2, respectively, with respect to noncore vertices having null spread -1, then the kernel eigenspace E of $G_1 \circ G_2$ satisfies $E \supset E_1^{'} \oplus E_2^{'}$.*

Proof: The proof for obtaining $E_1^{'} + E_2^{'} \subseteq E$ follows the same steps as in the proof of Theorem 3.2. By Theorem 1.1, we have $dimE = \eta_1 + \eta_2 + 1 = dimE_1 + dimE_2 + 1$. Additionally, $E_1^{'} \cap E_2^{'} = \phi$. Thus, it follows that $E \supset E_1^{'} \oplus E_2^{'}$. □

When noncore vertices with null spread -1 are identified, the dimension of $E_1^{'} \oplus E_2^{'}$ is one less than the dimension of E. Consequently, some noncore vertices in G_1 and G_2 become core vertices in the graph $G_1 \circ G_2$. The following theorem describes the preservation of vertex roles from G_1 and G_2 in $G_1 \circ G_2$.

Theorem 3.9 *If $G_1 \circ G_2$ is the coalescence of two singular graphs G_1 and G_2 with nullities η_1 and η_2, respectively, with respect to noncore vertices having null spread -1, then all core vertices of G_1 and G_2 remain core vertices in $G_1 \circ G_2$.*

Proof: Using Definition 2.1 and Theorem 3.8, we can directly prove this theorem. □

In the final part of this section, we define the subspaces $E_1^{''}$ and $E_2^{''}$ of $\mathbb{R}^{n+m-1}$ using subspaces $W_1 \subset E_1$ and $W_2 \subset E_2$, respectively, where:

- W_1 consists of all vectors in E_1 with the entry corresponding to a core vertex v_i equal to zero.
- W_2 consists of all vectors in E_2 with the entry corresponding to the core vertex u_j equal to zero.

The subspaces $E_1^{''}$ and $E_2^{''}$ are defined as follows:

- $E_1^{''}$ is an $\eta_1 - 1$ dimensional subspace of $\mathbb{R}^{n+m-1}$ in which the first n entries of each vector are the same as those of the vectors in W_1 and the last $m-1$ entries are zero.
- $E_2^{''}$ is an $\eta_2 - 1$ dimensional subspace of $\mathbb{R}^{n+m-1}$ in which the first n entries of each vector are zero, and the last $m-1$ entries are obtained from W_2 by removing the entry corresponding to the core vertex u_j.

We present an example to illustrate the nature of the vertices in the coalesced graph $G_1 \circ G_2$, obtained by identifying the noncore vertex and core vertex.

Example 3.10 *Consider the same graphs G_1 and G_2 in the Example 3.4. In Figure 4, $G_1 \circ G_2$ is the coalescence of G_1 and G_2 with respect to noncore vertex u_6 with null spread -1 and core vertex v_3.*

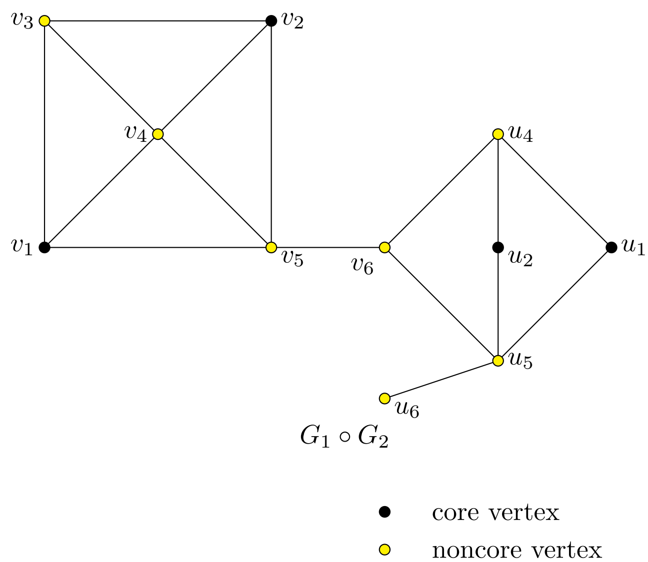

Figure 4: $G_1 \circ G_2$ with respect to noncore vertex with null spread -1 and core vertex

Here, $E_1 = span\{(-1,1,0,0,0,0)^T\}$
$E_2 = span\{(-1,1,0,0,0,0)^T, (-1,0,1,0,0,0)^T\}$
$E = span\{(-1,1,0,0,0,0,0,0,0,0,0)^T, (0,0,0,0,0,0,-1,1,0,0,0)^T\}$
Then, as noted earlier in this section, we construct the subspace W_2 of E_2. Thus we get $W_2 = span\{(-1,1,0,0,0,0)^T\}$. *Hence* $E_2^{''} = span\{(0,0,0,0,0,0,-1,1,0,0,0)^T\}$.
Using E_1 we get $E_1^{'} = span\{(-1,1,0,0,0,0,0,0,0,0,0)^T\}$.
Also $dimE_1^{'} + dimE_2^{''} = dimE$. *Hence we get* $E_1^{'} \oplus E_2^{''} = E$.

The above example gives rise to the following theorem.

Theorem 3.11 *If $G_1 \circ G_2$ is the coalescence of two singular graphs, G_1 with nullity η_1 and G_2 with nullity η_2, with respect to a noncore vertex in G_1 and a core vertex u_j in G_2, then the kernel eigenspace, E of $G_1 \circ G_2$ satisfies $E = E_1^{'} \oplus E_2^{''}$.*

Proof: Let the vertices $v_1, v_2, \ldots, v_N$ and $u_1, u_2, \ldots, u_M$ represent the core vertices of G_1 and G_2, respectively. All remaining vertices in both graphs are considered noncore. To form the coalesced graph $G_1 \circ G_2$, we identify a noncore vertex v_n of G_1 with a core vertex u_j of G_2; this choice does not affect generality. The adjacency matrix A_1 of G_1 can be expressed in block form, similar to its representation in the proof of Theorem 3.2:

$$A_1 = \begin{bmatrix} P_1 & P_2 & S_1 \\ P_2^T & P_3 & S_2 \\ S_1^T & S_2^T & 0 \end{bmatrix}.$$

Likewise, the adjacency matrix A_2 of G_2 can be partitioned as

$$A_2 = \begin{bmatrix} Q_1 & T_1 & Q_2 & Q_3 \\ T_1^T & 0 & T_2^T & T_3^T \\ Q_2^T & T_2 & Q_4 & Q_5 \\ Q_3^T & T_3 & Q_5^T & Q_6 \end{bmatrix},$$

where Q_1 is a $(j-1)\times(j-1)$ matrix, Q_2 is a $(j-1)\times(M-j)$ matrix, Q_3 is a $(j-1)\times(m-M)$ matrix, Q_4 is a $(M-j)\times(M-j)$ matrix, Q_5 is a $(M-j)\times(m-M)$ matrix, Q_6 is a $(m-M)\times(m-M)$ matrix, and T_1, T_2, T_3 are column matrices of corresponding orders.

The adjacency matrix A of the coalesced graph $G_1 \circ G_2$ is then

$$A = \begin{bmatrix} P_1 & P_2 & S_1 & 0 & 0 & 0 \\ P_2^T & P_3 & S_2 & 0 & 0 & 0 \\ S_1^T & S_2^T & 0 & T_1^T & T_2^T & T_3^T \\ 0 & 0 & T_1 & Q_1 & Q_2 & Q_3 \\ 0 & 0 & T_2 & Q_2^T & Q_4 & Q_5 \\ 0 & 0 & T_3 & Q_3^T & Q_5^T & Q_6 \end{bmatrix}.$$

Since G_1 contains N core vertices, every vector in its kernel eigenspace E_1 has its last $(n-N)$ components equal to zero. Hence, an arbitrary vector in E_1 can be represented as $\begin{bmatrix} X \\ O_{n-N} \end{bmatrix}$, where X is an $N \times 1$ column vector.

Then $\begin{bmatrix} X \\ O_{n-N-1} \\ 0 \\ O_M \\ O_{m-M-1} \end{bmatrix} = \begin{bmatrix} X \\ O_{n-N} \\ O_{m-1} \end{bmatrix} \in E_1^{'}$, since $A \begin{bmatrix} X \\ O_{n-N-1} \\ 0 \\ O_M \\ O_{m-M-1} \end{bmatrix} = 0$. Hence,

$$E_1^{'} = span\left\{ \begin{bmatrix} X \\ O_{n-N} \\ O_{m-1} \end{bmatrix} \middle| \begin{bmatrix} X \\ O_{n-N} \end{bmatrix} \in E_1 \right\} \subset E \tag{3.3}$$

.

Consider the subspace W_2 of E_2 which contains all vectors in E_2 having jth entry zero. Since G_2 has M core vertices, the last $m-M$ entries of every vector in E_2 are zero. Let $\begin{bmatrix} Y_1 \\ 0 \\ Y_2 \\ O_{m-M} \end{bmatrix}$ be an arbitrary vector in W_2 , where Y_1 is a column matrix with $(j-1)$ entries, Y_2 is a column matrix with $(M-j)$ entries.

Then $\begin{bmatrix} O_N \\ O_{n-N-1} \\ 0 \\ Y_1 \\ Y_2 \\ O_{m-M} \end{bmatrix} = \begin{bmatrix} O_n \\ Y_1 \\ Y_2 \\ O_{m-M} \end{bmatrix} \in E$, since $A \begin{bmatrix} O_N \\ O_{n-N-1} \\ 0 \\ Y_1 \\ Y_2 \\ O_{m-M} \end{bmatrix} = 0$. Hence,

$$E_2^{''} = span\left\{ \begin{bmatrix} O_n \\ Y_1 \\ Y_2 \\ O_{m-M} \end{bmatrix} \middle| \begin{bmatrix} Y_1 \\ 0 \\ Y_2 \\ O_{m-M} \end{bmatrix} \in E_2 \right\} \subset E \tag{3.4}$$

From (3.3) and (3.4), $E_1^{'}, E_2^{''} \subseteq E$ and $E_1^{'} + E_2^{''} \subseteq E$. By Theorem 2.2, $dim(E) = \eta_1 + \eta_2 - 1 = dim(E_1^{'}) + dim(E_2^{''})$. Hence, $E = E_1^{'} \oplus E_2^{''}$. □

Theorem 3.12 *If $G_1 \circ G_2$ is the coalescence of two singular graphs G_1 and G_2 with respect to a noncore vertex in G_1 and a core vertex u_j in G_2, then all core vertices in G_1 remain core vertices in $G_1 \circ G_2$ and all noncore vertices in G_1 and G_2 remain noncore vertices in $G_1 \circ G_2$.*

Proof: Using Definition 2.1 and Theorem 3.11, we can directly prove this theorem. □

4. Conclusion

In this paper, we studied the structure of the kernel eigenspace of the coalescence of two singular graphs under different conditions on the null spread of the involved vertices. We established that when the coalescence is taken with respect to noncore vertices having null spread 0, the kernel eigenspace of the resulting graph decomposes as the direct sum of the vector spaces constructed from the corresponding eigenspaces, and the classification of core and noncore vertices is preserved. A similar preservation of eigenspaces and vertex classification was obtained when the coalescence is taken with respect to noncore vertices having null spreads 0 and -1 (or vice versa).When the coalescence is performed with respect to noncore vertices having null spread -1, we determined a relation of the kernel eigenspaces of the graphs and that of the coalesced graph, and proved that the coalesced graph preserves the roles of core vertices. Furthermore, when a noncore vertex of one graph is coalesced with a core vertex of the other, we established a relation among the eigenspaces, and verified that noncore vertices remain unchanged in the coalesced graph.Overall, these results provide a detailed characterization of how kernel eigenspaces and the classification of core and noncore vertices behave under coalescence. They offer insight into the algebraic structure of singular graphs and open further directions for investigating spectral properties of more general graph operations. Since singular graphs and their nullities have interpretations in molecular orbital theory, particularly in the study of nonbonding orbitals, the present result may be applied to construct or analyze molecular structures with desired spectral characteristics.

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Reeja S. B.,
Department of Mathematics,
University College,
University of Kerala,
India.
E-mail address: reejasurijabeegum@gmail.com

and

John K. Rajan,
Department of Mathematics,
University College,
University of Kerala,
India.
E-mail address: johnkrajan@yahoo.co.in

and

Sreekumar K. G.,
Department of Mathematics,
University of Kerala,
Thiruvananthapuram
India.
E-mail address: sreekumar3121@gmail.com