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Certain Aspects of Analytic Function Subclasses Associated with Laguerre Polynomials

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ABSTRACT: In this research article, two novel subclasses of analytic functions denoted by $\mathcal{R}_{lag}$ and $\mathcal{C}_{lag}$ are defined through subordination involving Laguerre polynomials. The initial coefficients of the functions belonging to these classes are determined and the corresponding Fekete- szegö inequalities are derived. In addition, analogous results are obtained for the inverse function h^{-1}. As an application to the main results, we examine their connection with the Pòlya- Eggenberger distribution.

Key Words: Analytic functions, univalent functions, Laguerre polynomials, Pólya -Eggenberger distribution, coefficient bounds, Fekete-szegö inequality.

Contents

1. Introduction and Preliminaries

Consider the class $\mathfrak{A}$ of analytic functions in the open unit disk $\mathfrak{D}$ such that $\{\zeta \in \mathfrak{C} : |\zeta| < 1\}$ normalized by the expansion

$$h(\zeta) = \zeta + \sum_{k=2}^{\infty} c_k \zeta^k, \zeta \in \mathfrak{D} \tag{1.1}$$

Functions in $\mathfrak{A}$ that are injective in $\mathfrak{D}$ form a subclass $\mathfrak{S}$ of normalized univalent functions. By the Koebe one quarter theorem, every $h \in \mathfrak{S}$ has an inverse h^{-1} analytic in a disk of radius atleast $1/4$. This inverse is expressed as

$$l(\xi) = h^{-1}(\xi) = \xi - c_2\xi^2 + (2c_2^2 - c_3)\xi^3 - (5c_3^2 - 5c_2c_3 + c_4)\xi^4 + \dots \tag{1.2}$$

Denote by $\mathfrak{P}$ the class of analytic functions with $\Re(r(\zeta)) > 0$ in the Taylor series representation

$$r(\zeta) = 1 + \sum_{k=1}^{\infty} r_k \zeta^k \quad (\zeta \in \mathfrak{D}) \tag{1.3}$$

Two analytic functions g and h in $\mathfrak{D}$ are said to be subordinate to each other denoted by $g \prec h$ if there exists a Schwarz function $\varphi(\zeta)$ satisfying the conditions $\varphi(0) = 0$ and $\varphi(\zeta) < 1$ such that [8]

$$g(\zeta) = h(\varphi(\zeta)) \quad \zeta \in \mathfrak{D}.$$

If g is univalent in $\mathfrak{D}$, then subordination relation is equivalent to

$$g(0) = h(0) \quad \text{and} \quad g(\mathfrak{D}) \subset h(\mathfrak{D}).$$

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The concept of generalized subclasses of analytic functions was introduced by Ma and Minda [6] in 1992 through the use of subordination as follows

$$S^*(\psi) = \left\{h \in \mathfrak{A} : \frac{\zeta h'(\zeta)}{h(\zeta)} \prec \psi(\zeta) \quad (\zeta \in \mathfrak{D})\right\}$$

and

$$C(\psi) = \left\{h \in \mathfrak{A} : 1 + \frac{\zeta h''(\zeta)}{h'(\zeta)} \prec \psi(\zeta) \quad (\zeta \in \mathfrak{D})\right\}.$$

In these definitions ,the comparison function $\psi(\zeta)$ is analytic in the unit disk $\mathfrak{D}$ satisfying the conditions $\psi(0) = 1$ and $\psi'(0) > 0$. This transforms $\mathfrak{D}$ onto a domain that is starlike with respect to 1 and symmetric about the real axis. Different selctions of the comparison function ψ give rise to numerous Ma-Minda type subclasses of starlike and convex functions which have been widely examined in the literature. [4,7,17]

The determination of the initial coefficients c_2 and c_3 is a well-known problem in Geometric Function Theory. Two fundamental tools in this direction are the Fekete-szegö inequality and the Hankel determinant. The sth Hankel determinant was first introduced by Pommerenke [13] as follows,

$$H_{s,k} = \begin{bmatrix} c_k & c_{k+1} & c_{k+2} & \cdots & c_{k+s-1} \\ c_{k+1} & c_{k+2} & c_{k+3} & \cdots & c_{k+s} \\ c_{k+2} & c_{k+3} & c_{k+4} & \cdots & c_{k+s+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{k+s-1} & c_{k+s} & c_{k+s+1} & \cdots & c_{k+2(s-1)} \end{bmatrix}. \tag{1.4}$$

Noor [11] studied the asymptotic behavior of $H_{s,k}(h)$ as $k \to \infty$, while Pommerenke [14] highlighted its applications in detecting singularities. Later, numerous researchers investigated the Hankel determinant for distinct subclasses of $\mathfrak{A}$ [1,9,10,15].

For different choices of s and k, we get the Hankel determinant as

$$H_{2,1}(c) = \begin{vmatrix} 1 & c_2 \\ c_2 & c_3 \end{vmatrix} = c_3 - c_2^2 \tag{1.5}$$

and

$$H_{2,2}(c) = \begin{vmatrix} c_2 & c_3 \\ c_3 & c_4 \end{vmatrix} = c_2 c_4 - c_3^2. \tag{1.6}$$

The functionals $H_{2,1}(c)$ and $H_{2,2}(c)$ reduce to the Fekete-Szegö inequality $(\nu = 1)$ and the Hankel determinant respectively which are well-known results in this field.

The Fekete-Szegö functional

$$|c_3 - \nu c_2^2|, \quad \gamma \in \mathfrak{C} \tag{1.7}$$

was introduced in the study of univalent functions by Fekete and Szegö, this provides valuable information about the growth and distortion of analytic mappings. Sharp bounds for this functional have been established for several subclasses of $\mathfrak{S}$ and later was extended to bi-univalent classes. This inequality serves as a refinement of the classical Bieberbach conjecture and remains central in coefficient theory.

Recently, considerable interest has been focused on developing subclasses of analytic functions with geometric properties through the use orthogonal polynomials and special functions. In this context, we employ the Laguerre polynomial $L_k(\lambda, \tau)$ which satisfies the differential equation

$$ny'' + (1 + \lambda - n)y' + ry = 0$$

where $1 + \lambda > 0, n \in \mathbb{R}$ and $k \geq 0$ [5]. The solution of this differential equation is said to be the generalized Laguerre polynomial, written as $\mathfrak{L}_k^{\lambda}(q)$. Applications of Laguerre polynomials arise in several branches of mathematical physics notably in solving the Helmholtz equation in paraboloidal coordinates and examining electromagnetic waves propagation. The generating function of Laguerre polynomial [16] is given by

$$G_\lambda(q,\zeta)=\sum_{k=0}^{\infty}L_k^{\lambda}(q)\zeta^k=\frac{e^{\frac{-\lambda\zeta}{1-\zeta}}}{(1-\zeta)^{\lambda+1}} \tag{1.8}$$

and the recurrence relation of Laguerre polynomial is given by

$$\mathfrak{L}_{k+1}^{\lambda}(q)=\frac{2k+1+\lambda-q}{k+1}\mathfrak{L}_k^{\lambda}(q)+\frac{k+\lambda}{k+1}\mathfrak{L}_{k-1}^{\lambda}(q)\quad \text{for}\quad k\geq 1 \tag{1.9}$$

with the initial terms as

$$\mathfrak{L}_0^{\lambda}(q)=1,\mathfrak{L}_1^{\lambda}(q)=1+\lambda-q\quad\text{and}\quad \mathfrak{L}_2^{\lambda}(q)=\frac{q^2}{2}-(\lambda+2)q+\frac{(\lambda+1)(\lambda+2)}{2}. \tag{1.10}$$

The recurrence relation of the Laguerre polynomial when $\lambda=0$ becomes $L_k^0(q)=L_k(q)$.
Motivated by the work of the researchers in this field and using the concept of subordination between two analytic functions, we introduce two novel subclases $\mathcal{R}_{lag}$ and $\mathcal{C}_{lag}$ satisfying the following subordination conditions.

Definition 1.1 *Let $0\leq\eta\leq 1$ and $h\in\mathfrak{A}$. Then $h\in\mathcal{R}_{lag}(G,\eta,\zeta)$ if the following condition is satisfied*

$$[h'(\zeta)]^{\eta}\left[\frac{\zeta h'(\zeta)}{h(\zeta)}\right]^{1-\eta}\prec G_\lambda(q,\zeta),\quad (\zeta\in\mathfrak{D}). \tag{1.11}$$

Note that

$$\mathcal{R}_{lag}^{*}(G,\zeta)=\mathcal{R}_{lag}^{*}(G,0,\zeta)=\left\{h\in\mathfrak{A}:\frac{\zeta h'(\zeta)}{h(\zeta)}\prec G_\lambda(q,\zeta),\quad (\zeta\in\mathfrak{D})\right\}$$

and

$$\mathcal{R}_{lag}^{**}(G,\zeta)=\mathcal{R}_{lag}^{*}(G,1,\zeta)=\left\{h\in\mathfrak{A}:h'(\zeta)\prec G_\lambda(q,\zeta),\quad (\zeta\in\mathfrak{D})\right\}.$$

Definition 1.2 *Let $0\leq\eta\leq 1$ and $h\in\mathfrak{A}$. Then $h\in\mathcal{C}_{lag}(G,\eta,\zeta)$ if the following condition is satisfied*

$$[h'(\zeta)]^{\eta}\left[1+\frac{\zeta h''(\zeta)}{h'(\zeta)}\right]^{1-\eta}\prec G_\lambda(q,\zeta),\quad (\zeta\in\mathfrak{D}) \tag{1.12}$$

Note that

$$\mathcal{C}_{lag}^{*}(G,\zeta)=\mathcal{C}_{lag}^{*}(G,0,\zeta)=\left\{h\in\mathfrak{A}:1+\frac{\zeta h''(\zeta)}{h'(\zeta)}\prec G_\lambda(q,\zeta),\quad (\zeta\in\mathfrak{D})\right\}$$

and

$$\mathcal{C}_{lag}^{**}(G,\zeta)=\mathcal{C}_{lag}^{*}(G,1,\zeta)=\left\{h\in\mathfrak{A}:h'(\zeta)\prec G_\lambda(q,\zeta),\quad (\zeta\in\mathfrak{D})\right\}.$$

The primary objective of this research article is to determine the coefficient estimates and derive bounds for the Fekete-szegö functional $|c_3-\nu c_2^2|$ with respect to both real and complex values of the parameter ν. Furthermore, coefficient inequalities are obtained for the inverse function h^{-1} associated with the considered classes. An application of the results is also provided in connection with the Polya- Eggenberger distribution.
To investigate further, we are in need of the following lemmas:

Lemma 1.1 *[2] Let $r\in\mathfrak{P}$ be of the form (1.3). Then $\forall\, k\in\mathbb{N}$,*

$$|r_k|\leq 2 \tag{1.13}$$

Lemma 1.2 *[3] Let $r \in \mathfrak{P}$ be of the form (1.3). If ν is any complex number then we have*

$$|r_3 - \nu r_2^2| \leq \max\{1, |\nu|\} \tag{1.14}$$

Lemma 1.3 *[6] Let $r \in \mathfrak{P}$ be of the form (1.3).If ν is any real number then we have*

$$|r_3 - \nu r_2^2| \leq \begin{cases} -4\nu + 2 & \nu \leq 0, \\ 2 & 0 \leq \nu \leq 1, \\ 4\nu - 2 & \nu \geq 1. \end{cases} \tag{1.15}$$

2. Coefficient Estimates and Fekete- Szegö Inequalities

Theorem 2.1 *Let h of the form (1.1) belong to the class $\mathcal{R}_{lag}(G, \eta, \zeta)$. Then we have,*

$$|c_2| \leq \frac{2|g_1|}{|1+\eta|} \tag{2.1}$$

$$|c_3| \leq \frac{|g_1|}{|2+\eta|} \max\left\{1, \left|\frac{(2+\eta)(\eta-1)g_1}{2(1+\eta)^2} - \frac{g_2}{g_1}\right|\right\}. \tag{2.2}$$

Proof: Let $h \in \mathcal{R}_{lag}(G, \eta, \zeta)$.Then there exists a Schwarz function $\varphi(\zeta)$ analytic in the unit disk $\mathfrak{D}$, such that

$$[h'(\zeta)]^\eta \left[\frac{\zeta h'(\zeta)}{h(\zeta)}\right]^{1-\eta} = G_\lambda(q, \varphi(\zeta)), \quad (\zeta \in \mathfrak{D}). \tag{2.3}$$

Consider a function $r \in P$. By the subordination principle and the Schwarz function $\varphi(\zeta)$, we have

$$r(\zeta) = \frac{1+\varphi(\zeta)}{1-\varphi(\zeta)} = 1 + r_1\zeta + r_2\zeta^2 + r_zeta^3 + \dots \tag{2.4}$$

$$\implies \varphi(\zeta) = \frac{r_1\zeta + r_2\zeta + r_3\zeta_3 + \dots}{2 + r_1\zeta + r_2\zeta^2 + \dots} \tag{2.5}$$

$$= \frac{r_1}{2}\zeta + (\frac{r_2}{2} - \frac{r_1^2}{4})\zeta^2 + (\frac{r_1^3}{8} - \frac{1}{2}r_1r_2 - \frac{r_3}{2})\zeta^3 + \dots. \tag{2.6}$$

Using the representation of $G_\lambda(q, \varphi(\zeta))$

$$\begin{aligned} G_\lambda(q, \varphi(\zeta)) &= 1 + g_1\varphi(\zeta) + g_2\varphi^2(\zeta) + \dots \\ &= 1 + g_1r_1\zeta + [g_1r_2 + g_2r_1^2]\zeta^2 + \dots \end{aligned} \tag{2.7}$$

where $g_1 = \mathfrak{L}_0^\lambda(q), g_2 = \mathfrak{L}_1^\lambda(q)$ and $g_3 = \mathfrak{L}_2^\lambda(q)$.
On the other hand, from (1.1), we can expand the left- hand side of (2.3) as

$$\begin{aligned} [h'(\zeta)]^\eta \left[\frac{\zeta h'(\zeta)}{h(\zeta)}\right]^{1-\eta} &= 1 + (1+\eta)c_2\zeta + (2+\eta)\left[c_3 - \frac{(1-\eta)}{2}c_2^2\right]\zeta^2 \\ &+ \frac{(3+\eta)}{6}[6c_4 - 6(1-\eta)c_2c_3 + (1-\eta)(2-\eta)c_2^3]\zeta^3 + \dots. \end{aligned} \tag{2.8}$$

Comparing coefficients of ζ and ζ^2 in (2.7) and (2.8), we deduce the following

$$(1+\eta)c_2 = g_1r_1 \tag{2.9}$$

$$(2+\eta)(c_3 - \frac{(1-\eta)c_2^2}{2}) = g_1r_2 + g_2r_1^2 \tag{2.10}$$

$$\implies \quad c_2 = \frac{g_1r_1}{(1+\eta)} \tag{2.11}$$

$$c_3 = \frac{g_1}{(2+\eta}\left(r_2 - \left(\frac{(2+\eta)(\eta-1)g_1}{2(1+\eta)^2} - \frac{g_2}{g_1}\right)r_1^2\right). \tag{2.12}$$

We may write c_3 as

$$c_3 = \frac{g_1}{(2+\eta}(r_2 - \lambda_1 r_1) \tag{2.13}$$

where

$$\lambda_1 = \frac{(2+\eta)(\eta-1)g_1}{2(1+\eta)^2} - \frac{g_2}{g_1}.$$

Using Lemma 1.1 and Lemma 1.2, the required result is obtained. □

Theorem 2.2 *Let h of the form (1.1) belong to the class $\mathcal{C}_{lag}(G,\eta,\zeta)$. Then we have*

$$|c_2| \leq |g_1| \tag{2.14}$$

$$|c_3| \leq \frac{|g_1|}{3|2-\eta|} \max\left\{1, \left|2(\eta-1)g_1 - \frac{g_2}{g_1}\right|\right\}. \tag{2.15}$$

Proof: Let $h \in \mathcal{C}_{lag}(G,\eta,\zeta)$.Then there exists a Schwarz function $\varphi(\zeta)$ analytic in the unit disk $\mathfrak{D}$, such that

$$[h'(\zeta)]^{\eta}\left[1 + \frac{\zeta h"(\zeta)}{h'(\zeta)}\right]^{1-\eta} = G_{\lambda}(q,\varphi(\zeta)), \quad (\zeta \in \mathfrak{D}) \tag{2.16}$$

From, (1.1) we have

$$\begin{aligned} [h'(\zeta)]^{\eta}\left[1 + \frac{\zeta h"(\zeta)}{h'(\zeta)}\right]^{1-\eta} =& 1 + 2c_2\zeta + [3(2-\eta)c_3 - 4(1-\eta)c_2^2]\zeta^2 \\ &+ [8(1-\eta)c_2^3 - 18(1-\eta)c_2c_3 - 4(3-2\eta)c_4]\zeta^3 + \ldots. \end{aligned} \tag{2.17}$$

Using (2.7) and (2.17) in (2.16) and by comparing the coefficients of ζ and ζ^2, we obtain

$$2c_2 = g_1 r_1 \tag{2.18}$$

$$[3(2-\eta)c_3 - 4(1-\eta)c_2^2] = g_1 r_2 + g_2 r_1^2. \tag{2.19}$$

$$\Longrightarrow \; c_3 = \frac{g_1}{3(2-\eta)}\left(r_2 - \left(2(\eta-1)g_1 - \frac{g_2}{g_1}\right)r_1^2\right)$$

The desired result is a consequence of Lemmas 1.1 and 1.2. □

The following results establish the Fekete- Szegö inequality for the subclasses $\mathcal{R}_{lag}(G,\eta,\zeta)$ and $\mathcal{C}_{lag}(G,\eta,\zeta)$ considering both real and complex values of the parameter ν.

Theorem 2.3 *Suppose that $h \in \mathfrak{A}$ is in the class $\mathcal{R}_{lag}(G,\eta,\zeta)$. Then for every complex number ν, the following inequality holds*

$$|c_3 - \nu c_2^2| \leq \frac{|g_1|}{(2+\eta)} \max\left\{1, \left|\frac{(2+\eta)(\eta+2\nu-1)g_1}{2(1+\nu)^2} - \frac{g_2}{g_1}\right|\right\}. \tag{2.20}$$

Proof: From (2.11) and (2.12), we get

$$|c_3 - \nu c_2^2| = \frac{|g_1|}{(2+\eta)}|r_2 - \lambda_2 r_1^2| \tag{2.21}$$

where

$$\lambda_2 = \frac{(2+\eta)(\eta+2\nu-1)g_1}{2(1+\nu)^2} - \frac{g_2}{g_1} \tag{2.22}$$

Applying Lemma 1.2 to (2.21) yields

$$|c_3 - \nu c_2^2| \leq \frac{|g_1|}{(2+\eta)} \max\left\{1, \left|\frac{(2+\eta)(\eta+2\nu-1)g_1}{2(1+\nu)^2} - \frac{g_2}{g_1}\right|\right\} \tag{2.23}$$

□

Theorem 2.4 *Let $h \in \mathcal{R}_{lag}(G,\eta,\zeta)$ then for any real ν the following condition is satisfied*

$$|c_3-\nu c_2^2| \leq \begin{cases} \frac{-2(2+\eta)(\eta+2\nu-1)g_1^2+2(1+\eta)^2(2g_2+g_1)}{g_1(1+\eta)^2} & ,\nu \leq \frac{-2(1+\eta)^2(g_1-g_2)-(2+\eta)(\eta-1)g_1^2}{2(2+\eta)g_1^2} \\ \frac{2g_1}{(2+\eta)} & ,\frac{-2(1+\eta)^2(g_1-g_2)-(2+\eta)(\eta-1)g_1^2}{2(2+\eta)g_1^2} \leq \nu \leq \frac{2(1+\eta)^2(g_1+g_2)-(2+\eta)(\eta-1)g_1^2}{2(2+\eta)g_1^2} \\ \frac{2(2+\eta)(\eta+2\nu-1)g_1^2-2(1+\eta)^2(2g_2+g_1)}{g_1(1+\eta)^2} & ,\nu \geq \frac{2(1+\eta)^2(g_1+g_2)-(2+\eta)(\eta-1)g_1^2}{2(2+\eta)g_1^2}. \end{cases} \tag{2.24}$$

Proof: The desired condition follow directly from applying Lemma 1.3 to (2.21). □

Theorem 2.5 *Suppose $h \in \mathfrak{A}$ is in the class $\mathcal{C}_{lag}(G,\eta,\zeta)$. Then for every complex number ν, the following inequality holds*

$$|c_3-\nu c_2^2| \leq \frac{|g_1|}{3(2-\eta)} \max\left\{1, \left|\frac{(\eta-8(1-\nu))g_1}{4} - \frac{g_2}{g_1}\right|\right\}. \tag{2.25}$$

Proof: From (2.18) and (2.19), we get

$$|c_3-\nu c_2^2| = \frac{|g_1|}{3(2-\eta)}|r_2-\lambda_3 r_1^2| \tag{2.26}$$

where

$$\lambda_3 = \frac{(\eta-8(1-\nu))g_1}{4} - \frac{g_2}{g_1} \tag{2.27}$$

Applying Lemma 1.2 to (2.26) yields the required result. □

Theorem 2.6 *Let $h \in \mathcal{C}_{lag}(G,\eta,\zeta)$ then for any real ν the following condition is satisfied*

$$|c_3-\nu c_2^2| \leq \begin{cases} \frac{-(\nu-2)g_1+8(1-\eta)g_1^2-4g_2}{g_1} & ,\nu \leq \frac{-4(g_1-g_2)+8(1-\eta)g_1^2}{g_1} \\ \frac{2g_1}{3(2-\eta)} & ,\frac{-4(g_1-g_2)+8(1-\eta)g_1^2}{g_1} \leq \nu \leq \frac{4(g_1+g_2)+8(1-\eta)g_1^2}{g_1} \\ \frac{(\nu-2)g_1-8(1-\eta)g_1^2-4g_2}{g_1} & ,\nu \geq \frac{4(g_1+g_2)+8(1-\eta)g_1^2}{g_1}. \end{cases} \tag{2.28}$$

Proof: The desired condition is obtained by applying Lemma 1.3 to relation (2.26). □

3. Coefficient Inequalities for h^{-1}

Theorem 3.1 *Let $h \in \mathcal{R}_{lag}(G,\eta,\zeta)$ be given by (1.1) and let the inverse of h be expressed as*

$$l(\xi) = h^{-1}(\xi) = \xi + \sum_{k=2}^{\infty} b_k \xi^k$$

which represents the analytic continuation of $\mathfrak{D}$. The inverse function of h is valid for $|\xi| \leq r_0$, where $r_0 > \frac{1}{4}$ (the Koebe radius). Then for any complex parameter ν, the following inequalities are satisfied:

$$|d_2| \leq \frac{2|g_1|}{|1+\eta|} \tag{3.1}$$

$$|d_3| \leq \frac{|g_1|}{|2+\eta|} \max\left\{1, \left|\frac{(2+\eta)(3+\eta)g_1}{2(1+\eta)^2} - \frac{g_2}{g_1}\right|\right\} \tag{3.2}$$

and

$$|d_3-\nu d_2^2| \leq \frac{|g_1|}{|2+\eta|} \max\left\{1, \left|\frac{(2+\eta)(3+\eta)g_1-2\nu}{2(1+\eta)^2} - \frac{g_2}{g_1}\right|\right\} \tag{3.3}$$

Proof: Since $l(\xi) = h^{-1}(\xi)$ is the inverse of h, we may write

$$l(\xi) = h^{-1}(\xi) = \xi + \sum_{k=2}^{\infty} d_k \xi^k. \tag{3.4}$$

By definition of the inverse function,

$$h^{-1}(h(\xi)) = h(h^{-1}(\xi)) = \xi \tag{3.5}$$

$$\implies h^{-1}\left(\xi + \sum_{k=2}^{\infty} c_k \xi^k\right) = \xi. \tag{3.6}$$

On expansion we get

$$\xi + (c_2 + d_2)\xi^2 + (c_3 + 2c_2 b_2 + d_3)\xi^3 + \cdots = \xi. \tag{3.7}$$

Comparing the coefficients we obtain

$$d_2 = -c_2, \tag{3.8}$$

$$d_3 = -c_3 - 2c_2 d_2 = 2c_2^2 - d_3. \tag{3.9}$$

Substituting for c_2 and c_3 from (2.11) and (2.12) in (3.8) and (3.9) we get

$$d_2 = \frac{-g_1 r_1}{(1+\eta)} \tag{3.10}$$

$$d_3 = \frac{-g_1}{(2+\eta)}\left(r_2 - \left(\frac{(2+\eta)(3+\eta)g_1}{2(1+\eta)^2} - \frac{g_2}{g_1}\right) r_1^2\right). \tag{3.11}$$

The estimate of d_2 follows directly from Lemma 1.1. To obtain the bound for d_3, we apply Lemma 1.2 , which gives

$$|d_3| \leq \frac{|g_1|}{|2+\eta|} \max\left\{1, \left|\frac{(2+\eta)(3+\eta)g_1}{2(1+\eta)^2} - \frac{g_2}{g_1}\right|\right). \tag{3.12}$$

For any complex number ν, we have

$$|d_3 - \nu d_2^2| \leq \frac{|g_1|}{|2+\eta|}\left| r_2 - \left(\frac{(2+\eta)(3+\eta)g_1 - 2\nu}{2(1+\eta)^2} - \frac{g_2}{g_1}\right) r_1^2 \right|. \tag{3.13}$$

Using Lemma 1.2, we get the desired result. □

Theorem 3.2 *Let $h \in \mathcal{C}_{lag}(G, \eta, \zeta)$ be given by (1.1) and let the inverse of h be expressed as*

$$l(\xi) = h^{-1}(\xi) = \xi + \sum_{k=2}^{\infty} b_k \xi^k$$

represents the analytic continuation of $\mathfrak{D}$. The inverse function of h is valid for $|\xi| \leq r_0$, where $r_0 > \frac{1}{4}$ (the Koebe radius). Then for any complex parameter ν the following inequalities are satisfied

$$|d_2| \leq |g_1| \tag{3.14}$$

$$|d_3| \leq \frac{|g_1|}{3|2-\eta|} \max\left\{1, \left|(4-\eta)g_1 - \frac{g_2}{g_1}\right|\right\} \tag{3.15}$$

and

$$|d_3 - \nu d_2^2| \leq \frac{|g_1|}{3|2-\eta|} \max\left\{1, \left|\frac{(4(4-\eta) + 3\nu(2-\eta))g_1}{4} - \frac{g_2}{g_1}\right|\right\}. \tag{3.16}$$

Proof: The proof is akin to Theorem 3.1 and subtituting the values of c_2 and c_3 from (2.18) and (2.19) in (3.8) and (3.9) , we get

$$d_2 = \frac{-g_1 r_1}{2} \tag{3.17}$$

and

$$d_3 = \frac{-g_1}{3(2-\eta)}\left(r_2 - \left((4-\eta)g_1 - \frac{g_2}{g_1}\right)r_1^2\right). \tag{3.18}$$

The upper bound of d_2 is the direct consequence of Lemma 1.1. Furthermore, employing Lemma 1.2 in connection with (3.18) produces the necessary bound for d_3.
For the case where ν is a complex parameter, it can further be deduced

$$|d_3 - \nu d_2^2| \leq \frac{|g_1|}{3|2-\eta|}\left|r_2 - \left(\frac{4(4-\eta)+3\nu(2+\eta)g_1}{4} - \frac{g_2}{g_1}\right)r_1^2\right|. \tag{3.19}$$

Substituting Lemma 1.2 into the relation (3.19) yields the required result. □

4. Application of the Pólya–Eggenberger Distribution

The Pólya–Eggenberger distribution, introduced by Eggenberger and Pólya in 1923 [12], generalizes the binomial law through an urn model with reinforcement. Motivated by its structural similarity to other discrete distributions such as the Poisson and binomial, we now introduce the corresponding analytic series expansion that will be used in defining subclasses of analytic functions.

Definition 4.1 *Let $a > 0$ and $t > 0$. The Pólya-Eggenberger distribution series is defined by*

$$P(a,t,\zeta) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!}(t\zeta)^k, \quad \zeta \in \mathfrak{D}, \tag{4.1}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the rising factorial (Pochhammer symbol). Equivalently the coefficient sequence is given by

$$a_k = \frac{(a)_k}{k!}t^k, k \geq 0$$

1. For $a = 1$, the sequence reduces to the binomial-type series

$$a_k = \frac{t^k}{k!}.$$

2. The normalized form of this series is written as

$$\tilde{P}(\zeta) = \zeta + \sum_{k=2}^{\infty}\gamma_k\zeta^k, \qquad \gamma_k = \frac{1}{S}\frac{(a)_k}{k!}t^k, \qquad S = \sum_{k=2}^{\infty}\frac{(a)_k}{k!}t^k,$$

3. This Pólya–Eggenberger distribution series serves as an analogue to the Poisson distribution series and provides a rich framework for investigating coefficient bounds and subordination results in analytic and bi-univalent function theory.

We introduce $E^a(t,\zeta)h(\zeta) : \mathfrak{A} \to \mathfrak{A}$, defined by

$$\begin{aligned} E^a h(\zeta) &= \tilde{P}(a,t,\zeta) * h(\zeta) \\ &= \zeta + \sum_{k=2}^{\infty}\gamma_k c_k \zeta^k \end{aligned} \tag{4.2}$$

where $\gamma_k = \frac{1}{S}\frac{(a)_k}{k!}t^k$. In this context, the symbol $*$ represents the Hadamard (or convolution) product between two holomorphic functions.
We now introduce the subclasses $\mathcal{R}_{lag}(G,\gamma,\eta,\zeta)$ and $\mathcal{C}_{lag}(G,\gamma,\eta,\zeta)$, which are given by

$$\mathcal{R}_{lag}(G,\gamma,\eta,\zeta) = \left\{h \in \mathfrak{A} : E^c h \in \mathcal{R}_{lag}(G,\eta,\zeta)\right\} \tag{4.3}$$

and

$$\mathcal{C}_{lag}(G,\gamma,\eta,\zeta) = \left\{h \in \mathfrak{A} : E^c h \in \mathcal{R}_{lag}(G,\eta,\zeta)\right\} \tag{4.4}$$

where the classes $\mathcal{R}_{lag}(G,\eta,\zeta)$ and $\mathcal{C}_{lag}(G,\eta,\zeta)$ are those described in Definitions 1.1 and 1.2, respectively.
Proceeding in the same manner as in Theorems 1.1 and 1.3, we can derive coefficient estimates and determine the Fekete - Szegö functional for the subclasses $\mathcal{R}_{lag}(G,\gamma,\eta,\zeta)$ and $\mathcal{C}_{lag}(G,\gamma,\eta,\zeta)$. These results follow directly by employing the estimates obtained for the subclasses $\mathcal{R}_{lag}(G,\gamma,\eta,\zeta)$ and $\mathcal{C}_{lag}(G,\gamma,\eta,\zeta)$.

Theorem 4.1 *Let $0 \le \eta \le 1$ and let $E^a h$ be defined as in (4.2). If $h \in \mathcal{R}_{lag}(G,\gamma,\eta,\zeta)$, then the initial coefficients satisfy the following estimates:*

$$|c_2| \le \frac{2|g_1|}{|1+\eta|\gamma_2} \tag{4.5}$$

$$|c_3| \le \frac{|g_1|}{|2+\eta|\gamma_3} \max\left\{1, \left|\frac{(\eta-1)(2+\eta)g_1}{2(1+\eta)^2} - \frac{g_2}{g_1}\right|\right\} \tag{4.6}$$

and for any $\nu \in \mathfrak{C}$, the Fekete-Szegö functional satisfies the inequality

$$|c_3 - \nu c_2^2| \le \frac{g_1}{|2+\eta|\gamma_3} \max\left\{1, \left|\frac{((\eta-1)\gamma_2^2 + 2\nu\gamma_3)(2+\eta)g_1}{2(1+\eta)^2\gamma_2^2} - \frac{g_2}{g_1}\right|\right\}. \tag{4.7}$$

Proof: Since $h \in \mathcal{R}_{lag}(G,\gamma,\eta,\zeta)$, by (4.3) we have

$$[(E^a h(\zeta))']^\eta \left[\frac{\zeta(E^a h(\zeta))'}{(E^a h(\zeta))}\right]^{1-\eta} = G_\lambda(q,\varphi(\zeta)), \quad (\zeta \in \mathfrak{D}). \tag{4.8}$$

From (4.2), it follows that

$$\begin{aligned}[(E^a h(\zeta))']^\eta \left[\frac{\zeta(E^a h(\zeta))'}{(E^a h(\zeta))}\right]^{1-\eta} &= 1 + (1+\eta)\gamma_2 c_2\zeta + (2+\eta)[\gamma_3 c_3 - \frac{1-\eta}{2}\gamma_2^2 c_2^2]\zeta^2 \\ &+ \frac{(3+\eta)}{6}[6\gamma_4 c_4 - 6(1-\eta)\gamma_2 c_2\gamma_3 c_3 - (1-\eta)(2-\eta)\gamma_2^3 c_2^3]\zeta^3 + \ldots \end{aligned} \tag{4.9}$$

Substituting (2.7) and (4.9) in (4.8) and then equating coefficients of ζ and ζ^2, we obtain

$$c_2 = \frac{g_1 r_1}{(1+\eta)\gamma_2} \tag{4.10}$$

and

$$c_3 = \frac{g_1}{(2+\eta)\gamma_3}\left(r_2 - \left(\frac{(\eta-1)(2+\eta)g_1}{2(1+\eta)^2} - \frac{g_2}{g_1}\right)r_1^2\right) \tag{4.11}$$

Combining (4.10) and (4.11),

$$c_3 - \nu c_2^2 = \frac{g_1}{(2+\eta)\gamma_3}\left(r_2 - \left(\frac{(\eta-1)(2+\eta)g_1}{2(1+\eta)^2} - \frac{g_2}{g_1}\right)r_1^2\right) \tag{4.12}$$

Lemmas 1.1 and 1.2 lead to the desired result. □

Theorem 4.2 *Let $h \in \mathcal{R}_{lag}(G,\gamma,\zeta,\eta)$ then for any real ν the following condition is satisfied*

$$|c_3 - \nu c_2^2| \leq \begin{cases} \frac{-2((\eta-1)\gamma_2^2+2\nu\gamma_3)(2+\eta)g_1^2+2(2g_2+g_1)(1+\eta)^2\gamma_2^2}{g_1(1+\eta)^2\gamma_2^2} & ,\nu \leq \mu_1 \\ \frac{2g_1}{(2+\eta)\gamma_3} & ,\mu_1 \leq \nu \leq \mu_2 \\ \frac{2((\eta-1)\gamma_2^2+2\nu\gamma_3)(2+\eta)g_1^2-2(2g_2+g_1)(1+\eta)^2\gamma_2^2}{g_1(1+\eta)^2\gamma_2^2} & ,\nu \geq \mu_2 \end{cases} \tag{4.13}$$

where

$$\mu_1 = \frac{-2(g_2-g_1)(1+\eta)^2\gamma_2^2 - (2+\eta)(\eta-1)g_1^2\gamma_2^2}{2(2+\eta)\gamma_3 g_1^2} \tag{4.14}$$

$$\mu_2 = \frac{2(g_2+g_1)(1+\eta)^2\gamma_2^2 - (2+\eta)(\eta-1)g_1^2\gamma_2^2}{2(2+\eta)\gamma_3 g_1^2}. \tag{4.15}$$

Proof: The proof of this theorem is a direct consequence of the application of Lemma 1.3 to the equation (4.12) to obtain the desired result. □

Theorem 4.3 *Let $0 \leq \eta \leq 1$ and let $E^a h$ be defined as in (4.2). If $h \in \mathcal{C}_{lag}(G,\gamma,\eta,\zeta)$, then the initial coefficients satisfy the following estimates*

$$|c_2| \leq \frac{|g_1|}{|\gamma_2|} \tag{4.16}$$

$$|c_3| \leq \frac{|g_1|}{3|2-\eta|\gamma_3} \max\left\{1, \left|(\eta-1)g_1 - \frac{g_2}{g_1}\right|\right\} \tag{4.17}$$

and for any $\nu \in \mathfrak{C}$, the Fekete - Szegö functional satisfies the inequality

$$|c_3 - \nu c_2^2| \leq \frac{g_1}{3|2-\eta|\gamma_3} \max\left\{1, \left|\frac{((\eta-1)4\gamma_2^2 + 3\nu(2-\eta)\gamma_3)g_1}{4\gamma_2^2} - \frac{g_2}{g_1}\right|\right\}. \tag{4.18}$$

Proof: Since $h \in \mathcal{C}_{lag}(G,\gamma,\eta,\zeta)$ by (4.4) we have

$$[(E^a h(\zeta))']^\eta \left[1 + \frac{\zeta(E^a h(\zeta))''}{(E^a h(\zeta))'}\right]^{1-\eta} = G_\lambda(q,\varphi(\zeta)), \quad (\zeta \in \mathfrak{D}). \tag{4.19}$$

From (4.2), it follows that

$$[(E^a h(\zeta))']^\eta \left[1 + \frac{\zeta(E^a h(\zeta))''}{(E^a h(\zeta))'}\right]^{1-\eta} = 1 + 2\gamma_2 c_2\zeta + [3(2-\eta)\gamma_3 c_3 - 4(1-\eta)\gamma_2^2 c_2^2]\zeta^2$$
$$+ [8(1-\eta)\gamma_2^3 c_2^3 - 18(1-\eta)\gamma_2 c_2 \gamma_3 c_3 - 4(3-2\eta)\gamma_4 c_4]\zeta^3 + \dots. \tag{4.20}$$

Substituting (2.7) and (4.20) in (4.19) and equating coefficients of ζ and ζ^2, we obtain

$$c_2 = \frac{g_1 r_1}{2\gamma_2} \tag{4.21}$$

and

$$c_3 = \frac{g_1}{3(2-\eta)\gamma_3}\left(r_2 - \left((1-\eta))g_1 - \frac{g_2}{g_1}\right)r_1^2\right) \tag{4.22}$$

Combining (4.21) and (4.22) we get

$$c_3 - \nu c_2^2 = \frac{g_1}{3(2-\eta)\gamma_3}\left(r_2 - \left(\frac{(4(1-\eta)\gamma_2^2 + 3\nu(2-\eta)\nu_3)g_1}{4\gamma_2^2} - \frac{g_2}{g_1}\right)r_1^2\right) \tag{4.23}$$

By Lemmas 1.1 and 1.2, the desired result is obtained. □

Theorem 4.4 *Let $h \in \mathcal{C}_{lag}(G, \gamma, \zeta, \eta)$, then for any real ν the following condition is satisfied*

$$|c_3 - \nu c_2^2| \leq \begin{cases} \frac{-(4(\eta-1)\gamma_2^2+3\nu(2-\eta)\gamma_3)g_1^2+2(2g_2+g_1)\gamma_2^2}{g_1\gamma_2^2} & ,\nu \leq \mu_3 \\ \frac{2g_1}{3(2-\eta)\gamma_3} & ,\mu_3 \leq \nu \leq \mu_4 \\ \frac{(4(\eta-1)\gamma_2^2+3\nu(2-\eta)\gamma_3)g_1^2-2(2g_2+g_1)\gamma_2^2}{g_1\gamma_2^2} & ,\nu \geq \mu_4 \end{cases} \tag{4.24}$$

where

$$\mu_3 = \frac{-4(g_2 - g_1)\gamma_2^2 - 4(1-\eta)\gamma_2^2 g_1^2}{3(2-\eta)\gamma_3 g_1^2} \tag{4.25}$$

$$\mu_4 = \frac{4(g_2 + g_1)\gamma_2^2 - 4(1-\eta)\gamma_2^2 g_1^2}{3(2-\eta)\gamma_3 g_1^2}. \tag{4.26}$$

Proof: The proof of this theorem is a direct consequence of the application of Lemma 1.3 to the equation (4.23) to determine the desired result. □

5. Conclusion

Based on the investigations into the novel subclasses $\mathcal{R}_{lag}$ and $\mathcal{C}_{lag}$ which was defined by subordination to Laguerre polynomial we have successfully established upper bounds for the initial coefficients c_2 and c_3 and derived the corresponding Fekete-Szegö inequalities for functions within these classes. Furthermore, analogous estimates were obtained for the coefficients of the inverse function h^{-1}. While the determination of the absolute sharpness of these bounds remains an open question for future work the obtained coefficient estimates were directly connected to parameters of the Pólya-Eggenberger distribution thereby demonstrating the probabilistic significance of the $\mathcal{R}_{lag}$ and $\mathcal{C}_{lag}$ function classes. This work therefore not only introduces and characterizes new subclasses in geometric function theory but also reveals a novel and meaningful bridge between analytic functions and discrete probability theory. The established bounds provide a rigorous foundation for further exploration into extremal problems and the exact coefficient behavior within these Laguerre-associated function classes.

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