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Fractional-Order Modeling and Analysis of Secondhand Smoking with Sensitivity and Control Strategies

H. Imo Mani Singha*, Anuradha Devi

ABSTRACT: This study develops a fractional-order mathematical model to describe the transmission dynamics of secondhand smoking, employing Caputo derivatives to capture memory and hereditary effects in smoking behavior. The model incorporates initiation, cessation, relapse, and awareness processes, with analytical results establishing the existence, uniqueness, positivity, boundedness, and stability of equilibria. The basic reproduction number R_0, obtained using the next-generation matrix approach, serves as a key threshold for smoking-free and smoking-present states. Sensitivity analysis highlighted contact and recruitment rates as dominant contributors to R_0, while cessation and awareness reduced its magnitude. An optimal control problem, formulated via Pontryagin's Minimum Principle, evaluates intervention strategies for minimizing smoking exposure. Numerical simulations using the Adams–Bashforth–Moulton method confirmed the theoretical results. The novelty of this study lies in the validation of the fractional-order model with WHO and national Indian data, demonstrating that fractional dynamics yield more realistic behavioral trends than classical integer-order models.

Keywords: Fractional-order model, secondhand smoking, Caputo derivative, basic reproduction number, sensitivity analysis, optimal control, fractional dynamics, numerical simulation.

Contents

* Corresponding author.
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1. Introduction

Secondhand smoke is a harmful mixture of smoke from the burning end of a cigarette and smoke exhaled by a smoker. When it contaminates the air, particularly in confined spaces, both smokers and non-smokers are exposed to its toxic effects. It has been observed to cause lung cancer in non-smokers and increase the risk of coronary heart disease. While most smokers are men, it can be seen that women and children are also significantly affected by secondhand smoke. Globally, second-hand smoke is responsible for an estimated 600,000 premature deaths annually, with 47% of these deaths occurring in women and 28% in children. In the Eastern Mediterranean Region, 38% of students aged 13-15 are exposed to second-hand smoke at home, and in many countries, only about a quarter of households are smoke-free. Furthermore, only approximately 50% of schools enforce a ban on tobacco use by teachers [31]. Secondhand smokers (also known as passive or involuntary smokers) are non-smokers exposed to tobacco smoke released by others. Thus, studying the dynamics of smoking behavior among people and its impact on secondhand smokers is of great importance. To obtain a proper picture of this study, we use fractional-order differential equations in the sense of Caputo. The reason for the use of fractional order derivatives lies in their ability to capture memory effects, which provides a more accurate approach to understanding real-world problems, which is not possible with classical integer order derivatives.

Tobacco smoking and its adverse health consequences have been a topic of significant research in the past decades, particularly through mathematical modeling approaches to study its spread, cessation, and relapse. Various researchers have investigated the behaviors of both smokers and secondhand smokers using classical and fractional-order derivatives.

Fekede et al. [1] analyzed a mathematical model for secondhand smoking and conducted a sensitivity analysis to identify the key parameters affecting the spread and impact of tobacco exposure. Alalhareth et al. [2] introduced a fractional-order model incorporating relapse behavior to understand the dynamics of quitting smoking, emphasizing the long-term memory effects captured by the fractional derivatives. Montoya et al. [3] studied the dynamics of smoking through a deterministic model and discussed various factors influencing smoking initiation and cessation. Peinjing et al. [4] applied fractional calculus to analyze a smoking model, emphasizing the utility of fractional derivatives in capturing hereditary effects and long-term dependencies in tobacco-related behavior. Rahman et al. [5] proposed a model that introduced compartments such as quit smokers and included relapse mechanisms, offering a more realistic description of smoking cessation efforts. Huo and Zhu [6] developed a model that differentiates between light and heavy smokers and incorporates the possibility of one or two relapses after quitting, highlighting the complex nature of smoking behavior. Erturk et al. [7] used Caputo fractional derivatives to examine the decline in smoking habits, showing that fractional-order models better capture the persistence of the behavioral traits. Lubin and Caporaso [8] studied the epidemiological relationship between cigarette smoking and lung cancer, reinforcing the need for preventive strategies. Khalid et al. [9] studied a fractional smoking cessation model using perturbation techniques, while Singh et al. [10] proposed a new smoking model based on fractional calculus with nonsingular kernels, demonstrating improved modeling accuracy. Ahmad et al. [11] utilized the Atangana–Baleanu (AB) derivative with Mittag–Leffler kernel and the Atangana–Toufik method (ATM) to study the smoking epidemic, revealing the enhanced sensitivity and adaptability of the model to real-world scenarios. Melkamu and Mebrate [12] developed a fractional-order smoking model using the Caputo–Fabrizio derivative to incorporate memory effects in tobacco use dynamics. Addai et al. in two studies [13,14] introduced fractional-order models based on the Atangana–Baleanu–Caputo and fractal–fractional Caputo–Fabrizio derivatives, respectively. These studies considered age structures and government interventions to provide a more comprehensive outlook on smoking dynamics across different population groups. On the theoretical side, Agrawal et al. [15] explored fractional optimal control problems involving multiple state and control variables, forming a foundational base for control-based interventions in the health models. Additionally, Diekmann et al. [16] proposed a rigorous method to compute the basic reproduction number R_0, which serves as a critical threshold parameter for analyzing the stability of equilibria in epidemic and behavioral models. These studies collectively contribute to the growing body of research on tobacco modeling, highlighting the importance of fractional-order systems in capturing realistic and complex smoking behaviors, relapse phenomena, and control strategies for smoking cessation.

The remainder of this paper is organized as follows. Section 2 introduces the basic definitions and mathematical preliminaries of the fractional calculus essential for the model formulation. Sections 3 and 4 present the fractional-order secondhand smoking model and explore its qualitative properties, including existence uniqueness, positivity, and boundedness. Section 5 provides the computation and interpretation of the basic reproduction number and a detailed stability analysis, followed by a sensitivity analysis in Section 6. Section 7 formulates and analyzes the optimal control problem by incorporating awareness, cessation, and relapse prevention strategies. The numerical results and discussion are presented in Section 8. Numerical simulations and validation using empirical tobacco data are discussed in Section 9 of this paper. The conclusions are presented in Section 10.

2. Basic Definitions

Definition 2.1 *[18] Let $f \in C^n[a,b]$ be an n-times continuously differentiable function. The Caputo fractional derivative of order $\alpha \in (n-1,n)$ is defined by*

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, \qquad t > a, \tag{2.1}$$

where $\Gamma(\cdot)$ denotes the gamma function.

Definition 2.2 *[18] The Riemann–Liouville fractional integral of order $\alpha > 0$ is given by*

$$ I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{2.2}$$

Remark 2.1 *The Caputo derivative satisfies ${}^{C}D_t^{\alpha} c = 0$ for any constant c, and ${}^{C}D_t^{1} f(t) = f'(t)$, thus generalizing the classical derivative. This method was preferred in this study because it accommodates real-world initial conditions expressed in integer-order derivatives.*

Remark 2.2 *Other fractional derivatives, such as the Riemann–Liouville and Atangana–Baleanu operators, capture different types of memory kernels. However, the Caputo operator was chosen here for analytical simplicity and its well-established numerical implementation via the Adams–Bashforth–Moulton method.*

Definition 2.3 *[17] Let $f(\tau)$ be a function of τ specified for $\tau > 0$. Then the Laplace transform of the function $f(\tau)$ is denoted and defined as*

$$ L\{f(\tau)\} = g(s) = \int_0^{\infty} e^{-s\tau} f(\tau) d\tau \quad \text{where} \;\; s \in \mathbb{C} \tag{2.3}$$

Definition 2.4 *[18] The Laplace transform of the Caputo fractional derivative is defined as*

$$ L[{}^{C}D_t^{m\alpha} f(t)] = s^{m\alpha} L[f(t)] - \sum_{j=0}^{m-1} s^{m\alpha-j-1} f^{(j)}(0) \;\; , \;\; m-1 < m\alpha \le m, \;\; m \in \mathbb{N} \tag{2.4}$$

3. Model formulation

The following assumptions were made to formulate the model:

1. Three compartments of the population have been considered for our study:
   (a) Secondhand smokers denoted by P.
   (b) Active smokers denoted by S.
   (c) Temporary smoking quitters denoted by Q.

2. The healthy population and the permanent smoking quitters are not taken into consideration in this study.

3. We assume $P(t) + S(t) + Q(t)$ is constant at a time t say $N(t)$.

4. For numerical simulation, $P(t), S(t), Q(t)$ are considered proportions of $N(t)$ such that $P(t)+S(t)+Q(t) = 1$.

5. The healthy people becomes secondhand smokers at a constant rate $\wedge$.

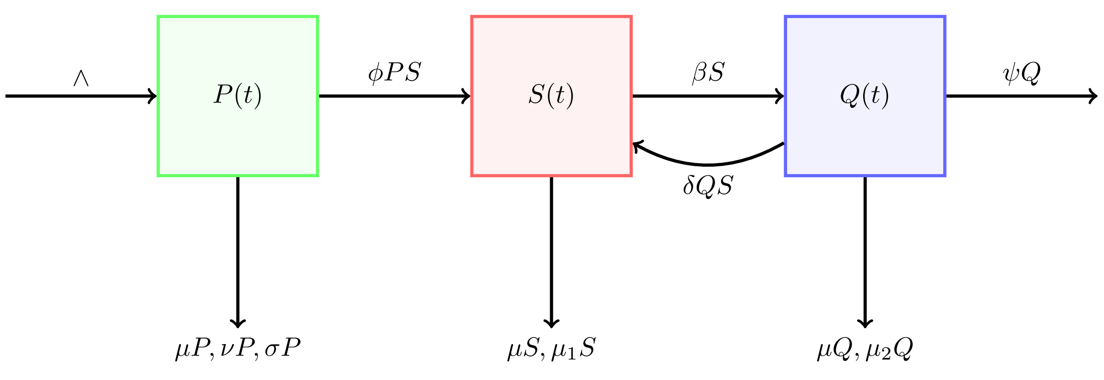
6. People leave compartments P or S or Q because of natural death or death due to smoking or permanently quit smoking.

In Table 1, we describe the parameters used in our system.

Table 1: Model parameter description

Parameter	**Description**
$\wedge$	recruitment rate (or, no. of healthy people becoming secondhand smoker)
ϕ	transmission rate of smoking habit
μ	natural death rate
μ_1	death rate of smokers due to smoking
μ_2	death rate of smoking quitters due to their earlier smoking habit
ν	death rate of secondhand smokers being secondhand smokers
σ	rate of secondhand smokers becoming healthy
δ	relapse rate of temporary smoking quitters to active smokers
β	rate of smokers quitting smoking
ψ	rate of temporary smoking quitters becoming permanent quitters

where all parameter values lie between 0 and 1. A schematic diagram of the model is as shown below.



In this model, we see from the above figure that healthy people become secondhand smokers at a constant rate of $\wedge$. Secondhand smokers leave their compartment due to natural death μP, death because of being secondhand smokers νP, becoming healthy σP and becoming smokers ϕPS. Smokers leave their compartment due to natural death μS, death because of smoking $\mu_1 S$ and becoming temporary smoking quitters βS. Temporary smoking quitters leave their compartment due to natural death μQ, death because of their earlier smoking habits $\mu_2 Q$, starting to smoke again δQS and becoming healthy ψQ.

The model proposed by Fekede et. al. [1] is as follows

$$\frac{dP}{dt} = \wedge - \phi PS - (\mu + \nu + \sigma)P$$
$$\frac{dS}{dt} = \phi PS + \delta QS - (\mu + \mu_1 + \beta)S$$
$$\frac{dQ}{dt} = \beta S - \delta QS - (\mu + \mu_2 + \psi)Q$$

We consider the fractional-order case of this system. Thus, our desired system is

$$\left.\begin{aligned} {}^cD^{\alpha}_{0,t}P &= \wedge - \phi PS - (\mu + \nu + \sigma)P \\ {}^cD^{\alpha}_{0,t}S &= \phi PS + \delta QS - (\mu + \mu_1 + \beta)S \\ {}^cD^{\alpha}_{0,t}Q &= \beta S - \delta QS - (\mu + \mu_2 + \psi)Q \end{aligned}\right\} \tag{3.1}$$

with initial conditions $P(0)$, $S(0)$ and $Q(0)$. Here, the fractional operator ${}^cD^{\alpha}_{0,t}$ is the Caputo derivative of order α with $0 < \alpha \le 1$.

3.1. Existence and Uniqueness of Solution

We now establish the existence and uniqueness of the solution of the system, which ensures that the model is well-posed.

Theorem 3.1 *[28] Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and locally Lipschitz. Then the Caputo fractional system*

$${}^CD^{\alpha}_{t}X(t) = F(X(t)), \quad 0 < \alpha \le 1, \quad X(0) = X_0,$$

admits a unique solution $X(t)$ on $[0, T]$, which can be extended to $t \ge 0$.

Theorem 3.2 *Let $X(0) = (P(0), S(0), Q(0))^T \in \mathbb{R}^3_+$. Then, the proposed model (3.1) admits a unique solution $(P(t), S(t), Q(t))$ for all $t \ge 0$.*

Proof: Let $X(t) = (P(t), S(t), Q(t))^T$ and define $F(X)$ as the right-hand side of the system (3.1). Because F consists of polynomial functions in P, S, and Q, it is continuous and locally Lipschitz on $\mathbb{R}^3_+$.

Therefore, the result follows directly from the theorem. □

4. Positivity and Boundedness of the model

We now establish the positivity and boundedness of the solutions, ensuring that the model remains biologically feasible and dynamically well behaved over time.

Let us define $\mathbb{R}^3_+ = \{X \in \mathbb{R}^3 \ : \ X \ge 0\}$ and $X(t) = (P(t), S(t), Q(t))^T$. By the generalized mean value theorem, we obtain the following lemma.

Lemma 4.1 *Let $X(t) \in C[a, b]$ and ${}^cD^{\alpha}_{0,t}X(t) \in (a, b]$ then $X(t) = X(a) + \frac{1}{\Gamma(\alpha)}\, {}^cD^{\alpha}_{0,t}X(r)(t-a)^{\alpha}$ where $a \le r \le t$ for all $t \in (a, b]$.*

Corollary 4.1 *[12] Let $X(t) \in C[a, b]$ and ${}^cD^{\alpha}_{0,t}X \in (a, b]$ where $0 < \alpha \le 1$. Then, from Lemma 4.1, if ${}^cD^{\alpha}_{0,t}X(t) \ge 0$ for all $t \in (a, b]$ then the function $X(t)$ is non-decreasing, and if ${}^cD^{\alpha}_{0,t}X(t) \le 0$ for all $t \in (a, b]$ then the function $X(t)$ is non-increasing for all $t \in [a, b]$.*

We now provide a theorem to prove the non-negativity and boundedness of the solution.

Theorem 4.1 *If the initial values $P(0), S(0)$ and $Q(0)$ are positive and bounded in $\mathbb{R}^3_+$ then $P(t), S(t)$ and $Q(t)$ are positive and bounded in $\mathbb{R}^3_+$ for all $t \ge 0$.*

Proof: We see from Equation (3.1) that

$$\begin{aligned} {}^{c}D^{\alpha}_{0,t}P(t)\big|_{P=0} &= \wedge \geq 0 \\ {}^{c}D^{\alpha}_{0,t}S(t)\big|_{S=0} &= 0 \geq 0 \\ {}^{c}D^{\alpha}_{0,t}Q(t)\big|_{Q=0} &= \beta S \geq 0 \end{aligned}$$

Applying Corollary 4.1, we can conclude that the solution of the proposed model is positive for all $t > 0$.

Adding all the equations of (3.1), we get

$$\begin{aligned} {}^{c}D^{\alpha}_{0,t}N(t) &= \wedge - (\mu + \nu + \sigma)P - (\mu + \mu_1)S - (\mu + \mu_2 + \psi)Q \\ &= \wedge - \mu N - (\nu + \sigma)P - \mu_1 S - (\mu_2 + \psi)Q \\ &\leq \wedge - \mu N \end{aligned}$$

Taking Laplace transform, we get

$$\begin{aligned} s^{\alpha}N(s) - s^{\alpha-1}N(0) &\leq \frac{\wedge}{s} - \mu N(s) \\ \implies N(s) &\leq \frac{\wedge}{s(s^{\alpha} + \mu)} + \frac{s^{\alpha-1}}{s^{\alpha} + \mu}N(0) \end{aligned}$$

Taking inverse Laplace transform, we get

$$N(t) \leq \wedge t\, E_{\alpha,2}(-\mu t^{\alpha}) + N(0)\, E_{\alpha}(-\mu t^{\alpha})$$

where $E_{\alpha}(-\mu t^{\alpha})$ and $E_{\alpha,2}(-\mu t^{\alpha})$ are the one- and two-parameter Mittag-Leffler functions, respectively. Thus, we can conclude that the solutions of our proposed model is bounded for all $t > 0$. □

5. Stability Analysis

5.1. Smoking free equilibrium point (E_0)

From system (3.1), equating ${}^{c}D^{\alpha}_{0,t}P = 0$, ${}^{c}D^{\alpha}_{0,t}S = 0$ and ${}^{c}D^{\alpha}_{0,t}Q = 0$, we obtain $P = \dfrac{\wedge}{\mu + \nu + \sigma}$, $S = 0$, $Q = 0$. Thus, $E_0 = \left(\dfrac{\wedge}{\mu + \nu + \sigma}, 0, 0\right)$.

5.2. Basic reproduction number (R_0)

The basic reproduction number (R_0) represents the average number of secondhand smokers who become active smokers due to the influence of one active smoker. If $R_0 > 1$ then each active smoker, on average, causes more than one new case, leading to potential disease spread (i.e., an increase in the number of active smokers) and an epidemic. If $R_0 = 1$ then the disease will remain stable in the population but will not lead to an outbreak. If $R_0 < 1$ then the disease will eventually decline and likely disappear from the population. The next-generation matrix method [16] was used to calculate R_0.

Our system (3.1) can be written as $\dfrac{dX}{dt} = F(X) - V(X)$

where $X = (P, S, Q)$, $F(X) = \begin{bmatrix} \wedge \\ \phi PS + \delta QS \\ \beta S \end{bmatrix}$ and $V(X) = \begin{bmatrix} \phi PS + (\mu + \nu + \sigma)P \\ (\mu + \mu_1 + \beta)S \\ \delta QS + (\mu + \mu_2 + \psi)Q \end{bmatrix}$

The Jacobian matrices of $F(X)$ and $V(X)$ at E_0 are $DF(E_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{\phi\wedge}{\mu + \nu + \sigma} & 0 \\ 0 & \beta & 0 \end{bmatrix}$ and $DV(E_0) =$

$$\begin{bmatrix} \mu+\nu+\sigma & \frac{\phi\wedge}{\mu+\nu+\sigma} & 0 \\ 0 & \mu+\mu_1+\beta & 0 \\ 0 & 0 & \mu+\mu_2+\psi \end{bmatrix}.$$

Now, $(DF)(DV)^{-1} = \frac{1}{det(DV)}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\phi\wedge}{\mu+\nu+\sigma}C & 0 \\ 0 & \beta C & 0 \end{bmatrix}$ where $C = (\mu+\nu+\sigma)(\mu+\mu_2+\psi)$.

which has eigen values 0 and $\frac{\phi\wedge C}{(\mu+\nu+\sigma)\, det(DV)}$.

Hence, we obtain $R_0 = \frac{\phi\wedge}{(\mu+\nu+\sigma)(\mu+\mu_1+\beta)}$.

5.3. Smoking present equilibrium point (E^*)

Here, we consider $S \neq 0$, thus, we get from our system (3.1),

$$^cD^{\alpha}_{0,t}S = 0 \implies \phi P + \delta Q = \mu+\mu_1+\beta \tag{5.1}$$

and

$$^cD^{\alpha}_{0,t}P = 0 \implies P = \frac{\wedge}{\mu+\nu+\sigma+\phi S} \tag{5.2}$$

and

$$^cD^{\alpha}_{0,t}Q = 0 \implies Q = \frac{\beta S}{\mu+\mu_2+\psi+\delta S} \tag{5.3}$$

Using (5.2) and (5.3) in (5.1), we get a quadratic equation in S as follows

$$S = \frac{-E \pm \sqrt{E^2 - 4DF}}{2D}$$

where $E = \delta BC(R_0 - 1) + \delta\beta B - \phi AC$, $D = \delta\beta\phi - \delta\phi C$, $F = CAB(R_0 - 1)$ and $A = \mu+\mu_2+\psi$, $B = \mu+\nu+\sigma$, $C = \mu+\mu_1+\beta$.

If $R_0 > 1$ then we have

$$S = \frac{-E - \sqrt{E^2 - 4DF}}{2D} \tag{5.4}$$

Using (5.4) in (5.2) and (5.3), we can find the values of P and Q.

Also, if $R_0 < 1$ then $-E > 0$ and $F < 0$.
$\therefore$ S is either a negative real number or a complex number.
$\therefore$ S does not exist, and thus, P and Q.

Finally, if $R_0 = 1$ then $E = \delta\beta B - \phi AC$ and $F = 0$.

$$\therefore \quad S = \frac{-E - \sqrt{E^2}}{2D} = \begin{cases} 0 \;, & E \leq 0 \\ -ve \;, & E > 0 \end{cases}$$

Thus, we can present the above discussions in the form of the following theorem:

Theorem 5.1 *The smoking present equilibrium point (E^*) of the system exists if $R_0 > 1$ or $R_0 = 1$ and $\delta\beta B > \phi AC$ (that is, $E > 0$) and does not exist if $R_0 < 1$.*

Remark 5.1 *Due to the nonlinear and algebraically complex structure of the model, obtaining a closed-form expression for E^* is not feasible. Therefore, E^* is computed numerically for the parameter values used in the simulations in Section 8.*

5.4. Local stability of smoking free equilibrium point (E_0)

To discuss the local stability of E_0, we find the Jacobian matrix of the system at $E_0 = \left(\frac{\wedge}{\mu+\nu+\sigma}, 0, 0\right)$ as follows

$$J(E_0) = \begin{bmatrix} -(\mu+\nu+\sigma) & \frac{-\phi\wedge}{\mu+\nu+\sigma} & 0 \\ 0 & (\mu+\mu_1+\beta)(R_0-1) & 0 \\ 0 & \beta & -(\mu+\mu_2+\psi) \end{bmatrix}$$

whose eigen values are $\lambda_1 = -(\mu+\nu+\sigma)$, $\lambda_2 = (\mu+\mu_1+\beta)(R_0-1)$ and $\lambda_3 = -(\mu+\mu_2+\psi)$. Clearly, $|arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ and $|arg(\lambda_3)| = \pi > \frac{\alpha\pi}{2}$.

If $R_0 < 1$ then $\lambda_2 < 0$, that is, $|arg(\lambda_2)| = \pi > \frac{\alpha\pi}{2}$; hence, the smoking-free equilibrium point E_0 is locally asymptotically stable [25]. However, if $R_0 > 1$ then $\lambda_2 > 0$, that is, $|arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$; hence, E_0 is unstable. Thus, we obtain the following theorem:

Theorem 5.2 *If $R_0 < 1$ then the smoking-free equilibrium point E_0 is locally asymptotically stable, but if $R_0 > 1$ then E_0 is unstable.*

5.5. Local stability of smoking present equilibrium point (E^*)

Due to the complexity in calculating E^*, we demonstrate the stability of the endemic equilibrium solution E^* through numerical simulations by assigning specific values to the parameters and variables in Section 8.

5.6. Global stability of smoking free equilibrium point (E_0)

To discuss the global stability of the equilibrium points, we consider the following lemma:

Lemma 5.1 *[19,20] Let $f(t) \in \mathbb{R}^+$ be a continuously differentiable function. For given any time $t \geq 0$, we have*

$$^cD^\alpha_{0,t}\left(f(t) - f^* - f^* ln\frac{f(t)}{f^*}\right) \leq \left(1 - \frac{f^*}{f(t)}\right) {^cD^\alpha_{0,t}} f(t)$$

and

$$\frac{1}{2}\, {^cD^\alpha_{0,t}} f^2(t) \leq f(t)\, {^cD^\alpha_{0,t}} f(t)$$

Theorem 5.3 *The smoker-free equilibrium point $E_0 = (P_0, S_0, Q_0) = \left(\frac{\wedge}{\mu+\nu+\sigma}, 0, 0\right)$ of the system is globally asymptotically stable if $R_0 \leq 1$.*

Proof: Let us consider a Lyapunov function $L(t)$ as

$$L(t) = \left(P(t) - P_0 - P_0\, ln\frac{P(t)}{P_0}\right) + S(t) + Q(t) \tag{5.5}$$

which is clearly continuous, differentiable, and positive $\forall\, t \geq 0$. Then applying Caputo derivative on (5.5) and using Lemma (5.1), we get

$$\begin{aligned} {^cD^\alpha_{0,t}} L(t) &\leq \left(1 - \frac{P_0}{P(t)}\right) {^cD^\alpha_{0,t}} P(t) + {^cD^\alpha_{0,t}} S(t) + {^cD^\alpha_{0,t}} Q(t) \\ &= \left(1 - \frac{P_0}{P(t)}\right)(\wedge - \phi PS - (\mu+\nu+\sigma)P) + (\phi PS + \delta QS - (\mu+\mu_1+\beta)) \\ &+ (\beta S - \delta QS - (\mu+\mu_2+\psi)Q) \end{aligned}$$

Thus, we obtain

$${^cD^\alpha_{0,t}} L(t) \leq \wedge\left(2 - \frac{P}{P_0} - \frac{P_0}{P}\right) + \frac{1}{\mu+\nu+\sigma}(\phi \wedge - (\mu+\mu_1)(\mu+\nu+\sigma))S - (\mu+\mu_2+\psi)Q$$

We know, $2 - \frac{P}{P_0} - \frac{P_0}{P} \leq 0$ (By arithmetic-geometric means)

$\therefore$ If we have $\phi \wedge - (\mu + \mu_1)(\mu + \nu + \sigma) \leq 0$, that is, $R_0 \leq 1$ then we get to see that ${}^cD^{\alpha}_{0,t}L(t) \leq 0$.

In addition, we have ${}^cD^{\alpha}_{0,t}L(t) = 0$ iff $(P, S, Q) = E_0 = \left(\frac{\wedge}{\mu + \nu + \sigma}, 0, 0\right)$.

$\therefore$ The maximum invariant set for $\{(P, S, Q) \in \mathbb{R}^3_+ : {}^cD^{\alpha}_{0,t}L(t) = 0\}$ is the set $\{E_0\}$.
Thus, by LaSalle's invariable principle [21,22], the smoking free equilibrium point E_0 is globally asymptotically stable. □

5.7. Global stability of smoking present equilibrium point (E^*)

Theorem 5.4 *Let $R_0 > 1$ and let $Q^* > Q$ and $S > S^*$ (or, $Q^* < Q$ and $S < S^*$) then the smoking-present equilibrium point $E^* = (P^*, S^*, Q^*)$ is globally asymptotically stable.*

Proof: Let us consider a Lyapunov function $L^*(t)$ as

$$\begin{aligned} L^*(t) = &\left(P(t) - P^* - P^* \, ln\frac{P(t)}{P^*}\right) + \left(S(t) - S^* - S^* \, ln\frac{S(t)}{S^*}\right) \\ &+ \left(Q(t) - Q^* - Q^* \, ln\frac{Q(t)}{Q^*}\right) \end{aligned} \tag{5.6}$$

which is clearly continuous, differentiable, and positive $\forall\ t \geq 0$. Then applying Caputo derivative on (5.6) and using Lemma (5.1), we get

$$\begin{aligned} {}^cD^{\alpha}_{0,t}L^*(t) \leq &\left(1 - \frac{P^*}{P(t)}\right) {}^cD^{\alpha}_{0,t}P(t) + \left(1 - \frac{S^*}{S(t)}\right) {}^cD^{\alpha}_{0,t}S(t) \\ &+ \left(1 - \frac{Q^*}{Q(t)}\right) {}^cD^{\alpha}_{0,t}Q(t) \end{aligned} \tag{5.7}$$

At $E^* = (P^*, S^*, Q^*)$, we have

$${}^cD^{\alpha}_{0,t}P^*(t) = 0 \implies \mu + \nu + \sigma = \frac{\wedge}{P^*} - \phi S^* \tag{5.8}$$

$${}^cD^{\alpha}_{0,t}S^*(t) = 0 \implies \mu + \mu_1 + \beta = \phi P^* + \delta Q^* \tag{5.9}$$

$${}^cD^{\alpha}_{0,t}Q^*(t) = 0 \implies \mu + \mu_2 + \psi = \frac{\beta S^*}{Q^*} - \delta S^* \tag{5.10}$$

Substituting the values from (5.8),(5.9) and (5.10) in (5.7), we obtain

$${}^cD^{\alpha}_{0,t}L^*(t) \leq \wedge\left(2 - \frac{P}{P^*} - \frac{P^*}{P}\right) + \beta S\left(1 - \frac{Q^*}{Q}\right) + \beta S^*\left(1 - \frac{Q}{Q^*}\right)$$

We know, $2 - \frac{P}{P^*} - \frac{P^*}{P} \leq 0$ (By arithmetic-geometric means)

$\therefore$ If we have $\frac{Q^*}{Q} > 1$ and $S > S^*$ (or, $\frac{Q}{Q^*} > 1$ and $S < S^*$), then we see that ${}^cD^{\alpha}_{0,t}L^*(t) \leq 0$.

Also, we have ${}^cD^{\alpha}_{0,t}L^*(t) = 0$ iff $(P, S, Q) = E^* = (P^*, S^*, Q^*)$.

$\therefore$ The maximum invariant set for $\{(P, S, Q) \in \mathbb{R}^3_+ : {}^cD^{\alpha}_{0,t}L^*(t) = 0\}$ is the set $\{E^*\}$.
Thus, by LaSalle's invariable principle [21,22], the smoking free equilibrium point E^* is globally asymptotically stable. □

6. Sensitivity Analysis and Biological Interpretation of the Reproduction Number

For the present fractional-order model, R_0 was obtained using the next-generation matrix approach as

$$R_0 = \frac{\phi \Lambda}{(\mu + \nu + \sigma)(\mu + \mu_1 + \beta)}. \tag{6.1}$$

The disease-free equilibrium is locally asymptotically stable when $R_0 < 1$ and unstable when $R_0 > 1$. Therefore, controlling the value of R_0 below unity is critical for eradicating smoking behavior in the community.

6.1. Sensitivity Analysis of R_0

To quantify the influence of model parameters on R_0, we compute the normalized forward sensitivity index of R_0 with respect to a parameter p, defined by

$$\Gamma_p^{R_0} = \frac{\partial R_0}{\partial p} \times \frac{p}{R_0}. \tag{6.2}$$

This index measures the relative change in R_0 due to a relative change in parameter p. If $\Gamma_p^{R_0} > 0$, an increase in p leads to an increase in R_0, whereas $\Gamma_p^{R_0} < 0$ implies that R_0 decreases when p increases.

Using the expression for R_0, the sensitivity indices of the main parameters are obtained as follows:

$$\begin{aligned}
&\Gamma_\phi^{R_0} = +1, \qquad \Gamma_\Lambda^{R_0} = +1,\\
&\Gamma_\beta^{R_0} = -\frac{\beta}{\mu + \mu_1 + \beta}, \qquad \Gamma_\nu^{R_0} = -\frac{\nu}{\mu + \nu + \sigma},\\
&\Gamma_\sigma^{R_0} = -\frac{\sigma}{\mu + \nu + \sigma}, \qquad \Gamma_\mu^{R_0} = -\left[\frac{\mu}{\mu + \nu + \sigma} + \frac{\mu}{\mu + \mu_1 + \beta}\right].
\end{aligned}$$

The results show that R_0 is most sensitive to ϕ (contact rate) and Λ (recruitment rate), both of which have positive indices. Parameters β, ν, σ, and μ have negative sensitivity indices, indicating that enhancing smoking cessation, natural mortality, or transition out of susceptibility can effectively reduce R_0.

6.2. Biological Interpretation

The sensitivity results have meaningful biological implications. The strong positive dependence of R_0 on ϕ implies that increased social contact between smokers and susceptibles accelerates the initiation of smoking. Hence, reducing social exposure to secondhand smoke through public awareness or anti-smoking policies significantly lowers the potential spread of smoking habits.

The negative sensitivity of R_0 to β highlights the crucial role of cessation and rehabilitation programs. Strengthening quitting campaigns and increasing access to smoking cessation therapies can substantially reduce R_0, promote recovery, and prevent relapse.

Similarly, increasing the death rates μ and ν reduces the average number of secondary smokers. In contrast, the recruitment rate Λ positively contributes to R_0, meaning that a higher inflow of susceptible individuals (e.g., through birth or immigration) increases the long-term risk of smoking.

Overall, these findings underline that public health interventions targeting smoking initiation (reducing ϕ) and relapse (δ), while promoting cessation (β), are the most effective strategies for maintaining $R_0 < 1$ and stabilizing the smoke-free equilibrium.

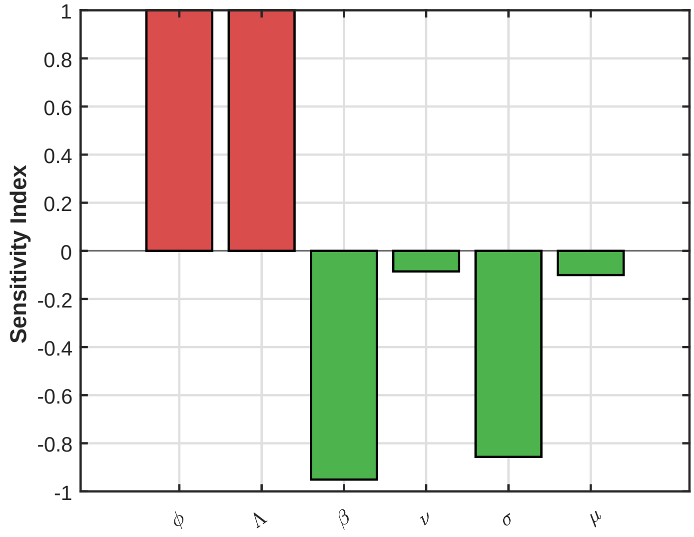


Figure 1: The graph shows the sensitivity indices of the parameters with respect to the basic reproductive number

Figure 1 shows the sensitivity indices of the basic reproduction number. The parameters ϕ (contact rate) and $\wedge$ (recruitment rate) have the highest positive impact, indicating that increased social exposure or new susceptible entries strongly promotes the spread of smoking.

7. Optimal Control

We now formulate an optimal control problem associated with the fractional-order secondhand smoking model given by (3.1). Three control functions, $u_1(t)$, $u_2(t)$, and $u_3(t)$ are introduced to represent public awareness, smoking cessation programs, and relapse prevention efforts, respectively. The controlled fractional-order system becomes

$$\begin{cases} {}^{C}D_t^{\alpha}P = \Lambda - \phi(1-u_1)PS - (\mu+\nu+\sigma)P, \\ {}^{C}D_t^{\alpha}S = \phi(1-u_1)PS + \delta QS - (\mu+\mu_1+\beta+u_2)S, \\ {}^{C}D_t^{\alpha}Q = \beta S - \delta QS - (\mu+\mu_2+\psi+u_3)Q, \end{cases} \tag{7.1}$$

with initial conditions $P(0) = P_0 > 0$, $S(0) = S_0 > 0$, and $Q(0) = Q_0 > 0$, and where $0 < \alpha \leq 1$.

The admissible control set is defined as

$$U = \left\{(u_1, u_2, u_3) \in [L^2(0,T)]^3 : 0 \leq u_i(t) \leq u_{i,\max},\ i = 1,2,3\right\}.$$

The objective functional to be minimized is

$$J(u_1,u_2,u_3) = \int_0^T \left[A_1P + A_2S + A_3Q + \frac{1}{2}\left(B_1u_1^2 + B_2u_2^2 + B_3u_3^2\right)\right] dt, \tag{7.2}$$

where A_1, A_2, A_3 are positive weights associated with the health impact of each compartment, and B_1, B_2, B_3 represent the implementation costs of the corresponding strategies.

Our goal is to determine the optimal controls (u_1^*, u_2^*, u_3^*) that minimize the cost functional J subject to the dynamic constraints (7.1).

To derive the necessary conditions for optimality, we define the Hamiltonian function

$$\begin{aligned} H = {} & A_1P + A_2S + A_3Q + \frac{1}{2}\left(B_1u_1^2 + B_2u_2^2 + B_3u_3^2\right) \\ & + \lambda_1\big[\Lambda - \phi(1-u_1)PS - (\mu+\nu+\sigma)P\big] \\ & + \lambda_2\big[\phi(1-u_1)PS + \delta QS - (\mu+\mu_1+\beta+u_2)S\big] \\ & + \lambda_3\big[\beta S - \delta QS - (\mu+\mu_2+\psi+u_3)Q\big], \end{aligned} \tag{7.3}$$

where $\lambda_i(t)$ $(i = 1, 2, 3)$ are the adjoint variables associated with $P(t)$, $S(t)$, and $Q(t)$, respectively.

According to the fractional version of Pontryagin's Minimum Principle for Caputo derivatives [23,24], there exist adjoint variables $\lambda_i(t)$ satisfying the following adjoint (co-state) system:

$$^{C}D_t^{\alpha}\lambda_i(t) = -\frac{\partial H}{\partial x_i}, \quad i = 1, 2, 3, \tag{7.4}$$

with the transversality (terminal) conditions

$$\lambda_i(T) = 0, \quad i = 1, 2, 3. \tag{7.5}$$

Explicitly, the adjoint equations are given by

$$^{C}D_t^{\alpha}\lambda_1 = -A_1 + \lambda_1[\phi(1-u_1)S + (\mu+\nu+\sigma)] - \lambda_2\phi(1-u_1)S, \tag{7.6}$$

$$^{C}D_t^{\alpha}\lambda_2 = -A_2 + \lambda_1\phi(1-u_1)P - \lambda_2[\phi(1-u_1)P + \delta Q - (\mu+\mu_1+\beta+u_2)] - \lambda_3\beta, \tag{7.7}$$

$$^{C}D_t^{\alpha}\lambda_3 = -A_3 - \lambda_2\delta S + \lambda_3[\delta S + (\mu+\mu_2+\psi+u_3)]. \tag{7.8}$$

The optimal controls $u_1^*(t)$, $u_2^*(t)$, and $u_3^*(t)$ minimize the Hamiltonian H pointwise, leading to the characterization

$$u_1^*(t) = \min\left(\max\left(0, \frac{\phi P(t)S(t)[\lambda_2(t) - \lambda_1(t)]}{B_1}\right), u_{1,\max}\right), \tag{7.9}$$

$$u_2^*(t) = \min\left(\max\left(0, \frac{\lambda_2(t)S(t)}{B_2}\right), u_{2,\max}\right), \tag{7.10}$$

$$u_3^*(t) = \min\left(\max\left(0, \frac{\lambda_3(t)Q(t)}{B_3}\right), u_{3,\max}\right). \tag{7.11}$$

The optimal control problem (7.1)–(7.2) is thus characterized by the system of state equations, adjoint equations (7.4)–(7.5), and the above optimality conditions, together with the boundary conditions on the state and adjoint variables.

For numerical implementation, the coupled system is solved iteratively using the forward–backward sweep method adapted to fractional differential equations with the Adams–Bashforth–Moulton predictor–corrector scheme.

8. Numerical results and discussions

8.1. Numerical Method (Adams–Bashforth–Moulton Scheme)

As the proposed smoking dynamics model involves a system of nonlinear differential equations, analytical solutions are generally difficult to obtain. Therefore, numerical methods are employed to approximate the solutions.

The system (3.1) is solved numerically using the Adams-Bashforth-Moulton (ABM) predictor-corrector scheme [26,27]. In general, for a system written in the form

$$^{C}D_t^{\alpha} X(t) = F(t, X(t)), \quad 0 < \alpha \leq 1,$$

the ABM method consists of two steps: a predictor (explicit Adams-Bashforth) and a corrector (implicit Adams-Moulton), which together improve the accuracy and stability of the numerical solution.

Specifically, the predictor step is given by

$$X_{n+1}^{p} = X_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} \, F(t_j, X_j), \tag{8.1}$$

while the corrector step is defined as

$$X_{n+1} = X_0 + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^{n} a_{j,n+1} \, F(t_j, X_j) + a_{n+1,n+1} \, F(t_{n+1}, X_{n+1}^{p}) \right), \tag{8.2}$$

where $a_{j,n+1}$ and $b_{j,n+1}$ are weight coefficients that depend on the step size and fractional order α.

In this study, we employed the MATLAB solver `fde12`, developed by **Garrappa** [27], which implements the Adams–Bashforth–Moulton predictor–corrector algorithm for fractional and integer-order differential equations. This method efficiently captures the dynamics of the system and is widely used in the literature to solve nonlinear models with memory effects. All numerical simulations presented in this paper were performed using this scheme for the parameter values.

8.2. Results and discussions

To validate the analytical results, biologically feasible parameters are chosen as $\alpha = 0.9$, $\mu = 0.0135$, $\beta = 0.3$, $\phi = 0.035$, $\mu_1 = 0.002$, $\delta = 0.5$, $\mu_2 = 0.02$, $\wedge = 0.5$, $\nu = 0.02$, $\sigma = 0.2$, $\psi = 0.1$.

With these parameters, the reproduction number $R_0 \approx 0.237548 < 1$. The smoking-free equilibrium point is $E_0 = (P_0, S_0, Q_0) = (2.14133, 0, 0)$. The simulation for different fractional orders $\alpha \in \{1, 0.9, 0.8, 0.7\}$ was performed. Figure 2 shows that the growth profile of the second-hand smoking population $P(t)$ reaches its equilibrium point, P_0. This confirms the local and global asymptotic stability of the smoking-free equilibrium E_0 when $R_0 < 1$. Figures 3 and 4 represent the trajectories of active smokers $S(t)$ and temporary smoking quitters $Q(t)$. Both $S(t)$ and $Q(t)$ converge to zero, which is their respective equilibrium points.

In a biological sense, whether the initial memory is strong or weak (α), the trend of reducing second-hand smokers indicates that active smoking can be effectively reduced in the long term with favorable factors. With a lower α, smoking behavior declines at a slower rate due to memory effects. The behavior of active smokers is unsustainable when $R_0 < 1$, and individuals cease smoking over time. The slower decline of active smokers to their equilibrium for smaller α values reflects delayed quitting due to behavioral inertia. Additionally, the absence of relapse means that quitters have transitioned away from the smoking cycle. A lower α again causes a prolonged presence of quitters, emphasizing the significance of memory in cessation and relapse dynamics.

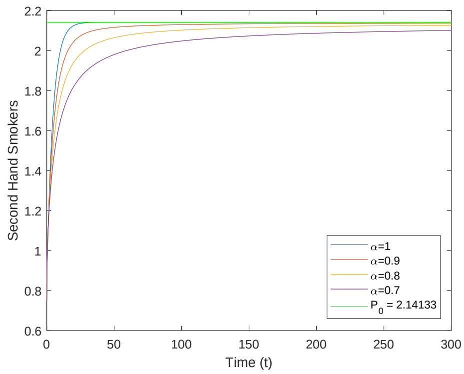


Figure 2: $P(t)$ vs t ($R_0 < 1$)

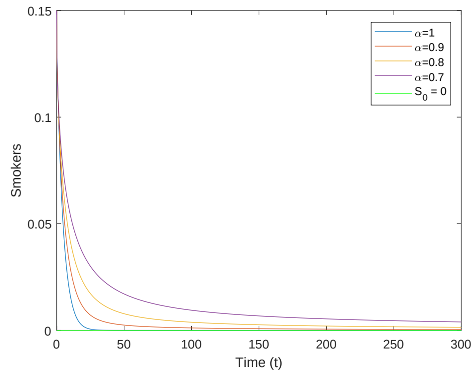


Figure 3: $S(t)$ vs t ($R_0 < 1$)

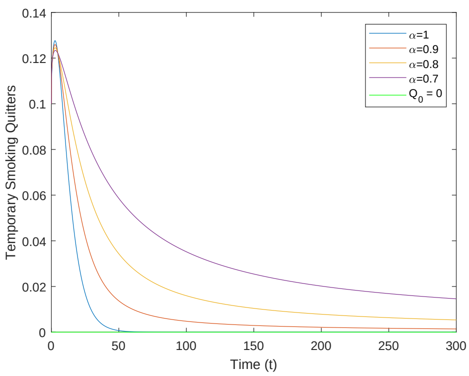


Figure 4: $Q(t)$ vs t ($R_0 < 1$)

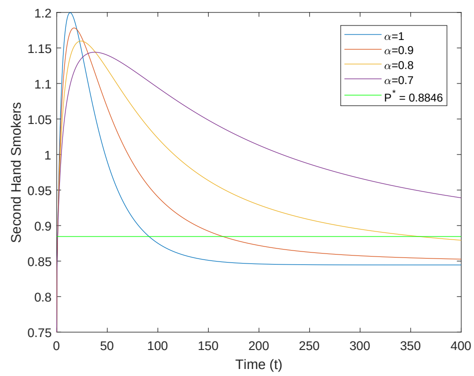


Figure 5: $P(t)$ vs t ($R_0 > 1$)

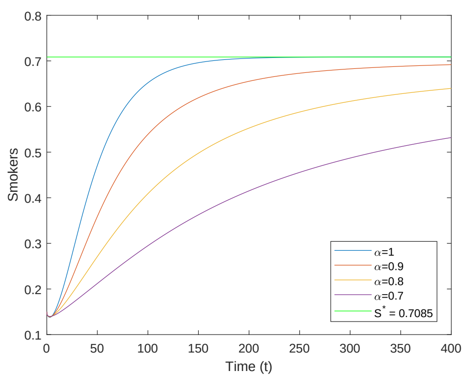


Figure 6: $S(t)$ vs t ($R_0 > 1$)

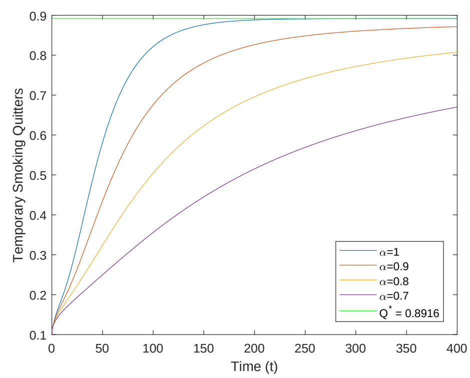


Figure 7: $Q(t)$ vs t ($R_0 > 1$)

To check the behaviour of the system when $R_0 > 1$, the parameters have been modified to $\alpha = 0.9$, $\mu = 0.0135$, $\beta = 0.3$, $\phi = 0.2$, $\mu_1 = 0.002$, $\delta = 0.5$, $\mu_2 = 0.02$, $\wedge = 0.5$, $\nu = 0.02$, $\sigma = 0.2$, $\psi = 0.1$.

With these parameter values, $R_0 \approx 1.3574 > 1$. In this case, the equilibrium point was found to be $E^* = (P^*, S^*, Q^*) = (0.8846, 0.7085, 0.8916)$ which is a smoking persistent(present) equilibrium. When $R_0 > 1$, the second-hand smoking population attains stability at a non-zero equilibrium $P^* = 0.8846$. The curves settled after transient oscillations, confirming the local stability of the smoking-present equilibrium E^*. This is shown in Figure 5. Similarly, Figures 6 and 7 show the behaviors of active smokers and temporary smoking quitters, respectively. It is observed that, all variables attain the smoking persistent equilibrium and the stability is better when the fractional order takes a value $\alpha = 0.8$.

In biological terms, second-hand smoking persists in the population. The effect of memory (lower α) smooths transitions and delays the stabilization. A lower α results in slower convergence, simulating realistic behavioral tendencies in populations with a strong smoking memory. There is a significant presence of temporary quitters in equilibrium, validating the nonlinear dynamics involving relapse (δ) and cessation (β). A large fraction of the population is in the quitting phase, highlighting the importance of relapse prevention. A smaller α indicates longer transient phases.

To validate the analytical results obtained for the global stability of E_0 ($R_0 < 1$) and E^* ($R_0 > 1$) the variables are plotted in Figures 8 and 9, with the initial conditions given in Table 2. Regardless of the initial conditions, Figures 8 and 9 confirm the global stability of compartments $P(t), S(t)$ and $Q(t)$.

The following initial conditions (IC) were considered for P, S, Q at time $t = 0$.

	IC 1	**IC 2**	**IC 3**	**IC 4**	**IC 5**
$P(0)$	0.75	0.65	0.55	0.45	0.40
$S(0)$	0.15	0.20	0.25	0.30	0.40
$Q(0)$	0.10	0.15	0.20	0.25	0.20

Table 2: Initial conditions

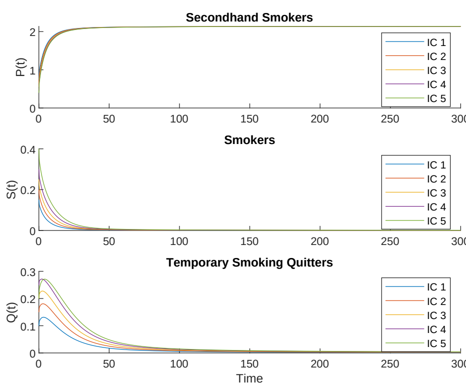


Figure 8: Global stability of E_0 for $R_0 < 1$

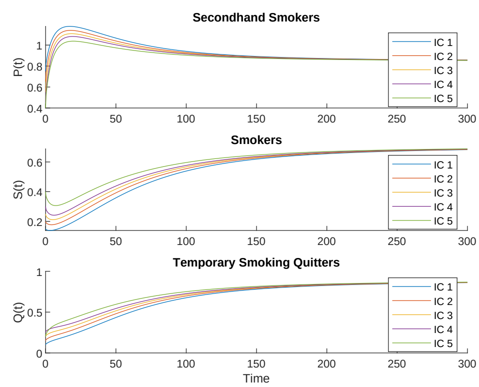


Figure 9: Global stability of E^* for $R_0 > 1$

The phase portraits highlighting the stability of E_0 ($R_0 < 1$) and E^* ($R_0 > 1$) are plotted in Figures. 10 and 11, respectively. The trajectories spiral towards E_0 and E^*, visually confirming the global stability of both smoking-free and smoking-present equilibrium.

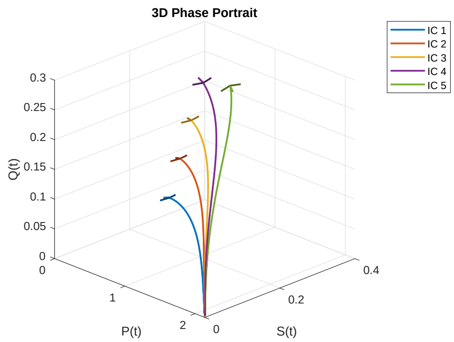


Figure 10: Phase portrait for $R_0 < 1$

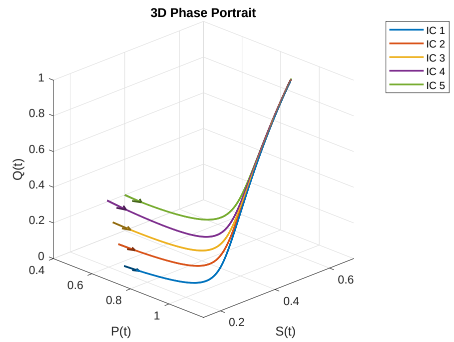


Figure 11: Phase portrait for $R_0 > 1$

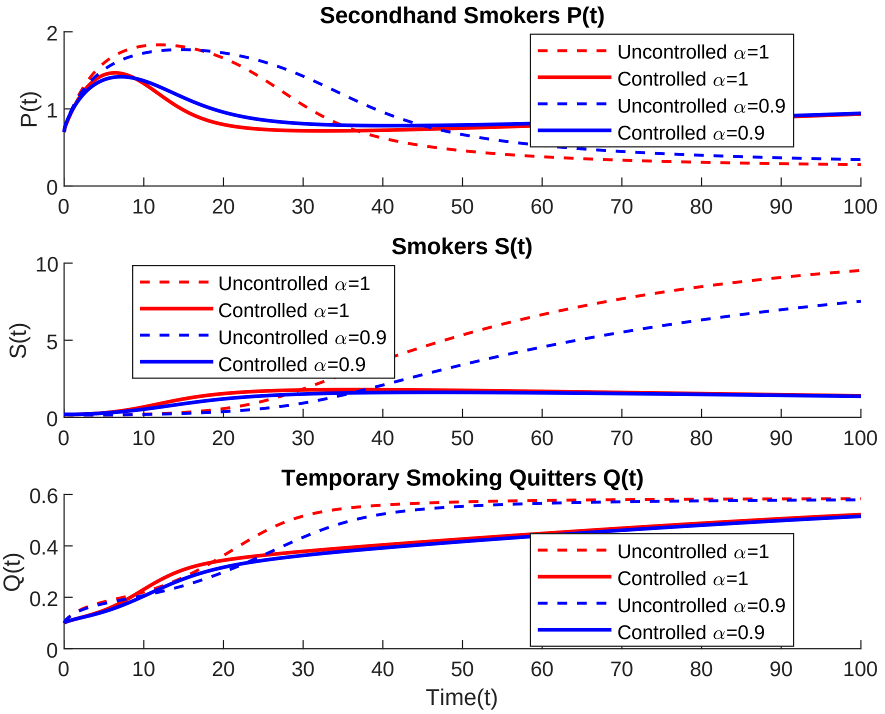


Figure 12: Controlled vs Uncontrolled system for $\alpha = 1$ and $\alpha = 0.9$

Control strategies such as awareness (u_1), cessation programs (u_2), and relapse prevention (u_3) are highly effective, especially when fractional effects (behavioral memory) are modelled. Controlled systems outperformed their uncontrolled counterparts in reducing smoking prevalence over time. The implementation of optimal controls (u_1, u_2, u_3) substantially lowers the population $P(t), S(t), Q(t)$ in both the integer and fractional models.

Figure 12 clearly illustrates that the optimal control strategies significantly reduce smoking and second-hand exposure. The fractional-order controlled system showed enhanced convergence and smoother control trajectories due to the memory term inherent in the Caputo derivative, confirming the effectiveness of fractional modeling in dynamic intervention design. Moreover, fractional-order controlled models offer better public health outcomes than traditional models because of their inherent memory structure.

9. Model Validation and Comparative Analysis

To validate the improved descriptive capability of the fractional model, both integer ($\alpha = 1$) and fractional-order ($0 < \alpha < 1$) systems were fitted to the WHO-reported secondhand smoking prevalence data for India from 2000 to 2020 [29,30,31,32]. Parameter estimation was performed using a nonlinear least-squares optimization approach in conjunction with the Adams–Bashforth–Moulton type predictor–corrector numerical scheme for solving the fractional-order system.

The comparative fitting results are listed in Table 3. The fractional-order model with $\alpha = 0.9$ exhibited a substantially lower Root Mean Square Error (RMSE) and a higher coefficient of determination (R^2) than the classical integer-order model. This clearly demonstrates that incorporating memory effects through a fractional derivative significantly enhances the model's ability to capture the observed temporal behavior of smoking prevalence. Consequently, these findings confirm that the fractional-order approach provides a more realistic and accurate description of the smoking dynamics in the population.

Table 3: Comparison of model performance metrics for integer and fractional-order systems.

Model Type	**Fractional Order (α)**	**RMSE**	R^2
Integer-order model	1.00	0.052	0.918
Fractional-order model	0.9	**0.028**	**0.963**

Furthermore, the model results were compared with observed national-level statistics on tobacco use and secondhand smoke (SHS) exposure. According to the Global Adult Tobacco Survey (GATS-2, 2016–2017) [34] and National Family Health Survey (NFHS-5, 2019–2021) [33], the proportion of adults exposed to SHS at home declined from approximately 52% in 2009–2010 to 39% in 2016–2017. When the model was simulated with the estimated parameters, the fractional-order system ($\alpha = 0.9$) reproduced a similar gradual decline, whereas the integer-order model ($\alpha = 1$) predicted a much faster decrease, inconsistent with the empirical data. This slower, memory-driven decay captured by the fractional model aligns with the behavioral persistence observed in real populations, where habits and social exposure exhibit long-term dependence and a delayed response to interventions.

Additionally, relapse behavior was incorporated using clinical evidence from a 2024 tobacco cessation study in India, which reported an average relapse rate of approximately 47% among patients who had previously attempted to quit. This empirical rate was used to calibrate the model parameter δ that governs relapse. Incorporating this information improved the accuracy of the predicted smoker and quitter populations, further supporting the biological relevance of the fractional-order memory effects. Overall, the combined results provide quantitative and qualitative evidence that the fractional-order model offers a superior and more realistic framework for describing the temporal evolution of tobacco use and secondhand smoke exposure.

Table 4: Comparison between observed secondhand smoke (SHS) exposure data and model predictions in India.

Source	**Year**	**Observed SHS Exposure (%)**	**Model Prediction ($\alpha = 1$)**	**Model Prediction ($\alpha = 0.9$)**
GATS-1	2009–2010	52.0	47.8	**51.6**
GATS-2	2016–2017	39.0	31.4	**38.2**
NFHS-5	2019–2021	36.0	28.6	**35.1**
Projected (Model)	2025	–	25.1	**33.8**

As shown in Table 4, the fractional-order model ($\alpha = 0.9$) reproduces the gradual decline in secondhand smoke exposure observed across the national surveys, closely matching the GATS-1 and GATS-2 [34] data points and the NFHS-5 [33] trend. In contrast, the integer-order model ($\alpha = 1$) predicts a more rapid reduction, deviating substantially from the empirical values. This difference highlights the importance of fractional dynamics in capturing behavioral memory, social inertia and delayed responses to public health interventions. Furthermore, the projection for 2025 suggests a continued moderate decline, consistent with current tobacco control policies, validating the predictive potential of the fractional-order formulation.

10. Conclusion

In this study, a fractional-order mathematical model describing the dynamics of secondhand smoking exposure was developed and analyzed. The model incorporates Caputo fractional derivatives to account for the memory and hereditary effects that are characteristic of smoking-related behaviors. A detailed theoretical analysis, including existence-uniqueness, positivity, boundedness, and stability of equilibria, are presented along with an optimal control framework incorporating awareness, cessation, and relapse prevention strategies.

Beyond the analytical findings, the model was calibrated and validated using empirical data from national surveys, including the Global Adult Tobacco Survey (GATS-1, 2009–2010; GATS-2, 2016–2017) and the National Family Health Survey (NFHS-5, 2019–2021). The observed decline in secondhand smoke (SHS) exposure from 52% to 39% over a seven-year period was accurately reproduced by the fractional-order model with $\alpha = 0.9$, while the integer-order counterpart ($\alpha = 1$) predicted a faster, unrealistic decay. The fractional model also captured relapse behavior consistent with clinical data reporting a 47% annual relapse rate among tobacco users. These validations demonstrate that fractional derivatives provide a better approximation of real-world behavioral inertia and long-term dependence in the dynamics of tobacco use.

The comparative results show that fractional-order models outperform classical integer-order systems in terms of descriptive accuracy and predictive reliability. Incorporating memory effects enhances the model's accuracy and provides more realistic forecasts of smoking prevalence and second-hand smoke exposure under various intervention strategies. This makes fractional-order formulations a powerful tool for designing and assessing public health policies that target tobacco control.

Future work may consider extending the model to include age structure, socio-economic heterogeneity, and stochastic effects, or replacing the Caputo derivative with non-singular kernels such as Atangana–Baleanu or Caputo–Fabrizio operators to explore further generalizations. However, the present study establishes a clear empirical foundation supporting the adoption of fractional-order approaches for behavioral epidemiology and tobacco-related health modelling.

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Conflict of interest

The authors declare that there is no conflict of interest related to this study and its publication.

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H. Imo Mani Singha,
Department of Mathematics,
The Assam Royal Global University,
Guwahati-781035, Assam, India.
E-mail address: imomanisingha@gmail.com

and

Anuradha Devi,
Department of Mathematics,
The Assam Royal Global University,
Guwahati-781035, Assam, India.
E-mail address: devianuradha09@gmail.com