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A Unified Approach to Half-Companion Sequences in Diophantine Triples with Centered (4m+2)-Gonal Numbers

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ABSTRACT: In this study, we develop a framework for generating generalized half-companion sequences linked to Diophantine triples that embed centered (4m+2)-gonal numbers across diverse ranks and structural configurations. We further examine the potential extension of these triples into Diophantine quadruples for specific ranks, employing a customized algebraic algorithm to facilitate the analysis. The investigation unveils a spectrum of intriguing numerical phenomena, which are systematically explored and illustrated through MATLAB-driven simulations and Python Programming.

Key Words: Diophantine triples, quadruples, perfect square, centered polygonal numbers, Half companion sequence.

Contents

1. Introduction

Number theory serves as the foundational philosophy of mathematics and deals with the essential properties of numbers that underpin every act of quantifying, labeling, and measuring. It is a highly specialized branch of pure mathematics where the constraints of seeking only integer solutions create intense complexity and focus. Among its most historically significant and enduring challenges are Diophantine equations, a compelling subject that traces its lineage back to the propositions of Diophantus of Alexandria regarding the construction of specific sets of numbers $\Omega_1 = \frac{1}{16}, \Omega_2 = \frac{33}{16}, \Omega_3 = \frac{68}{16}, \Omega_4 = \frac{105}{16}$ adhering to the constraint,

$$\Omega_i \Omega_j = \eta^2 - 1, \quad \forall\, i, j = 1, 2, 3, 4, \text{where } \eta \in \mathbb{Q}$$

Beyond the classical framework, extensive research has focused on constructing specific sequences of integers. Formally, a Diophantine h-tuple is defined as a sequence of integers $\{u_1, u_2, ...u_h\}$ governed by the property $D(\kappa)$, which means that the product of two distinct elements,u_k and u_m, when summed with κ equals a perfect square. That is,

$$u_k u_m + \kappa = \beta^2$$

where $k, m \in \{1, 2, ...h\}$ with $k \neq m$.

Research has subsequently extended this concept to polynomial rings. Specifically, a special Dio-3 tuple is formed from three distinct polynomials $\{d_1, d_2, d_3\}$ with integer coefficients and the property $D(\kappa)$ if $d_p d_q + d_p + d_q + \kappa =$, where $1 \leq p \neq q \leq 3$ and $p, q \in \mathbb{Z}$ always results in a perfect square for all pairwise combinations, where κ itself is a non-zero integer or a polynomial with integer coefficients.

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Initial research focused on characterizing the extension properties of well-known Diophantine families. The foundational work on the extensibility of Diophantine triples by Bugeaud et.al. [1] has often focused on specific parametric families to support the general conjectures concerning D(1) quadruples. A significant result was achieved regarding the family of triples that contain consecutive integers. Specifically, it has been conclusively proven that for any integer $k \geq 2$, if a D(1)- quadruple is formed by extending the triple $\{k-1, k+1, 16k^3 - 4k\}$ by a fourth positive integer element $'d'$, then d must take one of only two prescribed values: $d = 4k$ or $d = 64k^5 - 48k^3 + 8k$. This precise determination of the only two possible extensions has far-reaching consequences when combined with established theoretical results. When considered in conjunction with the work of Fujita on the nature of Diophantine extensions, this finding demonstrates unequivocally that all Diophantine quadruples of the form $\{k-1, k+1, c, d\}$ are regular. This regularity confirms that the largest element, d, is uniquely determined by the preceding three elements $\{k-1, k+1, c\}$ via the smallest possible extension formula, strongly supporting the conjecture that all Diophantine quadruples are regular.

Although the classical D(1) problem yields regularity for certain families, the literature also explores generalizations to different constants. The $D(4)-m-$tuple is a set of m positive integers for which the product of any two elements increased by four is a perfect square. Extending this line of inquiry, Bačić and Filipin [2] specifically addressed the question of whether a $D(4)-$quintuple can exist. By investigating the extensibility of a general D(4)-pair $\{a, b\}$, their work provided strong evidence and theoretical results supporting the widely held conjecture that there does not exist a $D(4)-$quintuple. This reinforced the notion that the limitations on the size of Diophantine tuples are not restricted to $n = 1$, but apply broadly to other square constants as well.To move toward a more general understanding of these tuples, research has focused on the necessary conditions for their existence, irrespective of the constant n. Zhang and Grossman [3] provided a significant generalized framework by considering a triple of integers $\{a, b, c\}$ and establishing the necessary and sufficient conditions for the existence of an integer n such that $\{a, b, c\}$ constitutes a $D(n)-$triple. This work provides a fundamental structural characterization, detailing precisely what relationships must hold among the elements of a three-tuple for it to satisfy the Diophantine property for some constant n. These main results were then applied through several illustrative examples to demonstrate their implications for both Diophantine triples and their extension to quadruples.

Building upon the initial classification of D(1)-triples of Fibonacci numbers, the analysis was extended to other constants. A separate, yet highly significant, stream of research involves restricting the elements of the tuples to terms from well-known number sequences. In this vein, He, Luca, and Togbe (2016) [4] provided a complete characterization of Diophantine triples of Fibonacci Numbers. Specifically, they determined that for a given pair of Fibonacci numbers $\{F_{2n}, F_{2n+2}\}$, the only possible third Fibonacci number, F_k, that can complete a D(1)-triple is $k = 2n + 4$ (for $n \geq 1$) or $k = 2n - 2$ (for $n \geq 2$), with the singular exception of $k = 1$ when $n = 2$. Rihane, Hernane, and Togbé [5] specifically characterized the D(4)-Diophantine triples of Fibonacci numbers. They focused on a specific pair of terms, $\{F_{2n+6}, 4F_{2n+4}\}$, and proved that the only third Fibonacci number, F_k, that completes this pair to a $D(4)-$ triple is $k = 2n$ (for $n \geq 1$), with the exception of $k = 1$ when $n = 1$. This specialized finding further confirms that even when the constant n is varied to 4, the constraints imposed by sequence membership are strong. Mihai Cipu, Yasutsugu Fujita, and Takafumi Miyazaki [6] focuses on improving the known upper bounds for the number of ways a Diophantine triple $\{a, b, c\}$ can be extended to a Diophantine quadruple $\{a, b, c, d\}$ by adjoining an element d greater than $\max\{a, b, c\}$.The long-standing conjecture is that this extension should be unique (corresponding to the 'regular' extension d^+. The authors utilize powerful number theory techniques, including the study of systems of Pellian equations and Baker's theory on linear forms in logarithms, and they also establish much tighter bounds (in some cases proving uniqueness) under specific conditions on the elements of the triple.

Earp-Lynch et.al [7] address the characterization of Diophantine m-tuples, a set of m distinct positive integers $\{a_1, \ldots, a_m\}$ such that $a_i a_j + q$ is a perfect square for all $i \neq j$. Specifically, they extend the work of Bacic and Filipin on $D(4)$-Diophantine triples by characterizing the solutions of Pellian equations that correspond to $\mathbf{D(l^2)}$-Diophantine triples subject to certain divisibility requirements. Their primary methodological tool involves employing the resulting characterization in tandem with bounds on linear forms in logarithms of algebraic numbers. This combined approach allows them to fully classify specific families of $D(9)$- and $D(64)$-Diophantine triples whose elements are defined by Fibonacci numbers, specifically those taking the forms $\{F_{2n+8}, 9F_{2n+4}, F_k\}$ and $\{F_{2n+12}, 16F_{2n+6}, F_k\}$, where F_i denotes the i-th Fibonacci number. Although previous research established classifications for $D(q)$-Diophantine triples related to Fibonacci numbers, a crucial parallel area of inquiry involves the extensibility of these sets—specifically, whether a triple can be extended to a quadruple. Kouèssi Norbert Adédji et.al [8] investigates the Diophantine triple $\{a, b, c\}$ under strict constraint $\mathbf{b} = \mathbf{3a}$. The authors show that triples satisfying this condition cannot be extended to an irregular Diophantine quadruple, leading to the corollary that any quadruple containing $\{a, 3a\}$ must be regular. This finding provides a significant structural constraint, demonstrating that specific initial conditions, such as $b = 3a$, impose regularity on the resulting Diophantine quadruples, a result that holds for similar conditions like $b = 8a$.

Investigating the extensibility and regularity of Diophantine triples naturally leads to the study of $\mathbf{D(n)}$-tuples with negative parameters. While the standard definition uses a positive integer q (or n), researchers have extensively studied $D(-k)$-tuples. For example, Nikola Adžaga [9] establishes a general structural result: for any positive integer k, if $\{k, k+1, c, d\}$ is a $D(-k)$-quadruple with $c > 1$, it must be $d = 1$. The methods employed are highly advanced, combining Baker's linear forms in logarithms and the hypergeometric method with recurrence arguments and integral points on hyperelliptic curves. In addition, subsequent research has focused on solidifying specific claims within this area. Ad´edji, K.N et.al [10] confirms a long-standing conjecture about the structure of the $D(-k)$-quadruple $\{k, k+1, c, d\}$, proving that the unique solution $(c, d) = (1, 4k+1)$ holds for the infinite family of parameters where k takes the form $k = l^2 - 1$ (for $l \geq 3$).

The previous works established both general structural constraints on Diophantine quadruples (such as the regularity imposed by $b = 3a$) and confirmed conjectures regarding $D(-k)$-tuples. Shanmuga Priya et.al [11] extends the investigation by focusing on constructive methods and specialized sequences. The authors generate the generalized half companion sequence from a specific Diophantine triple that incorporates centered triacontagonal numbers, examining the sequence's properties. Furthermore, they return to the theme of extensibility, analyzing the possibility of extending these specialized 3-tuples to quadruples for certain ranks using an algebraic algorithm, with the results and patterns demonstrated computationally. This transition highlights a shift from purely theoretical proofs and classifications to the study of sequences derived from specific number forms and the use of computational tools in Diophantine analysis. In this study, a generalized half companion sequence of Diophantine triple is produced by incorporating centered polygonal numbers and further, their maximality is checked using Python.

2. Preliminaries

The mathematical concept of centered k-gonal numbers has its origins in ancient Greece, specifically with the Pythagoreans in the 6th century BCE. Driven by their philosophical belief that "all is number," they developed the idea of figurate numbers, which represented integers through geometric arrangements of dots.The earliest focus was on standard polygonal numbers (like squares and triangles), which were constructed by adding L-shaped layers, or gnomons. The idea of a centered figure naturally followed, modifying the structure by beginning with a single central dot and surrounding it with complete concentric layers of the k-sided polygon.

Although Greek figures such as Nicomachus of Gerasa documented the geometric properties of these number sequences, precise mathematical formalization, including the general formula $C_m(b) = 1 + m.T_{b-1}$, was a later development. This formula relates centered b-gonal numbers to triangular numbers T_{n-1}, and was thoroughly explored during the Renaissance and modern eras by influential number theorists such as Pierre de Fermat and Leonhard Euler.

Definition 2.1: The B^{th} centered μ-gonal number, denoted by $C_\mu(B)$ is given by

$$C_\mu(B) = \frac{\mu b^2 + \mu b + 2}{2}$$

where $B = b + 1$ is also known as rank such that $b \in \mathbb{W}$ and $\mu \geq 3$ is the number of sides of the polygon. Also, $b = 0$ yields the central dot.

When $\mu = 4m + 2$, then it results in centered $(4m + 2)$-gonal number of rank $B = b + 1$, which is given by

$$C_{4m+2}(B) = (2m + 1)b^2 + (2m + 1)b + 1 \tag{2.1}$$

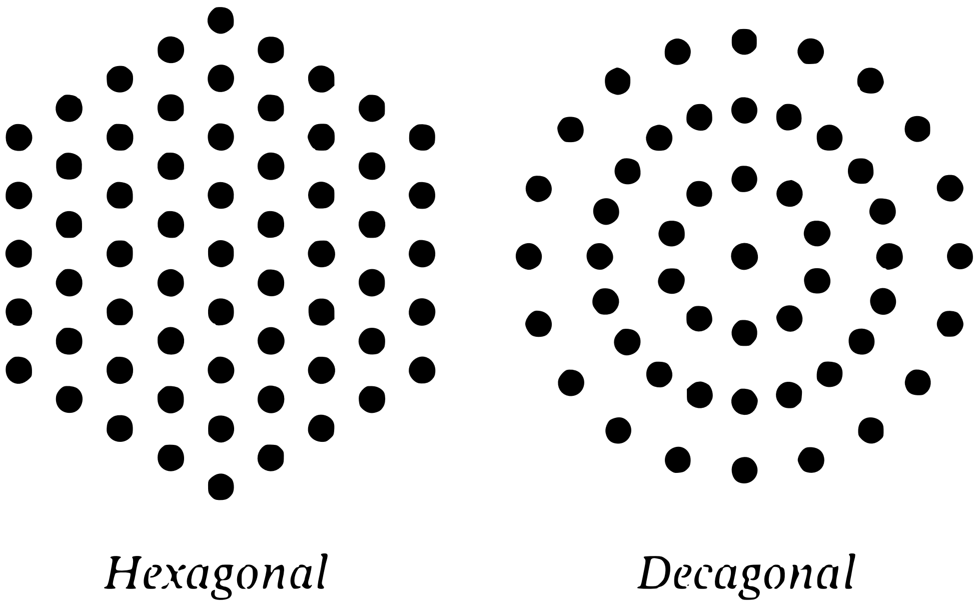

Figure 1: Examples for Centered (4m+2)-gonal number

3. Half Companion Sequences of Diophantine Triples

This section presents the sequence of Diophantine triples derived from distinct ranks of centered $(4m+2)$-gonal numbers.

Theorem 3.1:
Let $C_{4m+2}(B), C_{4m+2}(B+h)$ be the two centered $(4m+2)$-gonal numbers with $B = b+1$ and $b \in \mathbb{W}$, then $[(\eta_1, \eta_2, 2\eta_1 + 2\eta_2 - (2m+1)h^2 + h), (\eta_2, 2\eta_1 + 2\eta_2 - (2m+1)h^2 + h, 3\eta_1 + 6\eta_2 - (4m+2)h^2 + 2h), (2\eta_1 + 2\eta_2 - (2m+1)h^2 + h, 3\eta_1 + 6\eta_2 - (4m+2)h^2 + 2h, 10\eta_1 + 15\eta_2 - (12m+6)h^2 + 6h), ...]$ forms the half companion sequence of Diophantine triples generated by $\eta_1 = C_{4m+2}(B), \eta_2 = C_{4m+2}(B+h)$ with the property $D((2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h)$, where $b, m \in \mathbb{W}$ with $m \geq 1$.

Proof:
Let η_1 and η_2 be the centered $(4m+2)$-gonal numbers of the ranks B and $B+h$, respectively with $B = b+1$ and $b \in \mathbb{W}$, then

$$\eta_1 = C_{4m+2}(B) = (2m+1)b^2 + (2m+1)b + 1 \tag{3.1}$$
$$\eta_2 = C_{4m+2}(B+h) = (2m+1)(b+h)^2 + (2m+1)(b+h) + 1 \tag{3.2}$$

where $b, m \in \mathbb{W}$ and $m \geq 1$.

$$\begin{aligned}\eta_1\eta_2 = {} & 4b^4m^2 + 8b^3m^2 + 4b^2m^2 + 4b^4h^2m^2 + 4b^4hm^2 + 8b^3hm^2 \\ & + 12b^2hm^2 + 4bhm^2 + 4b^4m + 8b^3m + 8b^2m + 4bm + 4b^4h^2m \\ & + 4b^4h^2m + 2h^2m + 8b^3hm + 12b^2hm + 8bhm + 2hm + b^4 + 2b^3 \\ & + 3b^2 + 2b + b^2h^2 + bh^2 + h^2 + 2b^3h + 3b^2h + 3bh + h + 1\end{aligned} \tag{3.3}$$

It is observed that $\eta_1\eta_2 + (2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h$ is a perfect square. Thus, (η_1, η_2) form a Diophantine pair with the property $D((2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h)$.

To extend this to a Diophantine triple, it is necessary to find $\eta_3 \in \mathbb{Z}_+$ such that the following constraints hold with assumptions $\Omega_1 = \beta_1 + \eta_1\delta_1$ and $\Omega_2 = \beta_1 + \eta_2\delta_1$, where $\beta_1, \delta_1 \in \mathbb{Z}_+$.

$$\eta_1\eta_3 + (2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h = \Omega_1^2 \quad (3.4)$$
$$\eta_2\eta_3 + (2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h = \Omega_1^2 \quad (3.5)$$

The following pellian equation is resulted from (3.4) and (3.5),

$$\beta_1^2 = (2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h + \eta_1\eta_2\delta_1^2$$

Taking $\delta_1 = 1$ as a fundamental solution of pellian equation, the value of β_1 is derived as follows:

$$\beta_1 = (2m+1)b^2 + (2m+1)b + (2m+1)bh + (m+1)h + 1 \quad (3.6)$$

Utilizing (3.6) in $\Omega_1 = \beta_1 + \eta_1\delta_1$,

$$\Omega_1 = (4m+2)b^2 + (4m+2)b + (2m+1)bh + (m+1)h + 2 \quad (3.7)$$

With the help of (3.1) and (3.7), η_3 can be determined as below.

$$\eta_3 = (8m+4)b^2 + (8m+4)b + (8m+4)bh + (2m+1)h^2 + (4m+3)h + 4 \quad (3.8)$$
$$\Rightarrow \eta_3 = 2[C_{4m+2}(B) + C_{4m+2}(B+h)] - (2m+1)h^2 + h$$
$$\Rightarrow \eta_3 = 2\eta_1 + 2\eta_2 - (2m+1)h^2 + h \quad (3.9)$$

Thus, (η_1, η_2, η_3) forms a Diophantine triple. Now, $\eta_4 \in \mathbb{Z}_+$ is derived from η_2, η_3 with the following constraints with $\Gamma_1 = \alpha_1 + \eta_2\gamma_1$ and $\Gamma_2 = \alpha_1 + \eta_3\gamma_1$.

$$\eta_2\eta_4 + (2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h = \Gamma_1^2 \quad (3.10)$$
$$\eta_3\eta_4 + (2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h = \Gamma_1^2 \quad (3.11)$$

The above equations lead to the pellian equation of the form $\alpha_1^2 = (2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h + \eta_2\eta_3\gamma_1^2$. Applying the fundamental solution, the value of $\alpha_1 = (4m+2)b^2 + (4m+2)b + (6m+3)bh + (2m+1)h^2 + (3m+2)h + 2$. This will further lead to the value of Γ_1 as

$$\Gamma_1 = (6m+3)b^2 + (6m+3)b + (10m+5)bh + (4m+2)h^2 + (5m+3)h + 3$$

As a result, η_4 is obtained as,

$$\eta_4 = (18m+9)b^2 + (18m+9)b + (24m+12)bh + (8m+4)h^2 + (12m+8)h + 9$$
$$\Rightarrow \eta_4 = 3C_{4m+2}(B) + 6C_{4m+2}(B+h) - (4m+2)h^2 + 2h$$
$$\Rightarrow \eta_4 = 3\eta_1 + 6\eta_2 - (4m+2)h^2 + 2h$$

Hence, (η_2, η_3, η_4) becomes a Dio-triple. Similarly, the Diophantine triple (η_3, η_4, η_5) can be constructed by producing η_5 from η_3, η_4, which is given by

$$\eta_5 = 10C_{4m+2}(B) + 15C_{4m+2}(B+h) - (12m+6)h^2 + 6h$$
$$\eta_5 = 10\eta_1 + 15\eta_2 - (12m+6)h^2 + 6h$$

Proceeding in this way, $\eta_6, \eta_7, ...$ are generated. Thus, $\{(\eta_1, \eta_2, \eta_3), (\eta_2, \eta_3, \eta_4), (\eta_3, \eta_4, \eta_5), ...\}$ forms the half companion sequence.

Theorem 3.2:

Let $C_{4m+2}(B-h), C_{4m+2}(B+h)$ be the two centered $(4m+2)$-gonal numbers with $B = b+1$ and $b \in \mathbb{W}$, then $\{\zeta_1, \zeta_2, 2\zeta_1 + 2\zeta_2 - (8m+4)h^2\}, \{\zeta_2, 2\zeta_1 + 2\zeta_2 - (8m+4)h^2, 3\zeta_1 + 6\zeta_2 - (16m+8)h^2\}, \{2\zeta_1 + 2\zeta_2 - (8m+4)h^2, 3\zeta_1 + 6\zeta_2 - (16m+8)h^2, 10\zeta_1 + 15\zeta_2 - (48m+24)h^2\}, ...$ represents the half companion sequence of Diophantine triples generated by $\zeta_1 = C_{4m+2}(B-h), \zeta_2 = C_{4m+2}(B+h)$ with the property

$D((2m-3)(2m+1)h^2)$, where $b, m \in \mathbb{W}$ with $m \geq 1$ and $b \geq h$.

Proof:

Assign ζ_1 and ζ_2 as centered $(4m+2)$-gonal numbers of ranks $B-h$ and $B+h$, respectively, where $B = b+1$ and $b \in \mathbb{W}$, then

$$\zeta_1 = C_{4m+2}(B-h) = (2m+1)(b-h)^2 + (2m+1)(b-h) + 1 \quad (3.12)$$
$$\zeta_2 = C_{4m+2}(B+h) = (2m+1)(b+h)^2 + (2m+1)(b+h) + 1 \quad (3.13)$$

where $b, m \in \mathbb{Z}_+$ with $m \geq 1$ and $b \geq h$.

$$\begin{aligned}\zeta_1\zeta_2 = {} & 4b^4m^2 + b^4 + 4mb^4 + 4h^4m^2 + 4mh^4 + h^4 + 2b^3 + h^2b^2 - 2bh^2 - 3b^2h^2 + 4b^3m^2 \\ & + 8mb^3 - 4h^3m^2 - 12b^2h^2m^2 - 12mb^2h^2 + 4h^2b^2m^2 - 4b^2m^2h - 4mb^2h + 3b^2 \\ & + 8mb^2 - 12bm^2h^2 + 4m^2b^3 + 4m^2h^3 + 4b^2hm + 4m^2b^2 - 4m^2h^2 \\ & + 4bm - 8bmh^2 + 4bh^2m^2 + 2b + h^2 + 4m^2b^2h + 4mh^2b^2 + 1\end{aligned}$$

This $\zeta_1\zeta_2$ turns out to be a perfect square using the property $D((2m-3)(2m+1)h^2)$. Thus, $\{\zeta_1, \zeta_2\}$ becomes a Diophantine pair. To extend this to a Diophantine triple, a positive integer $\zeta_3 \in \mathbb{Z}_+$ is found such that it satisfies the system of equations $\zeta_i\zeta_3 + D((2m-3)(2m+1)h^2) = \Omega_i^2$ with the assumption of linear forms $\Omega_i = \beta_1 + \zeta_i\delta_1$, where $i = 1, 2$. Elimination of ζ_3 from the system generates a Pellian equation. To solve this, a fundamental solution of the Pellian equation is applied. As a result, Ω_1 is obtained as $\Omega_1 = (4m+2)b^2 + (4m+2)b - (2m+1)h - (4m+2)bh + 2$. This will also help in finding ζ_3 as,

$$\zeta_3 = (8m+4)b^2 + (8m+4)b + 4 \quad (3.14)$$
$$\Rightarrow \zeta_3 = 2\zeta_1 + 2\zeta_2 - (8m+4)h^2 \quad (3.15)$$

Hence, $(\zeta_1, \zeta_2, \zeta_3)$ is a Dio-triple. Using a similar approach, ζ_4 and ζ_5 are produced from the pairs $\{\zeta_2, \zeta_3\}$ and $\{\zeta_3, \zeta_4\}$, respectively.

$$\zeta_4 = 3\zeta_1 + 6\zeta_2 - (8m+4)h^2$$
$$\zeta_5 = 10\zeta_1 + 15\zeta_2 - (48m+24)h^2$$

Thus, each ζ_i can be derived from the pair $\{\zeta_{i-1}, \zeta_{i-2}\}$, where $i = 6, 7, 8, ...$).Through the systematic application of this governing principle, every Diophantine triple $\{\zeta_i, \zeta_{i+1}, \zeta_{i+2}\}$(for $i = 1, 2, ...$) is obtained so that the half companion sequence $\{\zeta_1, \zeta_2, 2\zeta_1 + 2\zeta_2 - (8m+4)h^2\}, \{\zeta_2, 2\zeta_1 + 2\zeta_2 - (8m+4)h^2, 3\zeta_1 + 6\zeta_2 - (16m+8)h^2\}, \{2\zeta_1 + 2\zeta_2 - (8m+4)h^2, 3\zeta_1 + 6\zeta_2 - (16m+8)h^2, 10\zeta_1 + 15\zeta_2 - (48m+24)h^2\}, ...$ is formed.

4. Non-existence of Quadruples

This section provides the theoretical framework establishing the maximal extent (non-extendibility) of Diophantine triples derived from the base pairs $\{\eta_1, \eta_2\}$ and $\{\zeta_1, \zeta_2\}$. Crucially, these non-extensions are determined by a dual-condition analysis using the distinct $D(\kappa)-$ properties, where $\kappa = (2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h$ and $\kappa = (2m-3)(2m+1)h^2$ respectively.

Theorem 4.1:

Let η_1, η_2, η_3 be defined as in (3.1),(3.2) and (3.8) respectively. Then, the Diophantine triple $\{\eta_1, \eta_2, \eta_3\}$ is non-extendible to quadruple under the property $D((2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h)$ for all values of $b \in \mathbb{W}$ and $m, h \in \mathbb{Z}_+$ with $1 \leq m \leq 2000$ and $0 < h \leq 2000$.

Proof:

By theorem 3.1, it is proved that $\{\eta_1, \eta_2, \eta_3\}$ forms $D(N)$- triple , where $N = (2m+1)b^2h + (2m+1)bh^2 + (2m+1)bh + m^2h^2 + h$ and $b \in \mathbb{W}, m, h \in \mathbb{Z}_+, m \geq 1$. To find whether this is extendable, it is required to find some $\eta_0 \in \mathbb{Z}_+$ such that the following conditions are kept with the assumptions $\Omega_j = \beta_1 + \eta_j \delta_1,$ where $j = 1, 2, 3$.

$$\eta_1\eta_0 + N = \Omega_1^2 \tag{4.1}$$
$$\eta_2\eta_0 + N = \Omega_2^2 \tag{4.2}$$
$$\eta_3\eta_0 + N = \Omega_3^2 \tag{4.3}$$

Removing η_0 from (4.1) and (4.3), yields a pellian equation of the form $\beta_1^2 = N + \eta_1\eta_3\delta_1^2$ for which the fundamental solution $\{\beta_1, 1\}$ is applied so that β_1 can be evaluated as

$$\beta_1 = (4m+2)b^2 + (4m+2)b + (2m+1)bh + (m+1)h + 2$$

This will give rise to the value of Ω_1 as $\Omega_1 = (6m+3)b^2 + (6m+3)b + (2m+1)bh + (m+1)h + 3$. Using all the known values in (4.1), it is found that

$$\eta_0 = (18m+9)b^2 + (18m+9)b + (12m+6)bh + (2m+1)h^2 + (6m+5)h + 9 \tag{4.4}$$

The above equation must satisfy (4.2). To verify that, three cases must be taken into account.
Case(i): $b > h$. Suppose that $b = h + 1$, then

$$\begin{aligned}\eta_2\eta_0 + N \;=\;& 256m^2h^4 + 256mh^4 + 960m^2h^3 + 1281m^2h^2 + 144m^2 + 980mh^3 + 1416mh^2 + 216m \\ &+64h^4 + 250h^3 + 388h^2 + 720m^2h + 912mh + 279h + 81\end{aligned} \tag{4.5}$$

Case (ii): $b = h$. Now,

$$\begin{aligned}\eta_2\eta_0 + N \;=\;& 256h^4m + 64h^4 + 340h^3m + 90h^3 + 210h^2m + 81h^2 \\ &+m^2h^2 + 60hm + 33h + 256h^4m^2 + 320h^3m^2 + 96h^2m^2 + 9\end{aligned} \tag{4.6}$$

Case(iii): $b < h$. Assume $b = h - 1$,then

$$\begin{aligned}\eta_2\eta_0 + N \;=\;& 256m^2h^4 + 256mh^4 - 320m^2h^3 - 300mh^3 + 188mh^2 + 64h^4 - 70h^3 \\ &+70h^2 + 97m^2h^2 - 60mh - 27h + 9\end{aligned} \tag{4.7}$$

A straightforward computational check (e.g., via Python Programming Software) confirms that equations (4.5), (4.6) and (4.7) are non-perfect squares for all $1 \le m \le 2000$ and $0 < h \le 2000$. Thus, there exists no $\eta_0 \in \mathbb{Z}_+$ such that $\{\eta_1, \eta_2, \eta_3, \eta_0\}$ forms a quadruple for all values of $m, b, h \in \mathbb{Z}_+$ with $m, h \in [1, 2000]$.

Theorem 4.2:
Let $\zeta_1, \zeta_2, \zeta_3$ be defined as in (3.12),(3.13) and (3.14) respectively. Then, no quadruples can be formed from the Diophantine triple $\{\zeta_1, \zeta_2, \zeta_3\}$ with the property $D((2m-3)(2m+1)h^2)$ for all positive integer values of m, b, h with $b \ge h$, $1 \le m \le 2000$ and $0 < h \le 2000$.

Proof:
According to theorem 3.2, $\{\zeta_1, \zeta_2, \zeta_3\}$ is a Dio-triple with the property $D((2m-3)(2m+1)h^2)$, where $m, b, h \in \mathbb{N}$ with $b \ge h$. Let the positive integer ζ_0 be determined from the following system of equations by considering the linear forms $\Omega_k = \beta_1 + \zeta_k\delta_1,$ where $k = 1, 2, 3$.

$$\zeta_1\zeta_0 + (2m-3)(2m+1)h^2 = \Omega_1^2 \tag{4.8}$$
$$\zeta_2\zeta_0 + (2m-3)(2m+1)h^2 = \Omega_2^2 \tag{4.9}$$
$$\zeta_3\zeta_0 + (2m-3)(2m+1)h^2 = \Omega_3^2 \tag{4.10}$$

By a similar procedure as in theorem 4.1, ζ_0 is determined as

$$\zeta_0 = (6m+3)b^2 + (6m+3)b - (12m+6)bh + (2m+1)h^2 - (6m+3)h + 9$$

This value of ζ_0 must satisfy (4.9). This verification requires a complete evaluation of two possible cases: $b > h$ and $b = h$. Python software analysis confirmed that the expression $\zeta_2\zeta_0 + (2m-3)(2m+1)h^2$ is never a perfect square when b is either $h + 1$ or h for $m, h \in [1, 2000]$.

5. Numerical Verification

To validate the theoretical results, a computational verification of Theorems 3.1 and 3.2 was conducted utilizing MATLAB software. Representative outputs from this analysis, which confirm the theorems, are tabulated in Tables 1 and 2.

m	h	b	(η_1,η_2,η_3)	(η_2,η_3,η_4)	(η_3,η_4,η_5)	$D(N)$
1	1	1	(7,19,50)	(19,50,131)	(50,131,343)	$D(11)$
		2	(19,37,110)	(37,110,275)	(110,275,733)	$D(26)$
		3	(37,61,194)	(61,194,473)	(194,473,1273)	$D(47)$
		4	(61,91,302)	(91,302,725)	(302,725,1963)	$D(74)$
2	1	1	(11,31,80)	(31,80,211)	(80,211,551)	$D(20)$
		2	(31,61,180)	(61,180,451)	(180,451,1201)	$D(45)$
		3	(61,101,320)	(101,320,781)	(320,781,2101)	$D(80)$
		4	(101,151,500)	(151,500,1201)	(500,1201,3251)	$D(125)$

Table 1: A few representative cases of Theorem 3.1

m	h	b	$(\zeta_1,\zeta_2,\zeta_3)$	$(\zeta_2,\zeta_3,\zeta_4)$	$(\zeta_3,\zeta_4,\zeta_5)$	$D(\kappa)$
1	1	1	(1,19,28)	(19,28,93)	(28,93,223)	$D(-3)$
		2	(7,37,76)	(37,76,219)	(76,219,553)	$D(-3)$
		3	(19,61,148)	(61,148,399)	(148,399,1033)	$D(-3)$
		4	(37,91,244)	(91,244,633)	(244,633,1663)	$D(-3)$
2	1	1	(1,31,44)	(31,44,149)	(44,149,355)	$D(5)$
		2	(11,61,124)	(61,124,359)	(124,359,905)	$D(5)$
		3	(31,101,244)	(101,244,659)	(244,659,1705)	$D(5)$
		4	(61,151,404)	(151,404,1049)	(404,1049,2755)	$D(5)$

Table 2: Illustrative examples of Theorem 3.2

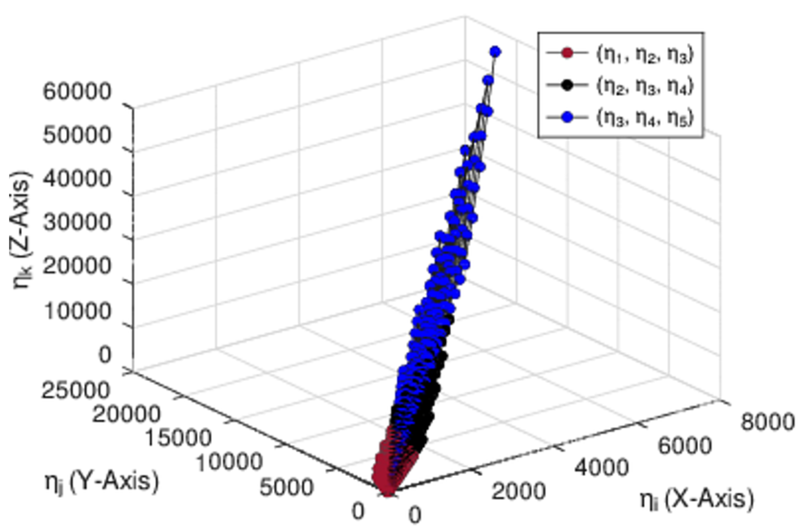

Figure 2: Scattered plot of Diophantine triples generated by $C_{4m+2}(B)$ & $C_{4m+2}(B+h)$.

6. Conclusion

This study concludes by providing a generalized framework for the half-companion sequence of Diophantine triples that incorporate centered $(4m+2)$-gonal numbers of ranks $B, B+h, B-h$, where $B = b+1$. More critically, the investigation computationally defined the limits of their extensibility: a comprehensive search utilizing Python programming software definitively established that the generated triples do not extend to quadruples with distinct properties $D((2m+1)b^2h+(2m+1)bh^2+(2m+1)bh+m^2h^2+h)$ and $D((2m-3)(2m+1)h^2)$ across the parameter space where m, h ranges from 1 to 2000.

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