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Morphic Uniquely Clean Rings

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ABSTRACT: In this research article it is observed that if a ring R is morphic uniquely clean in which every one sided ideal is principal then it is uniquely nil clean. We establish that every morphic uniquely clean ring has square stable range one. Also we provide a characterizations of Boolean rings.

Key Words: Jacobson radical, left morphic ring, uniquely clean ring.

Contents

1. Introduction

Ring theory is a key area within abstract algebra that studies mathematical structures called rings. This paper aims to explore the structural implications of imposing morphicity on uniquely clean rings. A result due to G. Ehrlich [5] says that an element α in the endomorphism ring of a module M is unit regular if and only if α is von Neumann regular and quotient module $M/Im(\alpha) \cong ker(\alpha)$. When $M =_R R$, Nicholson in [9] referred to a ring R as *left morphic* if $R/Ra \cong l(a)\ \forall a \in R$. Likewise, a ring R is termed left morphic when for each $a \in R$, $\exists b \in R$ such that $Ra = l(b)$ and $Rb = l(a)$.The notions of a right morphic element and a right morphic ring are introduced in a similar manner. A ring that satisfies the morphic property on both the left and right sides is referred to as a morphic ring. In [9] and [10], the authors proved many important results on left or right morphic rings. The idea of a clean ring, where every element is the sum of a unit and an idempotent, was originally formulated by W.K. Nicholson [8] in the late 1970s to explore properties related to exchange rings. For an element a in a ring R it is termed uniquely clean if there exists exactly one idempotent e in R such that the difference $a-e$ is invertible.The ring R itself is called uniquely clean when each of its elements is uniquely clean. In [2], [11] authors studied about uniquely clean rings. We know that $\mathbb{Z}_n$ is morphic but it is not uniquely clean. This indicates that morphic rings are not necessarily uniquely clean. If base ring is Boolean then $R[[x]]$ is not morphic. But by [11, Corollary 10] it is uniquely clean . Thus, uniquely clean ring need not be morphic. We are interested in the ring which is both morphic and uniquely clean.

2. Preliminaries

Throughout this work, every ring is associative and contain a multiplicative identity. The symbol J is used to denote the Jacobson radical of the ring R. The notation $Id(R)$ stands for the set of idempotent elements of the ring R, while $U(R)$ denotes its set of units. For any element a in a ring R, the notation $l_R(a)$ and $r_R(a)$ denote the sets of elements in R that annihilate a from the left and right respectively. We write $l(a)$ and $r(a)$ in place of $l_R(a)$ and $r_R(a)$ respectively.

For any field $\mathbb{F}$, the matrix ring $M_2(\mathbb{F})$ is an example for morphic uniquely clean ring. The direct product of rings that are each morphic and uniquely clean remains morphic uniquely clean. Artinian rings are morphic uniquely clean.

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3. Results

Consider a ring R, M be a bimodule over R. Let $R(\propto)M = \{(a, b) : a \in R, b \in M\}$ with $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$, $(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1b_2 + b_1a_2)$.

Proposition 3.1 *Let $S = R(\propto)R$ where R is commutative. For any $a \in R$ is unit regular then $(0, a) \in S$ is morphic uniquely clean.*

Proof: Consider an element a in a commutative unit regular ring R. Then there exists $f^2 = f \in Id(R)$ and $v \in U(R)$ such that $a = fv$. From this $(0, a) = (0, fv)$ which is morphic by [3, Proposition 20(1)]. Also $(0, a)$ is central nilpotent and by [11, Example 1] each nilpotent element that lies in the center of the ring has a unique clean decomposition. Therefore, $(0, a)$ is morphic uniquely clean in S. □

Proposition 3.2 *Assume R is morphic uniquely clean ring where each one-sided ideal is generated by a single element. Under these conditions, R qualifies as a uniquely nil clean ring.*

Proof: Let R be a morphic uniquely clean ring with the property that all one sided ideal is generated by a single element. Then by [10], R is P-morphic. Consequently, R is artinian by [10, Theorem 13] which implies, each prime ideal corresponds to a maximal ideal. Therefore, by [2, Theorem 4.1] ring R is uniquely nil clean . □

Theorem 3.1 *Each of these conditions is equivalent to the others.*

1. *R be a morphic uniquely clean ring with zero singular ideal.*
2. *R be a reduced morphic uniquely clean.*
3. *$R[[x]]/(x^n)$ is morphic uniquely clean $\forall n \geq 1$.*
4. *R is Boolean.*

Proof: (1) $\iff$ (4): Consider a ring R which is morphic uniquely clean with zero singular ideal. By [9, Theorem 24], $J = 0$. Then by [11, Theorem 19], R is Boolean. Converse is trivial.
(2) $\iff$ (4): Since R is reduced by [9, Proposition 28], $Z(_RR) = 0$. This implies that $J = 0$. Then by [11, Theorem 19], R is Boolean. Converse is trivial.
(3) $\iff$ (4): Let $R[[x]]/(x^n)$ be morphic uniquely clean. Using [7, Theorem 9], R is unit regular. From this we get $J = 0$. Since $R[[x]]/(x^n)$ is uniquely clean, base ring R is uniquely clean. Then using [11, Theorem 19] ring R is Boolean. Conversely, when ring R is Boolean then it is uniquely clean by [11, Corollary 2]. From [11, Corollary 10], $R[[x]]$ is uniquely clean. Therefore $R[[x]]/(x^n)$ is uniquely clean. Since R is Boolean it is unit regular. Then by [7, Theorem 9], $R[[x]]/(x^n)$ is strongly morphic. Hence morphic. Thus, $R[[x]]/(x^n)$ is morphic uniquely clean. □

Theorem 3.2 *Let R be a morphic uniquely clean ring. If every one sided ideal is principal then R is a basic ring.*

Proof: Given that R is morphic uniquely clean ring with every one sided ideal is generated by a single element. Clearly R is P-morphic. Then by [10, Theorem 13], R is artinian ring. Moreover R is quasi-duo by [11, Proposition 23]. Then by [1, Proposition 12], the quotient of the ring R by its jacobson radical $J(R)$ can be expressed as a finite direct product of division rings. Using [10, Theorem 13] and [6, Proposition 25.10] R is basic. □

Proposition 3.3 *In morphic uniquely clean ring, for any $\alpha \in R$, $\alpha = u^{-1}\alpha u$ where $u \in U(R), e \in Id(R), u = \beta - e$ and $R\alpha = l(\beta)$, $R\beta = l(\alpha)$.*

Proof: Since R is morphic, for any $\alpha \in R$, $\exists\ \beta \in R$ such that $R\alpha = l(\beta), R\beta = l(\alpha)$. Let $\beta = u_1 + e_1$, for some $u_1 \in U(R), e_1 \in Id(R)$. Then $\alpha\beta = \beta\alpha = 0$ which implies that $\alpha u_1 = -\alpha e_1$ and $u_1\alpha = -e_1\alpha = -\alpha e_1$. This implies that $\alpha u_1 = u_1\alpha$. Therefore, $\alpha = u_1^{-1}\alpha u_1$. □

Proposition 3.4 *Consider a morphic uniquely clean ring R. Let $\alpha, \beta \in R$ satisfy $R\alpha = l(\beta)$ and $R\beta = l(\alpha)$. Then $\beta = \sum_{i=1}^{2} u_i e_i$ where $u_i \in U(R), e_i \in Id(R)$.*

Proof: Take that R is morphic uniquely clean and $\alpha, \beta \in R$ satisfy $R\alpha = l(\beta)$ and $R\beta = l(\alpha)$. Since ring is clean, $\alpha = e_1 + u_1, \beta = f_1 + v_1$, where $e_1, f_1 \in Id(R)$ and $u_1, v_1 \in U(R)$. Given that R is morphic, which implies $\alpha\beta = 0$. Then we have $e_1f_1 + e_1v_1 + u_1f_1 + u_1v_1 = 0$. This implies that $g + e_1v_1 + u_1f_1 + u_1v_1 = 0$ where $g = e_1f_1$. Then we have $u_1^{-1}g + u_1^{-1}v_1e_1 + \beta = 0$. Thus $\beta = -u_1^{-1}g - v_1'e_1$ where $v_1' = u_1^{-1}v_1 \in U(R)$. □

A ring R said to possess square stable range one provided that whenever $\alpha R + \beta R = R$ $\exists$ some $x \in U(R)$ such that $\alpha^2 + \beta x \in U(R)$ for [4].

Theorem 3.3 *If R is a morphic uniquely clean ring, then it possesses the property of having square stable range one.*

Proof: Consider a ring R which is both morphic and uniquely clean. Clearly R_R is morphic module. Each uniquely clean ring is necessarily a clean ring.. Since, by [8, Theorem 2.1], the class of clean rings is contained within the class of exchange rings we get, by [12, Theorem 50], R_R is cancellable. Therefore, by [13, Theorem 9], R possesses stable range one. R is a quasi-duo ring on both the left and right sides by [11, Proposition 23]. Then using [4, Theorem 2.3], R possesses the property of having square stable range one. □

In [9], authors asked that if R is morphic and jacobson radical $J = 0$ is R (unit) regular? In case R is morphic uniquely clean with jacobson radical $J = 0$ then it is unit regular. This is because from [11, Theorem 19] R is regular. Using [9, Proposition 5] it is unit regular.

4. Conclusion

We have shown that every morphic uniquely clean ring has square stable range one. Also we provide a characterizations of Boolean rings. In summary, the study of morphic uniquely clean rings highlights significant structural connections between morphic properties and unique clean decompositions. This investigation clarifies how the interplay between morphicity and uniqueness in clean representations shapes the algebraic behavior of these rings. Several promising avenues remain open for future research. Further, examining how morphic uniquely clean rings interact with common ring-theoretic constructions—like extensions, matrix rings, or factor rings—may yield new insights about the preservation or emergence of these properties.

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