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Rainbow Colouring of Certain Product Graphs

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ABSTRACT: Graphs play an essential role in computer science and network analysis, representing complex relationships among entities. A rainbow colouring of a connected graph is an edge colouring process in which every pair of vertices is linked by at least one path with distinct coloured edges. This study examines the rainbow colouring properties of the cyclic product graph $G = C(2n, 1)$, determining its rainbow connection number $rc(G)$ and strong rainbow connection number $src(G)$. Additional analyses are carried out for corona product graphs to establish their vertex and strong rainbow connection numbers.

Keywords: Rainbow edge colouring, rainbow connection number, cyclic product of graphs.

Contents

1. Introduction

In a connected graph a rainbow edge colouring will assigns colours to edges so that minimum one path is present between two vertices. The minimum number of colours required to achieve this property is known as rainbow connection number, denoted as $rc(G)$. Chartrand et al. [1] first proposed the idea of a graph's rainbow connection number, or $rc(G)$, in 2008. Li and Sun examined the rainbow connection number of line graphs and upper bounds for the same in [8] and [9]. In [5] and [6] Syafrizal et al. determined the $rc(G)$ of the fan graph, sun graph, gear graph, book graph, and cycle-chain graph. [7], [2], [4], [10] [13] contains a variety of results regarding the rainbow connection number. A book by Li and Sun in [8] provides an overview of the rainbow connection number. Srinivasa Rao and Murali found results on the $rc(G)$ of brick product graphs and modified brick product graphs in [3] and [11]. Determining $rc(G)$ is an NP-hard problem, but is highly relevant for applications like secure data transmission, where different colours represent distinct communication paths.

2. Preliminaries

2.1. Rainbow Vertex Coloring

A graph G is said to be vertex rainbow-connected if there is a path which connects each pair of distinct vertices, where all the internal vertices are uniquely coloured [12]. The minimum number of colours needed for this is the rainbow vertex connection number, $rvc(G)$. When all vertices along the path, including endpoints, have distinct colours, the minimum number of colours required is called the strong rainbow vertex connection number, $srvc(G)$.

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2.2. Brick Product and Cyclic Product Graphs

The brick product of graphs, first introduced by Alspach et.al.[11],inspired further studies into related structures. The cyclic product graph, denoted $C(2n,m)$,extends this idea and has been analysed for Hamiltonian laceability. This section focuses on determining $rc(G)$ and $src(G)$ for the cyclic product graph $C(2n,1)$.

3. Main Results

Theorem 3.1 *The relation between the rainbow connection and strong rainbow connection number of the cyclic product graph $G = C(2n,1)$ where $n \in z^+$ is*

$$rc(G) = src(G) = \begin{cases} 1,2 & \text{for} \quad \text{n} = 2,3 \\ n & \text{for} \quad \text{n} \geq 4 \end{cases}$$

Proof: Consider the cyclic product graph $G = C(2n,1)$ with $2n$ vertices and $3n$ edges. Let us check $rc(G)$ =1 for $n = 2$.
Let us consider $G = C(4,1)$

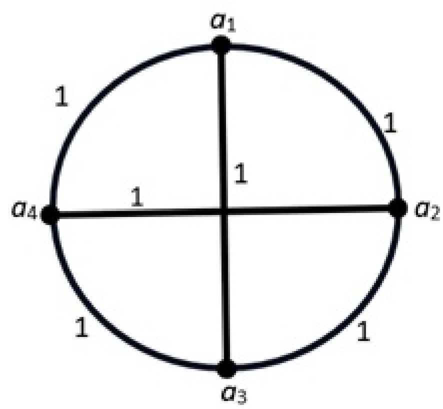

Figure 1: Graph $G = C(4,1)$ with $rc(G) = 1 = src(G)$

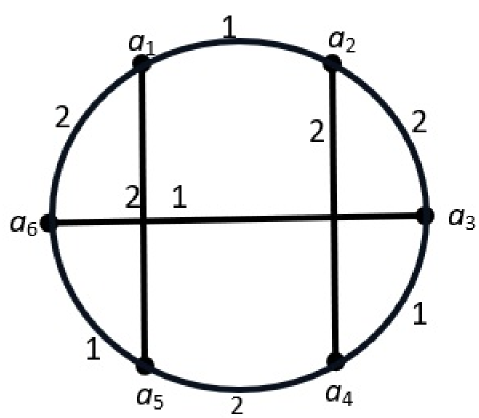

Figure 2: Graph $G = C(4,1)$ with $rc(G) = 2 = src(G)$

Between any two vertices (a_i, a_j), there exists at least one rainbow path.
Hence, $rc(C(2n,1)) = 1$, 2 for $n = 2$, 3. We now establish the result for $n \geq 4$.

Colouring Algorithm for G = C(2n,1)

1. Label the edges (a_i, a_{i+1}) by color i for $1 \leq i \leq n$
2. Label the edges (a_i, a_{i+1}) by color $i - n$ for $n + 1 \leq i \leq 2n$
3. Label the edges (a_j, a_{2n-k}) by color $j + 1$ for $1 \leq j \leq n - 1$ and $1 \leq k \leq n + 1$.
4. Finally, label the edge (a_n, a_{2n}) by colour 1.

Case Analysis

Case (i) In a graph G, consider any two distinct vertices a_i and a_j such that $|i-j| \leq n$, where $1 \leq i, j \leq 2n$ and $i \neq j$. Then the path $(a_i, a_{i+1}), (a_{i+2}, a_{i+3}), (a_{i+3}, a_{i+4}), \ldots, (a_{j-1}, a_j)$ is a rainbow path.

Case (ii) If $|i - j| > n$ and $n \leq i, j \leq 2n$ under modulo $2n$, then the path $(a_i, a_{i+1}), (a_{i+2}, a_{i+3}), (a_{i+3}, a_{i+4}), \ldots, (a_{j-1}, a_j)$ also forms a rainbow path.

Strong Rainbow Connection Number: For all vertices a_i and a_j, where $1 \leq i, j \leq 2n$ and $i \neq j$, each path between a_i and a_j forms the shortest rainbow path. Thus, $rc(C(2n, 1)) = src(C(2n, 1))$. □

Theorem 3.2 *The relation between the rainbow connection and strong rainbow connection number of the cyclic product graph $G = C(2n + 1, 1)$ where $n \in \mathbb{Z}^+$ is given by* $\text{rc}(C(2n + 1, 1)) = \text{src}(C(2n + 1, 1)) = \begin{cases} 1, 2, 3 & \text{for } n = 1, 2, 3 \\ n & \text{for } n \geq 4 \end{cases}$

Proof: Let the cyclic product graph $G = C(2n+1, 1)$ with $2n+1$ vertices and $3n+2$ edges. We establish $rc(G) = src(G) = 1$ for $n = 2, 3$. It is observed that between any two vertices, there exists a rainbow path in C(5,1) and C(7,1). Now for $n \geq 4$, labeling(colouring) algorithm as follows.

Algorithm for Coloring

Label the edges of the graph G as follows:

1. Label the edges of G (a_i, a_{i+1}) for $1 \leq i \leq n$ by the colour i.
2. Label the edges $(a_j, a_{(2n+1)-k}) = j + 1$ for $1 \leq j \leq n - 1$ and $1 \leq k \leq n + 1$ by the colour $j + 1$.
3. Finally, colour the edges (a_{2n+1}, a_n) and (a_{2n+1}, a_n) by the colour 1.

Case Analysis

Case (i) In G, consider any two distinct vertices a_i, a_j such that $|i - j| \leq n$ and $1 \leq i, j \leq 2n + 1$ with $i \neq j$. Then the path $(a_{i+2}, a_{i+3}), (a_{i+3}, a_{i+4}), \ldots, (a_{j-1}, a_j)$ is a rainbow path.

Case (ii) In G, consider any a_i, a_j such that $|i - j| > n$ under modulo $2n$, where $1 \leq i, j \leq 2n$ and $i \neq j$. Then the path $(a_{i+2}, a_{i+3}), (a_{i+3}, a_{i+4}), \ldots, (a_{j-1}, a_j)$ is a rainbow path.

□

Corona Product: Let $G = P_3$ be a path containing three vertices, and let $H = C_3$ be a cycle with three vertices. The corona product $G = P_3 \circ C_3$ is obtained by considering a copy of P_3 and three copies of C_3, where each vertex of P_3 is connected to every vertex in its corresponding copy of C_3.
It has been shown that the corona product of the rainbow vertex connection number of the $P_3 \circ C_3$ is $\text{rvc}(P_3 \circ C_3) = 6$, and the strong rainbow vertex connection number is $\text{src}(P_3 \circ C_3) = 3$.

Theorem 3.3 *The corona product $P_n \circ K_m)$ where $n \geq 2$ and $m \geq 1$: $rvcP_n \circ K_m) = m + n$ and $srvc(P_n \circ K_m) = n$.*

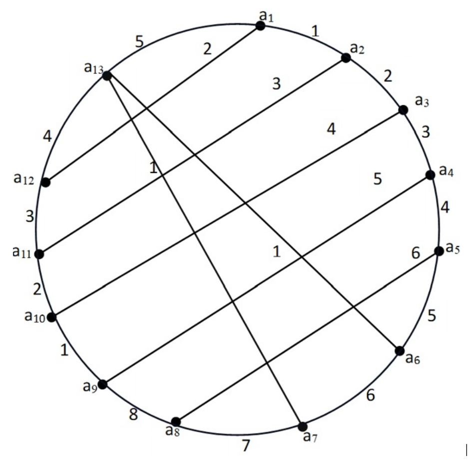

Figure 3: Graph $G = C(13, 1)$ with $rc(G) = 8$

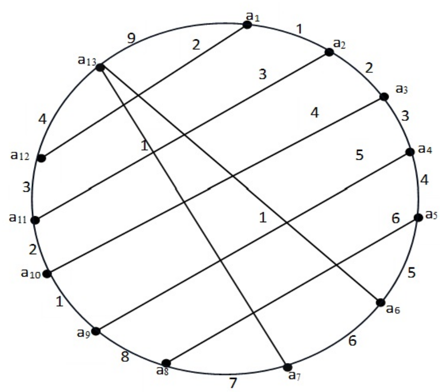

Figure 4: Graph $G = C(13, 1)$ with $src(G) = 9$

Proof: Consider any two adjacent vertices v_i and v_{i+1} in the path p_n. In the corona product, each is connected to its own complete graph K_m. To ensure rainbow connectivity between vertices in different K_m components, we need at least m colours for each complete graph, plus additional colours for the path vertices. Since adjacent path vertices cannot have the same colour and must be distinguishable from their K_m neighbourhoods, we require at least mn colours.
Using mn colours, we establish the following:

1. Color all vertices in each K_m component with colors 1, 2,...., m
2. Color path vertices alternately with colors $n + 1$ and n + 2, ..., $2n$

This colouring satisfies the rainbow vertex property. For strong rainbow vertex colouring, each couple of vertices must contain a rainbow path between them. Consider points in different K_m components. The path between them must traverse through the path structure, requiring distinct colours for all intermediate vertices. With n path vertices and nm vertices in total from K_m components, we need at least $2n$ colours to ensure all paths are rainbow. Assign distinct colours to all vertices: colours 1, 2, ..., m to the first K_m component, colors $m + 1, m + 2, ..., 2m$ to the second K_m component, and so on. This ensures all paths are rainbow, proving the result. □

Theorem 3.4 *For $m \geq 3, n \geq 2$, $rvc(P_n \circ C_m) = mn$ and $srvc(P_n \circ C_m) = n$.*

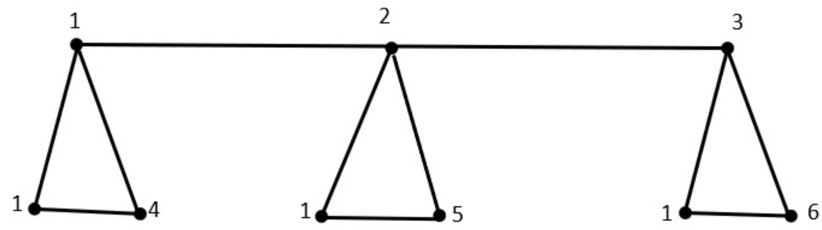

Figure 5: Graph $G = (P_3 \circ K_2)$

Proof: Consider vertices $v_1, v_2,$ and v_3 in different copies of K_3 attached to non-adjacent vertices of P_n. Any rainbow path between them must pass through the path, requiring at least $\lceil n/2 \rceil$ colours for the path, additional colours for the K_3 vertices. We construct a colour labelling using n colours. Label the path vertices with colours $1, 2, 3, \ldots, n$. Colour all vertices in the first copy of each K_3 with colour 1, and all remaining vertices in the $n+1, n+2, \ldots$ This ensures rainbow connectivity. $\text{rvc}(P_3 \circ C_4) = 12$ and $\text{srvc}(P_3 \circ C_4) = 3$. $\text{rvc}(P_3 \circ C_5) = 15$ and $\text{srvc}(P_3 \circ C_5) = 3$. $rvc(P_3 \circ C_6) = 18$ and $srvc(P_3 \circ C_6) = 3$. Therefore, $\text{rvc}(P_3 \circ C_n) = 3n$ and $\text{srvc}(P_3 \circ C_n) = 3$.
We note the following. For the corona product $P_n \circ K_3$, $\text{rvc}(P_n \circ K_3) = 3n$ and $\text{rvc}(P_n \circ K_m) = mn$.

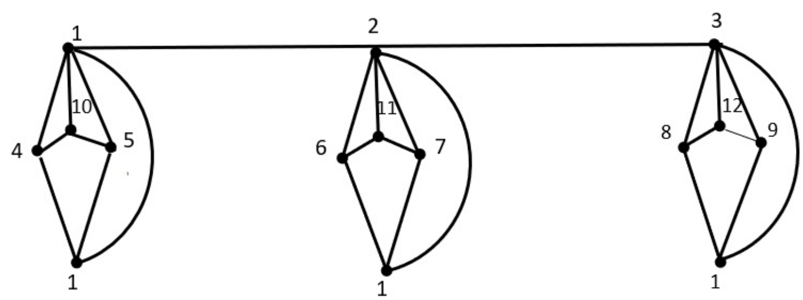

Figure 6: Graph $rvc(P_3 \circ C_4) = 12$ and $srvc(P_3 \circ C_4) = 3$.

For the corona product $P_3 \circ C_n$ with $n \geq 3$, rvc $P_3 \circ C_n = 3n$ and $P_3 \circ C_n = 3$. The expression is $\text{rvc}(P_n \circ C_n) = n^2$ and $\text{srvc}(P_n \circ C_n) = n$. For the corona product of $P_n \circ W_m$, $\text{rvc}(P_n \circ w_m) = mn$ $\text{srvc}(P_n \circ w_m) = n$. □

4. Conclusion

This concept is useful for network architecture and dependability, where $rc(G)$ represents the least number of colors needed to guarantee that a connection among any two nodes may still be made even if some edges are lost. In this work we have established significant results on rc and src of cyclic product graphs and rvc and $srvc$ of corona product graphs.

References

1. Chartrand, G., Johns, G. L. McKeon, K. A.,Zhang, P., *Rainbow connection in graphs.* Math. Bohem., 133(1), 85–98,(2008).

2. Arputhamary, M. A., Mercy, M. H., *Strong rainbow edge colouring of some interconnection networks*, Procedia Comput. Sci., 57, 338-347,(2015).

3. Girisha, A., Murali, R., *Hamiltonian laceability in the cyclic product and brick product of cycles*, J. Graph Theory, 1(1), 32–40, (2013).

4. Chakraborty, S., Fischer, E., Matsliah, A., Yuster, R., *Hardness and algorithms for rainbow connection*, J. Comb. Optim., 21(3), 330–347, (2011).

5. Faudree, R. J., Gyárfás, A., Lesniak, L.,Schelp, R. H., *Rainbow colouring the cube.* J. Graph Theory, 17(5), 607–612,(1993).

6. Ramya, N., Rangarajan, K., Sattanathan, R., *On rainbow colouring of some classes of graphs*, Int. J. Comput. Appl., 46(18), 36–38,(2012).

7. Arputhamary, A. D. *Rainbow colouring of certain classes of graphs*. International Journal of Advanced Information Science and Technology, 3(11), 53–56,(2014).

8. Li, X., Liu, S., Chandran, L. S., Mathew, R., Rajendra prasad, D. *Rainbow connection number and connectivity*, Electron. J. Comb., 19(1), 1–13, (2012).

9. Xueliang Li, Yuefang, Sun Yan Zhao, *Characterization of graphs with rainbow connection number m-2 and rainbow connection numbers of some graph operations*. Australas. J. Comb., 60(3), 306–313,(2014).

10. Li, X., Shi, Y., Sun, Y., *Rainbow connection of graphs: A survey*. Graphs Combin., 29(1), 1–38,(2013).

11. Alspach, B., Chen, C. C., and McAvaney, K, *On a class of Hamiltonian laceable 3-regular graphs*,Discrete Math., 151(1–3), 19–38,(1996).

12. M. Krivelevich, R. Yuster, *The Rainbow Connection of a Graph is (at most) Reciprocal to its Minimum Degree* J. Graph Theory, 63,185-191,(2010).

13. Chayapa Darayona, Wipawee Tangjaib, *Rainbow vertex-connection number on a small-world Farey graph*, AKCE Int. J. Graphs Comb., 19(1),54–60,(2022).

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