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Combinatorial Interpretations of Somos's Dedekind η-Function Identities

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ABSTRACT: Michael Somos used computer experimentation via the PARI/GP system to discover a large number of conjectural identities of the η-function type. He identified around 6200 such identities of varying levels. He did not provide rigorous proofs for them and they remained conjectural from the standpoint of the publication of his list. Among these he discovered nearly 15 Dedekind η-function identities. In the present work, we interpret them combinatorially by showing that they arise as generating functions for suitable colored partitions with suitable examples.

Key Words: Congruences, Dedekind η-function, partitions, q-identities, Theta functions.

Contents

1. Introduction

A positive integer n is said to have l colors if there are l distinct copies of n, each corresponding to a different color and treated as a separate object. Partitions of a positive integer in which parts may appear in different colors are called colored partitions. For instance, if the integer 1 is allowed to have two colors, then the colored partitions of 2 are $2, 1_r + 1_r, 1_g + 1_g$, and $1_r + 1_g$, where the subscripts r (red) and g (green) indicate the two available colors of 1. An important observation is that $(q^a; q^b)_\infty^{-k}$ is the generating function for the number of partitions of n, where all the parts are congruent to $a \pmod{b}$ having k colors.

For $|ab| < 1$, Ramanujan's theta function $f(a, b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

From Jacobi's triple product identity, it follows that

$$f(a,b) := (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

Ramanujan defined the following special case of $f(a, b)$ [4, p. 36]:

$$f(-q) \quad := \quad f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty,$$

where here and throughout the paper, we utilize the following definition:

$$(x;q)_\infty = \prod_{k=0}^{\infty} (1 - xq^k) \qquad |q| < 1.$$

Note that, if $q = e^{2\pi i \tau}$ then $f(-q) = e^{-\pi i \tau/12} \eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind η-function for $Im(\tau) > 0$. A theta function identity which relates $f(-q)$ to $f(-q^n)$ is called theta function identity

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of level n. Ramanujan recorded several identities which involve $f(-q), f(-q^2), f(-q^n)$ and $f(-q^{2n})$ in his second notebook [7] and 'Lost' Notebook [8]. For example [5, p.206],

$$f_1^4 f_2^4 f_5^2 f_{10}^2 + 5 f_1^2 f_2^2 f_5^4 f_{10}^4 = f_2^6 f_5^6 + f_1^6 f_{10}^6.$$

where $f_n = f(-q^n)$. After the publication of [5], several authors including Baruah [2,3], and Vasuki [17] have found many modular equations of the above type. In recent years, Michael Somos [9] has made a remarkable computational contribution to the study of modular forms and theta functions. Using the powerful number-theoretic capabilities of PARI/GP, a computer algebra system designed for fast computations in number theory, Somos generated nearly 6,200 distinct theta function identities. These identities span over various modular levels and involve complex combinations of theta functions, Dedekind eta-functions, and other modular expressions. Especially , these identities were found experimentally-through algorithmic exploration and pattern recognition-without accompanying formal proofs. his methodology exemplifies the expanding role of computational experimentation in mathematics, revealing intricate structures that challenge traditional proof techniques. Yuttanan [19] has rigorously proved several of Somos's theta identities, especially at levels 4 and 8, which relate to modular transformations and classical modular forms. His work not only confirms Somos's findings but also uncovers new partition identities derived from the q-series expansions of theta functions, offering deep combinatorial insights. Vasuki and Veeresha [18] have established proofs for all 24 level 14 theta function identities identified by Somos. Furthermore Srivatsa Kumar and Veeresha [10] have obtained partition identities for these theta-function identities. Srivatsa Kumar and his team have developed a proof of Somos identities of various levels and one can refer [10,11,12,13,14,15,16]. The concept of colored partitions was first introduced by Agarwal and Andrews [1]. Further, Huang [6] continued this work on establishing the modular relations between Göllnitz-Gordan functions. Inspired by the aforementioned contributions, this paper focuses on the validation of Michael Somos's theta function identities of level 12 by employing the framework of colored partitions. Specifically, we explore how these identities originally discovered through computational experimentation can be interpreted and verified using combinatorial techniques rooted in partition theory.

We conclude this section by presenting all fifteen theta function identities of level 12 that form the foundation of our analysis. In the following section, we translate these analytic expressions into partition-theoretic language, establishing new identities that count colored partitions under specific constraints. This approach not only validates the modular identities but also reveals deeper connections between theta functions and partition theory.

$$f_2^{20} f_3^8 f_{12}^8 + 8 f_1^9 f_2^2 f_3^5 f_4^9 f_6^6 f_{12}^5 - 9 f_1^8 f_4^8 f_6^{20} = 0, \tag{1.1}$$
$$f_1^6 f_4^3 f_6^3 f_{12} + f_2^9 f_3^2 f_{12}^2 - 2 f_1^3 f_3 f_4^6 f_6^3 = 0, \tag{1.2}$$
$$f_1^2 f_4^2 f_6^9 + 2q f_1 f_2^3 f_3^3 f_{12}^6 - f_2^3 f_3^6 f_4 f_{12}^3 = 0, \tag{1.3}$$
$$f_1^4 f_6^2 f_{12} + 3 f_2^2 f_3^4 f_{12} - 4 f_1 f_3 f_4^3 f_6^2 = 0, \tag{1.4}$$
$$f_2^{10} f_3^4 f_{12}^4 + 2 f_1^6 f_2 f_3^2 f_4^3 f_6^3 f_{12}^3 - 3 f_1^4 f_4^4 f_6^{10} = 0, \tag{1.5}$$
$$f_1^8 f_4^8 f_6^{20} + 8q f_1^5 f_2^6 f_3^9 f_4^5 f_6^2 f_{12}^9 - f_2^{20} f_3^8 f_{12}^8 = 0, \tag{1.6}$$
$$f_1^4 f_3^4 f_4^2 f_{12}^2 + 4q f_1^2 f_3^2 f_4^4 f_{12}^4 - f_2^6 f_6^6 = 0, \tag{1.7}$$
$$f_1^3 f_4^6 f_6^3 + 3q f_1^3 f_2^2 f_4^2 f_6 f_{12}^4 - f_2^9 f_3 f_{12}^2 = 0, \tag{1.8}$$
$$f_1 f_4^4 f_6^2 + q f_1 f_2^2 f_{12}^4 + f_2^2 f_3^3 f_4 f_{12} = 0, \tag{1.9}$$
$$f_1^3 f_4 f_6^2 f_{12} + 3q f_2^2 f_3 f_{12}^4 - f_3 f_4^4 f_6^2 = 0, \tag{1.10}$$
$$f_1^6 f_4^3 f_6^3 + 2 f_2^9 f_3^2 f_{12} - 3 f_1^2 f_2^2 f_3^4 f_4^3 f_6 = 0, \tag{1.11}$$
$$f_1^4 f_2 f_3^2 f_6^2 f_{12}^3 + f_2^3 f_3^6 f_{12}^3 - 2 f_1^2 f_4 f_6^9 = 0, \tag{1.12}$$
$$f_1^4 f_4^4 f_6^{10} + 4 f_1^3 f_2^3 f_3^3 f_4^2 f_6 f_{12}^4 - f_2^{10} f_3^4 f_{12}^4 = 0, \tag{1.13}$$
$$f_1^4 f_4^4 f_6^{10} + f_2^{10} f_3^4 f_{12}^4 - 2 f_1^2 f_2^3 f_3^6 f_4^3 f_6 f_{12}^3 = 0, \tag{1.14}$$
$$f_2^{10} f_3^4 f_{12}^4 + 3 f_1^4 f_4^4 f_6^{10} - 4 f_1^3 f_2 f_3^3 f_4^6 f_6^3 f_{12}^2 = 0. \tag{1.15}$$

2. Somos's η Identities: A Combinatorial View

In this section, we give a combinatorial demonstration for the Somos's identities of level 12. In sequel, for simplicity, we adopt the notation

$$(a_1, a_2, ..., a_n; q)_\infty = \prod_{j=1}^{n} (a_j; q)_\infty,$$

and define,

$$(q^{\pm r}; q^s)_\infty = (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integers and $r < s$. For example, $(q^{2\pm}; q^8)_\infty$ means $(q^2, q^6; q^8)_\infty$ which is $(q^2; q^8)_\infty$ $(q^6; q^8)_\infty$.

Theorem 2.1 *Let $\alpha(n)$ denote the total number of segments congruent to $\pm1, \pm5(mod\ 12)$ with 9 colors and $\pm3(mod\ 12)$ with 6 colors. Let $\beta(n)$ represent the total number of parts congruent to $\pm2, \pm4, +6(mod\ 12)$ with 9, 5 and 12 colors respectively. Let $\gamma(n)$ denote the total number of parts congruent to $\pm1, \pm5(mod\ 12)$ with 1 color each and $\pm2, \pm3, \pm4, +5(mod\ 12)$ with 12, 6, 9 and 20 colors respectiverly. Then, we have*

$$\alpha(n) + 8\beta(n) - 9\gamma(n) = 0, \qquad n \geq 0.$$

Proof: On rewriting (1.1) subject to the common base q^{12}, we have

$$\frac{1}{(q_9^{\pm1}, q_6^{\pm3}, q_9^{\pm5}; q^{12})_\infty} + \frac{8}{(q_9^{\pm2}, q_5^{\pm4}, q_{12}^{+6}; q^{12})_\infty} - \frac{9}{(q_1^{\pm1}, q_{12}^{\pm2}, q_6^{\pm3}, q_9^{\pm4}, q_1^{\pm5}, q_{20}^{+6}; q^{12})_\infty} = 0.$$

The quotients in the preceding identity can be interpreted as the respective generating functions $\alpha(n), \beta(n)$ and $\gamma(n)$. Accordingly, one may rewrite the above identity in the equivalent form

$$\sum_{n=0}^{\infty} \alpha(n)q^n + 8\sum_{n=0}^{\infty} \beta(n)q^n - 9\sum_{n=0}^{\infty} \gamma(n)q^n = 0,$$

where we set $\alpha(0) = \beta(0) = \gamma(0) = 1$. Upon equating the coefficients of q^n on both sides of the above equation, we are lead to the desired result. □

Example: The table below illustrates the verification of the case for $n = 2$ in the Theorem 2.1.

$\alpha(2) = 45:$	$1_y + 1_y, 1_o + 1_o, 1_r + 1_r,$ and 6 colors of the similar type, $1_y + 1 + o, 1_y + 1_r, 1_y + 1_b, 1_y + 1_{br}$ and 32 more colors of the similar type.
$\beta(2) = 9:$	$2_w, 2_y,$ and 7 more colors of the similar type.
$\gamma(2) = 13:$	$1 + r + 1_r, 2_w, 2_y,$ and 10 more colors of the similar type.

Theorem 2.2 *Let $\alpha(n)$ represent the total number of segments congruent to $\pm2, +6(mod\ 12)$ with 3 and 5 colors respectively. Let $\beta(n)$ denote the total number of parts congruent to $\pm1, \pm5(mod\ 12)$ with 6 colors each and $\pm3(mod\ 12)$ with 4 colors. Let $\gamma(n)$ represent the total number of parts congruent to $\pm1, \pm5(mod\ 12)$ with 3 colors each, and $\pm2, \pm3, \pm5(mod\ 12)$ with 6, 2 and 3 colors respectively. Then, we have*

$$\alpha(n) + \beta(n) - 2\gamma(n) = 0, \qquad n \geq 0.$$

Proof: On rewriting (1.2) subject to the common base q^{12}, we have

$$\frac{1}{(q_3^{\pm2}, q_5^{+6}; q^{12})_\infty} + \frac{1}{(q_6^{\pm1}, q_4^{\pm3}, q_6^{\pm5}; q^{12})_\infty} - \frac{2}{(q_3^{\pm1}, q_6^{\pm2}, q_2^{\pm3}, q_3^{\pm5}, q_7^{+6}; q^{12})_\infty} = 0.$$

The quotients in the above identity represent the generating functions $\alpha(n), \beta(n)$ and $\gamma(n)$ respectively. Thus the previous identity is equivalent to

$$\sum_{n=0}^{\infty} \alpha(n)q^n + \sum_{n=0}^{\infty} \beta(n)q^n - 2\sum_{n=0}^{\infty} \gamma(n)q^n = 0,$$

where we set $\alpha(0) = \beta(0) = \gamma(0) = 1$. Upon equating the coefficients of q^n on both sides of the above equation, we lead to the desired result.

□

Example: The following table confirms the case for $n = 2$ in the Theorem 2.2.

$\alpha(2) = 3:$	$2_r, 2_b, 2_m$
$\beta(2) = 21:$	$1_r + 1_r, 1_g + 1_g, 1_y + 1_y, 1_b + 1_b, 1_o + 1_0, 1_l + 1_l, 1_r + 1_g, 1_r + 1_y, 1_r + 1_b$ and 12 more colors of the similar type.
$\gamma(2) = 12:$	$1_r + 1_r, 1_g + 1_g, 1_y + 1_y, 1_r + 1_g, 1_r + 1_y, 1_g + 1_y, 2_r, 2_g, 2_y, 2_m, 2_b, 2_v.$

Theorem 2.3 *Let $\alpha(n)$ represent the total number of segments congruent to $\pm 2, \pm 3(mod\ 12)$ with 2 and 4 colors respectively. Let $\beta(n)$ denote the total number of parts congruent to $\pm 1, \pm 5(mod\ 12)$ with 1 color each and $\pm 3, +6(mod\ 12)$ with 2 and 12 colors respectively. Let $\gamma(n)$ represent the total number of parts congruent to $\pm 1, \pm 5, +6(mod\ 12)$ with 2 colors each, and $\pm 2(mod\ 12)$ with 1 color. Then we have*

$$\alpha(n) + 2\beta(n-1) - \gamma(n) = 0, \qquad n \geq 1.$$

Proof: On rewriting (1.3) subject to the common base q^{12}, we have

$$\frac{1}{(q_2^{\pm 2}, q_4^{\pm 3}; q^{12})_\infty} + \frac{2q}{(q_1^{\pm 1}, q_2^{\pm 3}, q_1^{\pm 5}, q_{12}^{+6}; q^{12})_\infty} - \frac{1}{(q_2^{\pm 1}, q_1^{\pm 2}, q_2^{\pm 5}, q_2^{+6}; q^{12})_\infty} = 0.$$

The quotients in the given identity are seen to correspond to the relevant generating functions $\alpha(n), \beta(n)$ and $\gamma(n)$ respectively. Hence we obtain

$$\sum_{n=0}^{\infty} \alpha(n)q^n + 2\sum_{n=0}^{\infty} \beta(n)q^{n+1} - 2\sum_{n=0}^{\infty} \gamma(n)q^n = 0,$$

where we set $\alpha(0) = \beta(0) = \gamma(0) = 1$. Upon equating the coefficients of q^n on both sides of the above equation, we lead to the desired result.

□

Example: To substantiate Theorem 2.3, the following table presents the verification for $n = 2$

$\alpha(2) = 2:$	$2_r, 2_g$
$\beta(1) = 1:$	1
$\gamma(2) = 4:$	$1_r + 1_r, 1_y + 1_y, 1_r + 1_y, 2.$

Theorem 2.4 *Let $\alpha(n)$ represent the total number of segments congruent to $\pm 1, \pm 5(mod\ 12)$ with 4 colors each, $\pm 2, \pm 4(mod\ 12)$ with 2 colors each and $\pm 3, +6(mod\ 12)$ with 1 color each. Let $\beta(n)$ denote the total number of parts congruent to $\pm 1, \pm 2, \pm 3, \pm 5, +6(mod\ 12)$ with 3 colors each. Then we have*

$$3\alpha(n) - 4\beta(n) = 0, \qquad n \geq 0.$$

Proof: On rewriting (1.4) subject to the common base q^{12}, we have

$$1+\frac{3}{(q_4^{\pm1},q_2^{\pm2},q_1^{\pm3},q_2^{\pm4},q_4^{\pm5},q_1^{+6};q^6)_\infty}-\frac{4}{(q_3^{\pm1},q_3^{\pm2},q_3^{\pm3},q_3^{\pm5},q_3^{+6};q^{12})_\infty}=0.$$

We observe that each quotient in the preceding identity corresponds respectively to the generating functions $\alpha(n)$ and $\beta(n)$ respectively. Hence we obtain

$$1+3\sum_{n=0}^{\infty}\alpha(n)q^n-4\sum_{n=0}^{\infty}\beta(n)q^n=0,$$

where we set $\alpha(0)=\beta(0)=1$. Upon equating the coefficients of q^n on both sides of the above equation, we lead to the desired result. □

Example: The subsequent table demonstrates the validity of Theorem 2.4 for $n=2$.

$\alpha(2)=12:$	$1_r+1_r, 1_y+1_y, 1_b+1_b, 1_g+1_g, 1_r+1_y, 1_r+1_b, 1_r+1_g, 1_y+1_b,$ $1_y+1_g, 1_b+1_g, 2_r, 2_y$
$\beta(1)=9:$	$1_r+1_r, 1_y+1_y, 1_b+1_b, 1_y+1_b, 1_y+1_r, 1_b+1_r, 2_y, 2_b, 2_r.$

Theorem 2.5 *Let $\alpha(n)$ denote the total number of segments congruent to $\pm1,\pm5(mod\ 12)$ with 6 colors and $\pm3,\pm4(mod\ 12)$ with 4 colors, and $+6(mod\ 12)$ with 14 colors. Let $\beta(n)$ represent the total number of parts congruent to $\pm2,+6(mod\ 12)$ with 3 and 4 colors respectively. Let $\gamma(n)$ denote the total number of parts congruent to $\pm1,\pm4\pm5(mod\ 12)$ with 2 color each and $\pm2,\pm3(mod\ 12)$ with 6 and 4 colors respectively. Then we have*

$$\alpha(n)+2\beta(n)-3\gamma(n)=0,\qquad n\geq 0.$$

Proof: On rewriting (1.5) subject to the common base q^{12}, we have

$$\frac{1}{(q_6^{\pm1},q_4^{\pm3},q_4^{\pm4},q_6^{\pm5},q_{14}^{+6};q^{12})_\infty}+\frac{2}{(q_3^{\pm2},q_4^{+6};q^{12})\infty}-\frac{3}{(q_2^{\pm1},q_6^{\pm2},q_4^{\pm3},q_2^{\pm4},q_2^{\pm5};q^{12})_\infty}=0.$$

It is observed that the quotients in the above identity correspond to the generating functions $\alpha(n),\beta(n)$ and $\gamma(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty}\alpha(n)q^n+2\sum_{n=0}^{\infty}\beta(n)q^n-3\sum_{n=0}^{\infty}\gamma(n)q^n=0,$$

where we set $\alpha(0)=\beta(0)=\gamma(0)=1$. Upon equating the coefficients of q^n on both sides of the above equation, we are lead to the desired result.

□

Example: The table below illustrates the verification of the case for $n=2$ in the Theorem 2.5.

$\alpha(2)=21:$	$1_r+1_r, 1_y+1_y, 1_b+1_b, 1_g+1_g, 1_w+1_w, 1_p, 1_p,$ $1_r+1_y, 1_r+1_b, 1_r+1_g, 1_r+1_w, 1_r+1_p,$ $1_y+1_b, 1_y+1_g1_y+1_w, 1_y+1_p, 1_b+1_g, 1_b+1_w, 1_b+1_p,$ $1_g+1_w, 1_g+1_p, 1_w+1_p.$
$\beta(2)=3:$	$2_r, 2_y, 2_b.$
$\gamma(2)=9:$	$1_r+1_r, 1_y+1_y, 1_r+1_y, 2_r, 2_y, 2_b, 2_g, 2_w, 2_p.$

Theorem 2.6 *Let $\alpha(n)$ denote the total number of segments congruent to $\pm2,\pm3,\pm4(mod\ 12)$ with 12, 3 and 4 colors respectively. Let $\beta(n)$ represent the total number of parts congruent to $\pm1,\pm5(mod\ 12)$ with 3 colors each and $\pm2,+6(mod\ 12)$ with 9 colors, and $\pm4(mod\ 12)$ with 1 color. Let $\gamma(n)$ denote the total number of parts congruent to $\pm1,\pm5(mod\ 12)$ with 8 colors each and $\pm3(mod\ 12)$ with 3 colors. Then we have*

$$\alpha(n)+8\beta(n-1)+r(n)=0,\qquad n\geq 1.$$

Proof: On rewriting (1.6) subject to the common base q^{12}, we have

$$\frac{1}{(q_{12}^{\pm 2}, q_3^{\pm 3}, q_4^{\pm 4}; q^{12})_\infty} + \frac{8q}{(q_3^{\pm 1}, q_9^{\pm 2}, q_1^{\pm 4}, q_3^{\pm 5}, q_9^{+6}; q^{12})_\infty} - \frac{1}{(q_8^{\pm 1}, q_3^{\pm 3}, q_8^{\pm 5}; q^{12})_\infty} = 0.$$

It is observed that the quotients in the above identity correspond to the generating functions $\alpha(n), \beta(n)$ and $\gamma(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} \alpha(n) q^n + 8q \sum_{n=0}^{\infty} \beta(n) q^n - \sum_{n=0}^{\infty} \gamma(n) q^n = 0,$$

where we set $\alpha(0) = \beta(0) = \gamma(0) = 1$. Upon equating the coefficients of q^n on both sides of the above equation, we are lead to the desired result. □

Example: The table below illustrates the verification of the case for $n = 2$ in the Theorem 2.6.

$\alpha(2) = 12:$	$2_r, 2_o, 2_y, 2_b$ and 8 more colors of the similar type.
$\beta(1) = 3:$	$1_r, 1_y, 1_b$
$\gamma(2) = 32:$	$1_r + 1_r, 1_y + 1_y$, and 6 colors of the similar type,$1_r + 1 + y, 1_r + 1_b$ and 26 colors of the similar type.

Theorem 2.7 *Let $\alpha(n)$ denote the total number of segments congruent to $\pm 2 (mod\ 12)$ with 2 color, $+6 (mod\ 12)$ with 4 color. Let $\beta(n)$ represent the total number of parts congruent to $\pm 1, \pm 5, +6 (mod\ 12)$ with 2 colors each and $\pm 2, \pm 3 (mod\ 12)$ with 4 colors each. Let $\gamma(n)$ denote the total number of parts congruent to $\pm 1, \pm 5 (mod\ 12)$ with 4 colors each and $\pm 3 (mod\ 12)$ with 8 colors. Then we have*

$$\alpha(n) + 4\beta(n-1) - \gamma(n) = 0, \qquad n \geq 1.$$

Proof: On rewriting (1.7) subject to the common base q^{12}, we have

$$\frac{1}{(q_2^{\pm 2}, q_4^{+6}; q^{12})_\infty} + \frac{4q}{(q_2^{\pm 1}, q_4^{\pm 2}, q_4^{\pm 3}, q_2^{\pm 5}, q_2^{+6}; q^{12})_\infty} - \frac{1}{(q_4^{\pm 1}, q_8^{\pm 3}, q_4^{\pm 5}; q^{12})_\infty} = 0.$$

It is observed that the quotients in the above identity correspond to the generating functions $\alpha(n), \beta(n)$ and $\gamma(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} \alpha(n) q^n + 4q \sum_{n=0}^{\infty} \beta(n-1) q^{n+1} - \sum_{n=0}^{\infty} \gamma(n) q^n = 0,$$

where we set $\alpha(0) = \beta(0) = \gamma(0) = 1$. Upon equating the coefficients of q^n on both sides of the above equation, we are lead to the desired result. □

Example: The table below illustrates the verification of the case for $n = 2$ in the Theorem 2.7.

$\alpha(2) = 2:$	$2_r, 2_o$
$\beta(1) = 2:$	$1_r, 1_0$
$\gamma(2) = 10:$	$1_r + 1_r, 1_o + 1_o, 1_w + 1_w, 1_b + 1_b, 1_r + 1_o, 1_r + 1_w, 1_r + 1_b,$ $1_o + 1_w, 1_0 + 1_b, 1_w + 1_b$

Theorem 2.8 *Let $\alpha(n)$ denote the total number of segments congruent to $\pm 2 (mod\ 12)$ with 6 color, $+6 (mod\ 12)$ with 4 color. Let $\beta(n)$ represent the total number of parts congruent to $\pm 2 (mod\ 12)$ with 4 color and $\pm 4, +6 (mod\ 12)$ with 2 colors each. Let $\gamma(n)$ denote the total number of parts congruent to $\pm 1, \pm 5 (mod\ 12)$ with 3 colors each and $\pm 3 (mod\ 12)$ with 2 color. Then we have*

$$\alpha(n) + 3\beta(n-1) - \gamma(n) = 0, \qquad n \geq 1.$$

Proof: On rewriting (1.8) subject to the common base q^{12}, we have

$$\frac{1}{(q_6^{\pm 2}, q_4^{+6}; q^{12})_\infty} + \frac{3q}{(q_4^{\pm 2}, q_2^{\pm 4} q_2^{+6}; q^{12})_\infty} - \frac{1}{(q_3^{\pm 1}, q_2^{\pm 3}, q_3^{\pm 5}; q^{12})_\infty} = 0.$$

It is observed that the quotients in the above identity correspond to the generating functions $\alpha(n), \beta(n)$ and $\gamma(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} \alpha(n) q^n + 3q \sum_{n=0}^{\infty} \beta(n-1) q^{n+1} - \sum_{n=0}^{\infty} \gamma(n) q^n = 0,$$

where we set $\alpha(0) = \beta(0) = \gamma(0) = 1$. Upon equating the coefficients of q^n on both sides of the above equation, we are lead to the desired result. □

Example: The table below illustrates the verification of the case for $n = 2$ in the Theorem 2.8.

$\alpha(2) = 6:$	$2_r, 2_o, 2_b, 2_w, 2_v, 2_p$
$\beta(1) = 0:$	
$\gamma(2) = 6:$	$1_r + 1_r, 1_o + 1_o, 1_b + 1_b, 1_r + 1_o, 1_r + 1_b, 1_o + 1_b$

3. Conclusion

In the present investigation, we have established the validity of eight of Somos's level-15 identities by constructing explicit combinatorial interpretations. These interpretations not only provide a deeper insight into the algebraic structure underlying the Somos sequences but also illustrate the intrinsic connections between partition theory and recurrence relations. Furthermore, analogous combinatorial formulations can be systematically derived for the remaining identities by employing the framework of colored partitions. The detailed exploration of these additional cases, while following similar principles, is left to the interested reader.

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