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Power Domination on Semi-Strong Product of Graphs

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ABSTRACT: An electrical power system can be monitored efficiently by placing the measurement device called Phase Measurement Unit (PMU) in the power network which can be effectively done by identifying the locations where the devices have to be placed, giving rise to the power domination (PD) concept in graphs. A set of vertices $S \subseteq V$ that monitors every vertex in the graph $G = (V, E)$ according to the rules of power domination is called as the power dominating set (PD-set). The power domination number (PD-number) of a graph G denoted by $\gamma_p(G)$, is the minimum number of vertices that are required to power dominate the entire graph. In this paper, we investigate the bounds for the semi-strong product (SSP) of two general graphs in terms of the power domination number $\gamma_p(G)$. Also, we establish the exact bounds for certain graphs based on their orders and PD-numbers.

Key Words: Domination, power domination, product graphs, semi strong product, power domination number.

Contents

1. Introduction

Let G = (V, E) be an undirected, finite and loopless graph, with the vertex set $V = V(G)$, and the edge set $E = E(G)$. Given a vertex a of a graph G , $N(a)$ denotes the open neighbourhood of $a \in V(G)$ [2]. If $N[a]$ equals $N(a) \cup \{a\}$, then $N[a]$ is said to be the closed neighbourhood of the vertex a. For $S \subseteq V(G)$, the set $N(S) = \bigcup_{a \in S} N(a)$ is called its open neighbourhood and the set $N[S] = N(S) \cup S$ is called its closed neighbourhood. The problem of monitoring the electrical power systems was introduced by Baldwin [1], asks for minimum number of PMUs to be placed for adequate monitoring of the systems. This problem of minimizing the monitoring devices in a power system was proposed as a graph based approach by Haynes and Hedetenemi [9].

Graph products have been a topic of interest in graph theory as it has various application in real life such as finding paths or routes in networks and for determining the Shannon capacity of graphs [15]. Among various products, the cartesian, strong, direct and lexicographic products are explored very frequently in the literature [8,16]. Specifically [11] builds on the foundational work providing exact values for $\gamma(G)$ of small grids. The PD in cartesian product of two paths were presented by Dorfling and Henning in the paper [5]. Gravier et al. found the domination number of the direct product of paths [7]. Further the lower bound for the strong product of paths, the PD-number for the direct product of two graphs and the PD-number for an arbitrary lexicographic product was investigated by Dorbec et al. [4].

The concept of SSP, initially denoted as strong tensor product where Garman et al. [6] studied the fundamental properties and explored its behaviour within the framework of graph embeddings on orientable surface. Kevin J McCall, in his dissertation [14] explored the non topological properties such

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as distance, domination number, matching, weiner index, colouring and clique. The investigation of the basis number of the SSP for different types of graphs is presented in [12,13]. Two conjectures on the SSP of paths and cycles and the upper bound for the domination number of SSP of any two graphs were presented in [3]. An important property of the SSP is that, it is neither associative nor commutative unlike cartesian, direct and strong products. This was shown by taking an example of $(P_2 \bowtie P_2) \times P_2$ and $P_2 \times (P_2 \bowtie P_2)$ [6].The SSP is structurally sparser than the strong product and denser than the cartesian product.

The study of PD-number of SSP of graphs is primarily motivated by its potential applications in network monitoring and its contribution to the broader field of structural graph theory. Investigating the graph parameters for graph products is a central theme helping to understand the how the structure of the individual factors influences the structure of the resulting graph. In this paper we discuss, the general bounds and the PD-number of certain classes of graphs obtained by performing the operation SSP.

2. Preliminaries

Definition 2.1 A set $D \subseteq V$ is a dominating set if every vertex in V, is either an element of D or is adjacent to at least one element of D. The number of elements in the set with least cardinality among all such dominating sets is known to be the domination number and is denoted by $\gamma(G)$ [10].

Definition 2.2 Given a graph G, a set $S \subseteq V$ is called a PD-set if every vertex in the graph is monitored by the below stated rules:
i) Domination step: All vertices in G, except the non-neighbours of vertices in the S are monitored,
ii) PD step (or) Propagation step: If all the neighbours of a vertex $a \in S$, except for single vertex b, are monitored, then b becomes monitored by a. In this case we see that a propagates to b. The PD-number depicted as $\gamma_p(G)$, is the least cardinality among all such PD-sets. The PD-set with least cardinality is said to be γ_p-set.

Definition 2.3 If the set D is dominating and the subgraph induced by D is connected, then D is called a connected dominating set (CD-set) and the connected domination number of the graph G is denoted as $\gamma_c(G)$ is the set with minimum cardinality among all such CD-sets.

3. General Bounds on SSP of Graphs

Definition 3.1 The SSP of G and H denoted by $G \bowtie H$ is a graph where $V(G \bowtie H) = V(G) \times V(H)$ is the set of vertices which are ordered pairs, and the set of edges is defined as follows: $(u_1, v_1)(u_2, v_2) \in E(G \bowtie H)$ if $u_1 = u_2$ and $v_1 \sim v_2 \in H$ or $u_1 \sim u_2 \in G$ and $v_1 \sim v_2 \in H$.
Note that, $|V(G \bowtie H)| = |V(G)||V(H)|$ and $|E(G \bowtie H)| = 2|E(G)||E(H)| + |E(G)||V(H)|$. Let u be a vertex in G and v be a vertex in H. In the SSP of $G \bowtie H$, a set of vertices $V(G) \times \{v\}$ are called horizontal fiber and $\{u\} \times V(H)$ are called vertical fiber.

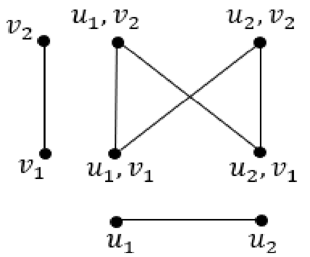

Figure 1: $G = K_2, H = K_2$

Example 3.1 We illustrate the construction of the operation SSP by considering the Figure 2. Let $G = P_4$ be a path with $m = 4$ vertices with vertex set $V(P_4) = \{u_1, u_2, u_3, u_4\}$ and $H = C_n$ with $n = 4$ vertices and the vertex set $V(C_4) = \{v_1, v_2, v_3, v_4\}$. Since $|V(G \bowtie H)| = |V(G)||V(H)|$, the total vertices in $P_4 \bowtie C_4$ is $V(P_4 \bowtie C_4) = 4 \times 4 = 16$. We aim to join the vertices according to the two rules.
Rule 1: Join the vertices if $u_1 = u_2$ and $v_1 \sim v_2 \in H$
Rule 2: Join the vertices if $u_1 \sim u_2 \in G$ and $v_1 \sim v_2 \in H$
Note that the Rule 1 facilitates $|V(G)|$ copies of H and the Rule 2 facilitates the formation of edges produced by direct product.

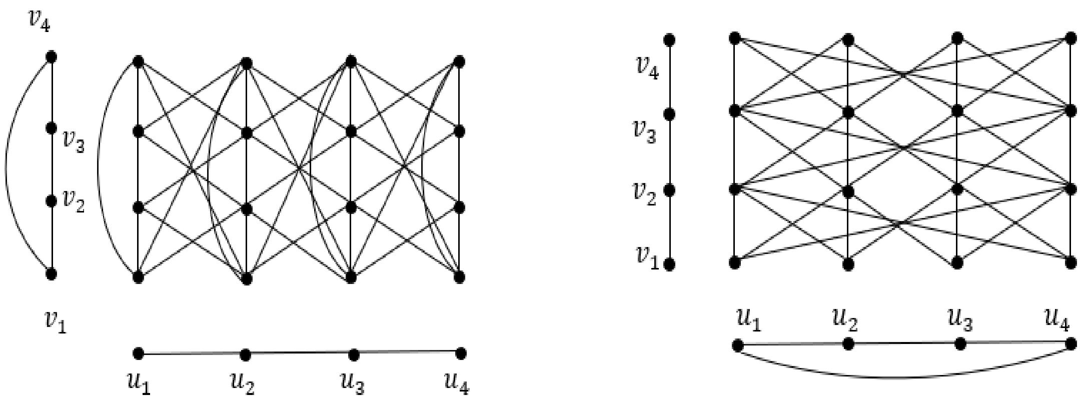

Figure 2: Construction of SSP for P_4 and C_4

Let us consider the vertex $(u_1, v_1) \in P_4 \bowtie C_4$, we see from the figure that the vertex (u_1, v_1) is joined with the vertices $(u_1, v_2), (u_1, v_4)$ according to the rule 1 and $(u_2, v_2), (u_2, v_4)$ by rule 2. Similarly we join other vertices in the graph. We also see that the SSP is not commutative.

Note: For any two graphs G and H,
$G \Box H$ denotes cartesian product, $G \boxtimes H$ denotes strong product, $G \times H$ denotes direct product. $E_{horizontal fibers}$ denotes the edges which forms copies of H over each vertex of G, that is for any two distinct vertices (u_1, v_1) and (u_2, v_2), the first components are equal $(u_1 = u_2)$ and second components are adjacent $v_1 v_2 \in E(H)$.

Theorem 3.1 *For any two graphs G and H, $\gamma_p(G \times H) \leq \gamma_p(G \bowtie H) \leq \gamma_p(G \boxtimes H)$*

Proof: Let $E(G \times H) = \{(u_1, v_1), (u_2, v_2) | u_1 \sim u_2 \text{ and } v_1 \sim v_2\}$ be the edge set of the direct product. The set of edges of SSP and strong product is defined as $E(G \bowtie H) = E_{horizontal fibers} \cup E(G \times H)$ and $E(G \boxtimes H) = E(G \Box H) \cup E(G \times H)$ respectively. As $E(G \times H) \subset E(G \bowtie H)$, any PD-set of $G \times H$ must be a PD-set of $G \bowtie H$. This brings out the fact that $\gamma_p(G \times H) \leq \gamma_p(G \bowtie H)$. Similarly, $E(G \bowtie H) \subset E(G \boxtimes H)$. Hence, any PD-set of $G \bowtie H$ is evidently a PD-set of $G \boxtimes H$, shows that $\gamma_p(G \bowtie H) \leq \gamma_p(G \boxtimes H)$. □

Remark 3.1 The above theorem depicts the relationship between the PD-numbers of direct product, SSP and strong product respectively.

Theorem 3.2 *Let G and H be two graphs with order m and n respectively. If $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$ then $\gamma_p(G \bowtie H) \leq \gamma(G)\gamma(H)$.*

Proof: Let the set of vertices of G be $V(G) = \{u_1, u_2, \ldots u_m\}$ and that of graph H be $V(H) = \{v_1, v_2, \ldots v_n\}$. By the definition of SSP, $G \bowtie H$ has the set of vertices to be $V(G \bowtie H) = V(G) \times V(H)$

and the set of edges as $E(G \bowtie H) = \{(u_i, v_i)(u_j, v_j) | u_i = u_j$ and $v_i \sim v_j \in H$ or $u_i \sim u_j \in G$ and $v_i \sim v_j \in H\}$. Consider the minimum dominating set of G as $D_G = \{u_1, u_2, \ldots, u_p : p < m\}$ and $\gamma(G) \neq 1$. Let the minimum dominating set of H be denoted as $D_H = \{v_1, v_2, \ldots, v_q : q < n\}$ and $\gamma(H) \neq 1$. We denote the PD-set of $G \bowtie H$ to be $D_G \times D_H = \{(u_a, v_b) : a = 1, 2, \cdots, p$ and $b = 1, 2, \cdots, q$ where $p < m, q < n\}$. To prove equality, consider that, suppose $D_G \times D_H$ is not a minimum PD-set, then there exists at least a set, say $(D_G \times D_H) - 1$, with cardinality less than $D_G \times D_H$ which is the γ_p-set of $G \bowtie H$. In this case, this set initially dominates vertices associated to it in accordance with the rules of SSP, but since these dominated vertices are adjacent to more than one vertex, it is not possible to proceed further to propagation step. This shows that the vertices in $(D_G \times D_H) - 1$ is not sufficient to power dominate the entire graph. Suppose if we select the γ_p-set to be $D_G \times D_H$, these vertices can dominate all the vertices except the vertices in the horizontal fiber corresponding to the vertex in $D_G \times D_H$. It is evident that the remaining vertices can be power dominated in the propagation step. Note that this is the minimum power dominating set for $G \bowtie H$. Thus $D_G \times D_H$ is the γ_p-set of $G \bowtie H$, implying $(D_G \times D_H) = \gamma(G)\gamma(H)$. The proof for inequality can be referred from Theorem 4.2. The equality holds if we consider $G = C_m$ and $H = C_n$. By combining the above arguments, we conclude that $\gamma_p(G \bowtie H) \leq \gamma(G)\gamma(H)$. □

Theorem 3.3 *If G is a graph with $m \geq 3$ vertices and if H is a graph with $n \geq 3$ vertices and $\gamma(H) = 1$ then $\gamma_p(G \bowtie H) \leq 2\gamma(G)$.*

Proof: Let G be a graph with the vertex set $V(G) = \{u_i : 1 \leq i \leq m$ where $m \geq 3\}$ and let H be any graph with $V(H) = \{v_j : 1 \leq j \leq n$ and $\gamma(H) = 1$. The γ- set of H contains only a vertex say $\{v_1\}$ as $\gamma(H) = 1$, denoted by $D_H = \{v_1\}$. Let the minimum dominating set of G be denoted by $D_G = \{u_1, u_2, \ldots, u_p : p < m\}$. According to the definition of SSP, $V(G) \times V(H) = V(G \bowtie H)$ forms the set of vertices and the set of edges be $E(G \bowtie H) = \{(u_i, v_i)(u_j, v_j) | u_i = u_j$ and $v_i \sim v_j \in H$ or $u_i \sim u_j \in G$ and $v_i \sim v_j \in H\}$. Note that $V(G \bowtie H) = mn$. Consider the minimum PD-set of $G \bowtie H$ to be $D_G \times D_H$. To prove the upper bounds, we consider the following cases:
Case(i): If $u_i \in V(G) - \gamma(G)$ is not adjacent to u_j:
Suppose that, the graph G in which a vertex $u_i \in V(G) - \gamma(G)$ is not adjacent to any other $u_j \in V(G) - \gamma(G)$. In this case, the vertices in $D_G \times D_H$ initially dominates every vertex in $G \bowtie H$ except for the vertices in the horizontal fiber corresponding to $v_1 \in H$. The rest of the unmonitored vertices are power dominated by the vertices in $D_G \times D_H$ in the power domination step. Thus $D_G \times D_H = \gamma(G)\gamma(H) = \gamma(G).1 < 2\gamma(G)$ vertices are sufficient.
Case(ii): $u_i \in V(G) - \gamma(G)$ is adjacent to u_j:
Suppose that, in graph G the vertices $v_i, v_j \in V(G) - \gamma(G)$ are adjacent, the vertices in $D_G \times D_H$ initially dominates every vertex in $G \bowtie H$ except the horizontal fiber corresponding to v_1. It is clear that the already dominated vertices cannot power dominate the remaining vertices. Thus we have to choose additional vertices to power dominate the entire graph. Therefore, $D_G \times D_H = \{\gamma(G)\gamma(H) + \gamma(G)\gamma(H)\} = 2\gamma(G)$. Hence we get $\gamma_p(G \bowtie H) \leq 2\gamma(G)$. In the following Figure 3, note that $\gamma(G_1) = 1, \gamma(H_1) = 1, \gamma(G_2) = 1, \gamma(H_2) = 1$ and $G_1 \bowtie H_1$ shows equality whereas $G_2 \bowtie H_2$ shows strict inequality. □

Theorem 3.4 *Let G be a graph with $\gamma(G) = 1$. Let H be a graph with $n > 3$ vertices and $\gamma(H) \neq 1$. If $\gamma(H)$-set is connected then $\gamma_p(G \bowtie H) = \gamma(H)$.*

Proof: Let the vertices of a graph G be $V(G) = \{u_1, u_2, \ldots u_m\}$ and let the vertices of graph H be $V(H) = \{v_1, v_2, \ldots v_n\}$. By the definition of SSP, $V(G \bowtie H) = V(G) \times V(H)$ and $E(G \bowtie H) = \{(u_i, v_i)(u_j, v_j) | u_i = u_j$ and $v_i \sim v_j \in H$ or $u_i \sim u_j \in G$ and $v_i \sim v_j \in H\}$ are the set of vertices and edges respectively. We also note that the considered graph G has only a vertex in its dominating set, say $D_G = \{u_1\}$ since $\gamma(G) = 1$. Let D_H be the minimum dominating set of H with vertices $D_H = \{v_1, v_2, \ldots, v_q : q < n\}$. Consider the PD-set of $G \bowtie H$ as $S = \{\{u_1\} \times \{v_1, v_2, \ldots, v_q : q < n\}\}$. The vertices in the set $S = \{(u_1, v_q) : 1 \leq q < n\}$ dominates initially all the vertices in $V(G \bowtie H)$ except the vertices in the corresponding horizontal fiber, which is power dominated in the propagation step. Note that, there is no set with elements less than $|S|$ can power dominate all the vertices in $G \bowtie H$.

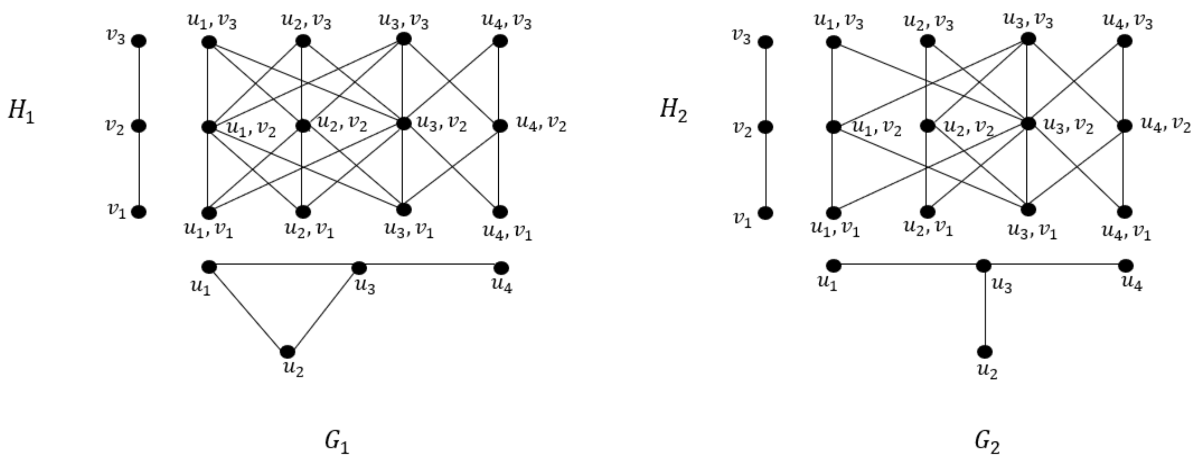

Figure 3: $\gamma_p(G_1 \bowtie H_1) = 2\gamma(G_1)$ and $\gamma_p(G_2 \bowtie H_2) = 1 < 2\gamma(G_2)$

Hence $S = D_G \times D_H = \{(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_q) : q < n\}$ is the required minimum PD-set. Thus $\gamma_p(G \bowtie H) = \gamma(H)$. □

Theorem 3.5 *Let G be a graph with $\gamma(G) = 1$ and H be a graph with $n \geq 3$ vertices and $\gamma(H) \neq 1$. If $\gamma(H)$-set is not connected then $\gamma_p(G \bowtie H) \leq \gamma_c(H)$.*

Proof: Let G be a graph with order $m \geq 2$ with vertex set $V(G) = \{u_1, u_2, \ldots u_m\}$ and $\gamma(G) = 1$. Let H be any graph of order $n \geq 3$ with vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$. We know that, $G \bowtie H$ has $V(G) \times V(H) = V(G \bowtie H)$ and $E(G \bowtie H) = \{(u_i, v_i)(u_j, v_j) | u_i = u_j$ and $v_i \sim v_j \in H$ or $u_i \sim u_j \in G$ and $v_i \sim v_j \in H\}$ as the set of vertices and edges respectively. Since $|\gamma(G)| = 1$, consider the vertex in the minimum dominating set denoted as $D_G = \{u_1\}$. Also the minimum dominating set of H to be $D_H = \{v_1, v_2, \ldots, v_q : q < n\}$ such that $\gamma(H)$ is not connected. To prove the equality, let us assume that $D_G \times D_H = \{u_1\} \times \{v_1, v_2, \ldots, v_q : q < n\} = \{(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_q) : q < n\}$. This set dominates all the vertices in $G \bowtie H$ except the vertices in the horizontal fiber corresponding to u_1 in the domination step. Suppose if $G = K_m$ or at least one $u_i \sim u_j : u_i, u_j \in V(G) - \gamma(G)$ then the remaining vertices cannot be power dominated by any of the dominated vertices as the degree of these vertices exceeds 1. Hence we add at least $\gamma_c(H)$ additional vertices in $D_G \times D_H$ to power dominate the entire graph. Thus $\gamma_p(G \bowtie H) \leq \gamma_c(H)$. Now for strict inequality, consider G such that no vertices of $u_i \in V(G) - \gamma(G)$ is adjacent to $u_j \in V(G) - \gamma(G)$. In this case, the vertices in $D_G \times D_H$ initially dominates all the vertices except the vertices in the horizontal fiber corresponding to u_1 in $G \bowtie H$. But since no $u_i \in V(G) - \gamma(G)$ is adjacent to $u_j \in V(G) - \gamma(G)$, it is clear that the remaining vertices can be power dominated by the already dominated vertices in $D_G \times D_H$. Hence, $\gamma_p(G \bowtie H) = D_G \times D_H = \gamma(H) < \gamma_c(H)$. This proves $\gamma_p(G \bowtie H) \leq \gamma_c(H)$. □

Corollary 3.1 *For any two graphs G and H, $\gamma(H) \leq \gamma_p(G \bowtie H) \leq \gamma_c(H)$.*

Theorem 3.6 *If G is a trivial graph (K_1) and H be any graph with order n, then $\gamma_p(G \bowtie H) = \gamma_p(H)$.*

Proof: Let G be a trivial graph with a vertex say $\{u_1\}$ and H be any graph of order $n \geq 2$ with vertex set $V(H) = \{v_1, v_2, \ldots v_n\}$. Since $K_1 \bowtie H \cong H$, we have $V(G)$ copies of H. Here $|V(G)| = 1$, therefore one copy of H is present in the construction of SSP. Hence $\gamma_p(H)$ vertices are sufficient to power dominate the entire graph. Thus, $\gamma_p(G \bowtie H) = \gamma_p(H)$. □

Remark 3.2 If $G = K_1$ and H be any graph with order $n \geq 2$, we have $\gamma(G \bowtie H) = \gamma(H)$.

Theorem 3.7 *Let G be any graph with order m and H be a trivial graph. Then, $\gamma_p(G \bowtie H) = m$.*

Proof: Let G be any graph with order $m \geq 1$ and $\{u_1, u_2, \cdots, u_m\}$ be the vertex set. Let H be a trivial graph with vertex say $\{v_1\}$. From the construction of SSP, we obtain $o(G) = m$ isolated vertices that is $G \bowtie K_1 \cong \overline{K}_{|V(H)|}$. Thus m vertices are required to dominate the entire graph. Hence, $\gamma_p(G \bowtie H) = m$. □

Remark 3.3 From the above stated theorem a similar proof can be derived for proving $\gamma(\ G \bowtie H) = m$.

4. Results on SSP of Certain Graph Classes

Theorem 4.1 *Let $G = K_m$ and $H = K_n$ be two complete graphs with $m \geq 1$ and $n \geq 2$ vertices respectively. Then*

$$\gamma_p(G \bowtie H) = \begin{cases} 2 & if\ m \geq 3 \\ 1 & if\ m = 1, 2. \end{cases}$$

Proof: Let G be a graph with $V(G) = \{u_1, u_2, \ldots, u_m\}$ and let H be a graph with vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$. For $m = 1$, according to the definition of SSP we have $G \bowtie H = H$. We know that $\gamma_p(K_n) = 1$. Therefore, $\gamma_p(G \bowtie H) = \gamma_p(H) = 1$.

If $m = 2$, then $D = \{u_1, v_1\}$ is a γ_p-set for $G \bowtie H$ because the vertices $V(G \bowtie H) - \{(u_2, v_1)\}$ are dominated initially, and the vertex (u_2, v_1) is dominated in the PD step (propagation step). So $\gamma_p(G \bowtie H) = \gamma_p(H) = 1$.

Now, we deal with $m \geq 3$.

Suppose, if we choose any one vertex $\{(u_i, v_1)\}$, where $1 \leq i \leq m - 1$ in $V(G \bowtie H)$ for the minimum power dominating set of $G \bowtie H$, then at the domination step, this vertex will dominate all the vertices in the graph except the vertices in horizontal fiber corresponding to the vertex $\{u_i, v_1\}$. Note that, the graph depicting the SSP of two complete graphs, has every vertex in G is adjacent to every other vertex in H except the vertices in the same horizontal fiber. Therefore, no vertices in the horizontal fiber corresponding to $\{u_i, v_1\}$ can power dominate in the propagation step. So, $\gamma_p(K_m \bowtie K_n) \geq 2$. Further, if we choose any one vertex from the same horizontal fiber $\{(u_{i+1}, v_1) : 1 \leq i \leq m - 1\}$ in addition, then the remaining vertices in $V(G \bowtie H) - N[(u_i, v_1)]$ are power dominated by chosen vertex. Therefore $D = \{(u_i, v_1), (u_{i+1}, v_1)\}$ is the minimum PD-set as the vertices in D can power dominate the entire graph. Hence, $\gamma_p(G \bowtie H) = 2$. □

Remark 4.1

If $n = 1$, in the above theorem, then the graph obtained by performing the semi strong product is collection of vertices which are isolated. Since, there are m-isolated vertices, $\gamma_p(K_m \bowtie K_n) = m$.

Theorem 4.2 *Let P_m and P_n be two graphs with $m \geq 2$ and $n = 2k + 1, k \geq 1$ vertices, respectively. Then, $\gamma_p(P_m \bowtie P_n) \leq \lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil - 1$.*

Proof: Let the vertices of a graph P_m be $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ and let the vertices of graph P_n be $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. By the definition of SSP $P_m \bowtie P_n$, the vertex set $V(P_m \bowtie P_n) = V(P_m) \times V(P_n)$ and the edge set $E(P_m \bowtie P_n) = \{(u_i, v_i)(u_j, v_j) | u_i = u_j$ and $v_i \sim v_j \in P_n$ or $u_i \sim u_j \in P_m$ and $v_i \sim v_j \in P_n\}$. Here note that $V(P_m \bowtie P_n) = mn$ and $E(P_m \bowtie P_n) = 2(m-1)(n-1) + m(n-1) = 3m(n-1) - 2(n-1)$. To construct the minimum PD-set for $(P_m \bowtie P_n)$, the below cases are under consideration:

Case (i): Let $m = 2$ and $n = 3$. In this case, $D = \{(u_2, v_2)\}$ is a γ_p-set for $P_m \bowtie P_n$ since the vertices $(u_1, v_1), (u_1, v_3), (u_2, v_1), (u_2, v_3)$ are dominated initially and the remaining vertex (u_1, v_2) of $P_2 \bowtie P_3$ is

power dominated by (u_2, v_2). Hence, $\gamma_p(P_2 \bowtie P_3) = 1$.
Case (ii): When $m \geq 3$ and $n = 3$, the set $D = \{(u_{3i-1}, v_2) : 1 \leq i \leq \lceil \frac{m}{3} \rceil\}$ is a γ_p-set for $P_m \bowtie P_3$ since the vertices $(u_{m-1}, v_1), (u_m, v_1), (u_{m+1}, v_1), (u_{m-1}, v_3), (u_m, v_3)$, (u_{m+1}, v_3) are dominated initially and the rest of the vertices of $P_m \bowtie P_3$ are power dominated by $\{(u_{3i-1}, v_2) : 1 \leq i \leq \lceil \frac{m}{3} \rceil\}$. Hence, $\gamma_p(P_m \bowtie P_3) = \lceil \frac{m}{3} \rceil$.
Now we deal with general case: When $m \geq 4$ and $n = 2k+1,\ k \geq 2$.
Consider the set $D = S_1 \cup S_2 \cup T$, where $S_1 = \{(u_2, v_{3j-1}) : 1 \leq j \leq \lfloor \frac{n-1}{3} \rfloor\}$,
$S_2 = \{(u_{3i-1}, v_{n-1}) : 1 \leq i \leq \lfloor \frac{m-1}{3} \rfloor\}$ and
$T = \Big\{(u_2, v_{n-1})|$ if $min\big\{d\big((u_2, v_{3j-1}), (u_2, v_n)\big), d\big((u_{3i-1}, v_{n-1}), (u_1, v_{n-1})\big)\big\} \geq 3\Big\}$. Here $T = \emptyset$, if $min\big\{d\big((u_2, v_{3j-1}), (u_2, v_n)\big), d\big((u_{3i-1}, v_{n-1}), (u_1, v_{n-1})\big)\big\} < 3$. By the definition of power domination, note that the vertices of $(P_m \bowtie P_n)$ are dominated and hence power dominated by the elements contained in the sets S_1, S_2 and T. Also note that, this is a minimum power dominating set and $\lfloor \frac{n-1}{3} \rfloor + \lfloor \frac{m-1}{3} \rfloor + 1 \leq \lceil \frac{n}{3} \rceil + \lceil \frac{m}{3} \rceil - 1$ vertices are required to power dominate all the vertices of $P_m \bowtie P_n$. Hence, $\gamma_p(P_m \bowtie P_n) = |D| \leq \lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil - 1$. □

Theorem 4.3 *Let P_m and P_n be two graphs with $m \geq 2$ and $n = 2k, k \geq 1$ vertices respectively. Then $\gamma_p(P_m \bowtie P_n) = \lceil \frac{n}{3} \rceil$.*

Proof: Let the vertices of a graph P_m be $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ and let the vertices of graph P_n be $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. By the definition of SSP, $P_m \bowtie P_n$ has the vertex set $V(P_m \bowtie P_n) = V(P_m) \times V(P_n)$ and the edge set $E(P_m \bowtie P_n) = \{(u_i, v_i)(u_j, v_j)|u_i = u_j$ and $v_i \sim v_j \in P_n$ or $u_i \sim u_j \in P_m$ and $v_i \sim v_j \in P_n\}$. To construct the minimum PD-set for $(P_m \bowtie P_n)$, we consider the following cases:
Case(i): When $m \geq 2$ and $n = 2$, the set $D = \{(u_2, v_1)\}$ is a γ_p set for $P_m \bowtie P_n$. Initially $\{(u_1, v_2), (u_2, v_2), (u_3, v_2)\}$ are dominated and subsequently the remaining vertices are power dominated in the propagation steps. $\implies \gamma_p(P_m \bowtie P_n) = 1$
Case(ii): When $m = 2$ and $n = 2k, k \geq 1$
In this case, $D =\{(u_2, v_{3i-1})$: $i = 1, 2, \ldots, n/3$ if $n \equiv 0 \ mod\ 3$; $i = 1, 2, \ldots, \lceil n/3 \rceil$ if $n \equiv 2 mod 3\}$ and $D = \{(u_2, v_n), (u_2, v_{3i-1}) : i = 1, 2, \ldots, \lfloor n/3 \rfloor$ if $n \equiv 1 mod 3\}$. It is clear that $\lceil n/3 \rceil$ vertices are sufficient to power dominate the entire graph.
Case(iii): We now deal with the general case where $m \geq 3$ and $n = 2k, k \geq 2$. In this case, we discuss the PD-set in following three sub cases.
Sub case(i): When $n \equiv 0 \ mod\ 3$, consider the set $S =\{(u_2, v_{3i-1})$: $i = 1, 2, \ldots n/3\}$ in the domination step, the vertices in S dominates $3n - (2n/3)$ vertices of $P_m \bowtie P_n$ and the remaining vertices are power dominated subsequently by the already dominated vertices. This proves, $\gamma_p(P_m \bowtie P_n) = n/3$
Sub case(ii): When $n \equiv 1 \ mod\ 3$, consider the set $S =\{(u_2, v_{3i-1}) \cup (u_2, v_j) : i = 1, 2, \ldots \lfloor n/3 \rfloor, j = n\}$. These vertices initially dominates $3n - \lfloor n/3 \rfloor + 1$ vertices and the remaining vertices are subsequently power dominated in the propagation steps. In this case, $\gamma_p(P_m \bowtie P_n) = \lfloor n/3 \rfloor + 1 = \lceil n/3 \rceil$.
Sub case(iii): When $n \equiv 2 \ mod\ 3$, we choose the set $S =\{(u_2, v_{3i-1})$: $i = 1, 2, \ldots \lceil n/3 \rceil\}$. The set D dominates $3n - 2\lceil n/3 \rceil$ vertices in the domination step and the remaining vertices are power dominated in the propagation steps.
Hence, combining all the above cases, we can easily conclude that the obtained PD-set S is the minimum PD-set which power dominates all the vertices of $V(P_m \bowtie P_n)$. Thus, $\gamma_p(P_m \bowtie P_n) = \lceil n/3 \rceil$. □

Theorem 4.4 *Let $G = P_m$ be a path of order $m \geq 3$ vertices and $H = K_n$ be a complete graph with order $n \geq 2$. Then, $\gamma_p(P_m \bowtie K_n) = \lceil m/3 \rceil$.*

Proof: Let the vertices of a graph $G = P_m$ be $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ and let the vertices of graph $H = K_n$ be $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. By the definition of SSP $P_m \bowtie K_n$, the vertex set $V(P_m \bowtie K_n) = V(P_m) \times V(P_n)$ and the edge set $E(P_m \bowtie K_n) = \{(u_i, v_i)(u_j, v_j)|u_i = u_j$ and $v_i \sim v_j \in K_n$ or $u_i \sim u_j \in P_m$ and $v_i \sim v_j \in K_n\}$.

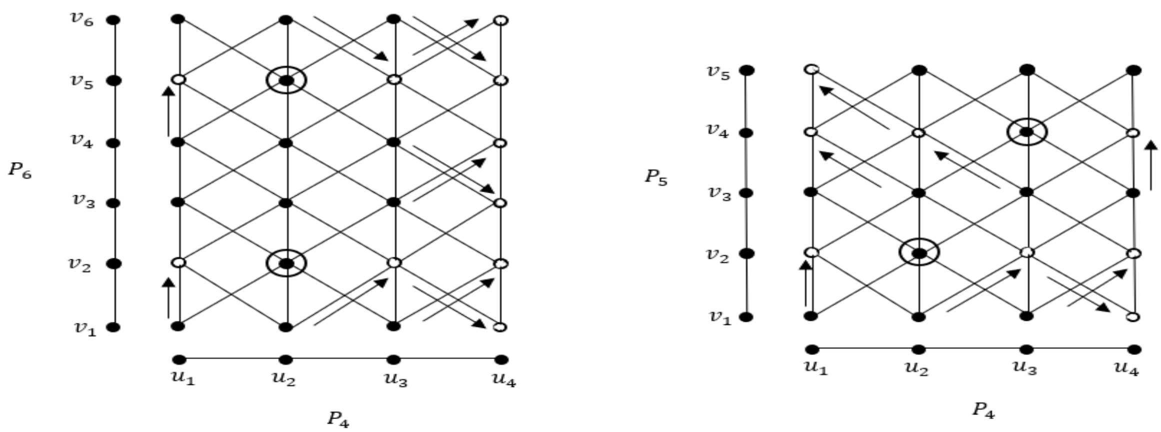

Figure 4: Illustration of $\gamma_p(P_4 \bowtie P_n)$ with $n = 2k$ and $n = 2k+1$ respectively

The minimum PD-set of $V(P_m \bowtie K_n)$ is found by considering the following cases:
Case(i): When $m \equiv 0 \ mod \ 3$, let the PD-set be $S = \{(u_{3i-1}, v_1) : i = 1, 2, \ldots m/3\}$. Here, D dominates $mn - 2(m/3)$ vertices initially. Note that all the vertices except $\{(u_i, v_1) : i \neq 3k-1, k = 0, 1, \ldots, m/3\}$ are dominated in the first step and the remaining vertices are power dominated in the propagation step.
Case(ii): When $m \equiv 1 \ mod \ 3$, let the power dominating set be $D = \{(u_{3i-1}, v_1) \cup (u_m, v_1) : i = 1, 2, \ldots, \lfloor m/3 \rfloor\}$. At the first step, $mn - 2\lfloor m/3 \rfloor$ vertices of $(P_m \bowtie K_n)$ are dominated. At the propagation step, $2\lfloor m/3 \rfloor$ vertices are power dominated. Thus, $mn - 2\lfloor m/3 \rfloor + 2\lfloor m/3 \rfloor = mn$ shows that all the vertices are power dominated.
Case (iii): When $m \equiv 2 \ mod \ 3$, consider the set $S = \{(u_{3i-1}, v_1) : i = 1, 2, \ldots, \lceil m/3 \rceil\}$. It is clear that these vertices are sufficient to dominate and power dominate the entire graph. Combining the above three cases, it can be easily verify that the obtained power dominating set S is a minimum PD-set for $P_m \bowtie K_n$. Hence, $\gamma_p(P_m \bowtie K_n) = \lceil m/3 \rceil$. □

Theorem 4.5 *Let $G(= S_m)$ and $H(= S_n)$ be star graphs with $n \geq 2$ and $m \geq 2$ vertices, respectively. Then, $\gamma_p(S_n \bowtie S_m) = 1$.*

Proof: Let the vertices of a graph $G = S_m$ be $V(S_m) = \{u_1, u_2, \ldots, u_m\}$ with u_1 as the central vertex and let the vertices of graph $H = S_n$ be $V(S_n) = \{v_1, v_2, \ldots, v_n\}$ with v_1 as the central vertex. By the definition of SSP, $S_m \bowtie S_n$ has the set of vertices $V(S_m \bowtie S_n) = V(S_m) \times V(S_n)$ and the set of edges $E(S_m \bowtie S_n) = \{(u_i, v_i)(u_j, v_j) | u_i = u_j$ and $v_i \sim v_j \in S_n$ or $u_i \sim u_j \in S_m$ and $v_i \sim v_j \in S_n\}$. Let $D = \{(u_1, v_1)\}$ be the γ_p-set of $S_m \bowtie S_n$. It is clear from the construction that (u_1, v_1) is adjacent to every vertex of $S_m \bowtie S_n$ except $\{(u_i, v_1) : 2 \leq i \leq n\}$, therefore $mn - (n-1) = mn - n + 1$ vertices are dominated at the domination step. The remaining $(n-1)$ vertices, $\{(u_i, v_1) : 2 \leq i \leq n\}$ are power dominated in the power domination step, hence monitoring the entire graph.
Thus $\gamma_p(S_m \bowtie S_n) = 1$. □

Theorem 4.6 *Let K_n and P_m be two graphs with $n \geq 3$ and $m \geq 3$ vertices, respectively. Then*

$$\gamma_p(K_n \bowtie P_m) = \begin{cases} \lceil \frac{m}{2} \rceil + 1 & \text{if } m = 4k+2, \text{where } k \geq 1 \\ \lceil \frac{m}{2} \rceil & \text{otherwise.} \end{cases}$$

Proof: Let the vertices of a graph K_n be $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and let the vertices of graph P_m be $V(P_m) = \{v_1, v_2, \ldots, v_m\}$. By the definition of semi strong product $K_n \bowtie P_m$, the vertex set $V(K_n \bowtie$

$P_m) = V(K_n) \times V(P_m)$ and the edge set $E(K_n \bowtie P_m) = \{(u_i, v_i)(u_j, v_j) | u_i = u_j$ and $v_i \sim v_j \in P_m$ or $u_i \sim u_j \in K_n$ and $v_i \sim v_j \in P_m\}$. The minimum PD-set of this graph can be concluded from the following cases:
Case (i): When $m = 4k, k \geq 1$
The γ_p-set S can be obtained by choosing the vertices $S = \{(u_2, v_{4i-2}), (u_2, v_{4i-1}) : 1 \leq i \leq \frac{m}{4}\}$. Note that the set S dominates the entire graph. Thus, $2(\frac{m}{4}) = \frac{m}{2} = \lceil \frac{m}{2} \rceil$ vertices are sufficient for this case.
Case(ii): When $m = 4k + 1, k \geq 1$
In this case, the number of vertices in $V(K_n \bowtie P_m)$ differs from the case (i). Therefore, the above PD-set, S as in case (i) is not sufficient to power dominate all the vertices in $(K_n \bowtie P_m)$, since the propagation step cannot power dominate any of the vertices as the adjacency of every monitored vertex is greater than 1. Hence the γ_p-set, $S = \{(u_2, v_{4i-2}), (u_2, v_{4i-1}) : 1 \leq i \leq \lfloor \frac{m}{4} \rfloor\} \cup \{(u_2, v_n)\}$. That is, $2(\frac{m}{4}) + 1 = \lceil \frac{m}{2} \rceil$.
Case(iii) When $m = 4k + 2, k \geq 1$
In this case, select the vertices $\{(u_2, v_{4i-2}), (u_2, v_{4i-1}) : 1 \leq i \leq \lfloor \frac{m}{4} \rfloor\}, (u_2, v_{n-1})$ and (u_2, v_n) in $V(K_n \bowtie P_m)$ to construct the minimum PD-set. That is, $2(\frac{m}{4}) + 2 = \lceil \frac{m}{2} \rceil + 1$ required to power dominate the entire graph.
Case (iv): When $m = 4k + 3, k \geq 1$
The γ_p-set in this case is same as the γ_p-set in case (i). Therefore, $2(\frac{m+1}{4}) = \lceil \frac{m}{2} \rceil$ vertices are needed to power dominate the entire graph.
Combining the above cases, we obtain the theorem. □

Remark 4.2 From Theorem 4.6, we also note that $\gamma(K_n \bowtie P_m) = \gamma_p(K_n \bowtie P_m)$.

Theorem 4.7 *Let $G = P_m$ and $H = C_n$ be path with $m \geq 3$ and cycle with $n = 2k + 1$ where $k \geq 1$ vertices, respectively. Then,*

$$\gamma_p(P_m \bowtie C_n) = \begin{cases} \lceil \frac{m}{3} \rceil & \text{if } m \geq 3, n = 3, \\ \lceil \frac{n}{3} \rceil & \text{if } m = 3, n = 2k+1 \text{ where } k \geq 1, \\ \lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil - 1 & \text{if } m \geq 4, n = 2k+1 \text{ where } k \geq 2. \end{cases}$$

Proof: Let the vertices of a graph $G = P_m$ be $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ let the set of vertices of $H = (C_n)$ be $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. By the definition of SSP $P_m \overline{\times} C_n$, the vertex set $V(P_m \overline{\times} C_n) = V(P_m) \times V(C_n)$ and the edge set $E(P_m \overline{\times} C_n) = \{(u_i, v_i)(u_j, v_j) | u_i = u_j$ and $v_i \sim v_j \in C_n$ or $u_i \sim u_j \in P_m$ and $v_i \sim v_j \in C_n\}$.
Case(i): When $m \geq 3$ and $n = 3$. Consider the set $S = \{(u_{3i-1}, v_2) : 1 \leq i \leq \lfloor \frac{m}{3} \rfloor\}$. It is clear that these vertices are not sufficient to power dominate the entire graph except the case where $m = 3k, k \geq 1$. Therefore we select one more vertex to power dominate the entire graph, thus $S = \{(u_{3i-1}, v_2) : 1 \leq i \leq \lfloor \frac{m}{3} \rfloor \} \cup \{(u_{3m}, v_2)\}$ for the case where $m \neq 3k, k \geq 1$. Hence, in both the cases, we observe that this is minimum set which power dominates the given graph $(P_m \bowtie C_n)$. Thus $\gamma_p(P_m \bowtie C_3) = \lceil \frac{m}{3} \rceil$
Case (ii): When $m = 3, n = 2k + 1$ where $k \geq 1$. The γ_p-set of $(P_3 \bowtie C_n)$ denoted by D is considered to be $D = \{(u_2, v_{3i-1}) : 1 \leq i \leq \frac{n}{3}\}$ if $n = 3k$ where $k \geq 1$ and $D = \{(u_2, v_{3i-1}) : 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \cup (u_2, v_{n-1})\}$. This proves $\gamma_p(P_3 \bowtie C_n) = \lceil \frac{n}{3} \rceil$.
We now deal with the general case where $m \geq 4, n = 2k + 1, k \geq 2$.
Let us consider the γ_p-set, $S = P_1 \cup P_2$. When $m = 3k, k \geq 2$, we select $P_1 = \{(u_{3i-1}, v_{n-1}) : 1 \leq i \leq \frac{m}{3}\}$ and when $n = 3k$ where k is odd, we select $P_2 = \{(u_2, v_{3i-1}) : 1 \leq i \leq \frac{n}{3}\}$. Otherwise, we choose $P_1 = \{(u_{3i-1}, v_{n-1}) : 1 \leq i \leq \lfloor \frac{m}{3} \rfloor \cup (u_{3m}, v_{n-1})\}$ and $P_2 = \{(u_2, v_{3i-1}) : 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \cup (u_2, v_{n-1})\}$. Therefore, by the above argument we get the γ_p-set of $(P_m \bowtie C_n)$ to be $\lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil$. But we observe that the vertex (u_2, v_{n-1}) is common for both P_1 and P_2. Thus, the minimum PD-set of $(P_m \bowtie C_n)$ reduces to $\lceil \frac{m}{3} \rceil + \lceil \frac{n}{3} \rceil - 1$. Hence, the theorem. □

5. Application and Conclusion

Large scale infrastructure networks such as telecommunication systems and sensor networks can often be modeled as products of smaller graphs. The SSP defines a specific complex connection pattern that may accurately model a large network built from a layered network of rings or paths connected in a grid like fashion. Studying $\gamma_p(G \bowtie H)$ provides a theoretical basis for determining the least number of sources needed to monitor the entire system. In this article, we have attempted to find the PD-number for certain combinations of basic graphs. The comparison table of $\gamma(G)$ as found in [3,14] and $\gamma_p(G)$ investigated in this paper are listed below:

Graphs	$\gamma(G)$	$\gamma_p(G)$
$G \bowtie H$	$\leq 2\gamma(G)\gamma(H)$	$\leq \gamma(G)\gamma(H)$
$P_m \bowtie P_n$	$\leq 2\left\lceil \frac{m}{4} \right\rceil \left\lceil \frac{n}{3} \right\rceil$	$\begin{cases} \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil - 1, & \text{if } \lvert V(H)\rvert \text{ is odd,} \\ \left\lceil \frac{n}{3} \right\rceil, & \text{if } \lvert V(H)\rvert \text{ is even.} \end{cases}$
$K_m \bowtie K_n$	2	$\begin{cases} 1, & \text{if } m = 1, 2, \\ 2, & \text{if } m \geq 3 \end{cases}$
$S_n \bowtie H$, $n \geq 2$, $H \neq K_1$	$\gamma_t(H)$	$\gamma(H)$
$K_n \bowtie H$, $H \neq K_1, n \geq 1$	$\gamma_t(H)$	$\leq \gamma_c(H)$
$S_n \bowtie S_m$	2	1

Power domination on SSP of grids, chordal and bipartite graphs are not yet characterized which are avenues for future research. Few problems to work on, are suggested below.
Question 1: To characterize the graphs for which $\gamma_p(G \bowtie H) = \gamma_p(G) + \gamma_p(H)$ and determine whether such equality implies structural constraints.
Question 2: Characterization of graphs with equal domination and PD-numbers are still open.

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