



## On The Study of Uniqueness of Entire and Meromorphic Functions Related to Linear Differential Polynomial and Differential-Difference Polynomials Sharing a Small Function

Jayashri Pattar, Shilpa N., Smita R. Kabbur

**ABSTRACT:** The results on the uniqueness of entire and meromorphic functions of finite order with two different finite complex numbers, developed for linear differential polynomial  $L(f) = \sum_{t=0}^k a_t f^{(t)}$  and a differential-difference polynomial  $W(f) = \sum_{j \in J} A_j(z) (\Delta_c f)^{(k_j)}$  are the primary focus of this study.

**Keywords:** Entire function, meromorphic function, uniqueness, difference polynomial, share, small function.

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### 1. Introduction

We investigate the uniqueness of the entire and meromorphic functions using the conventional notations of Nevanlinna theory (see [3,15]). Let  $f(z)$  and  $g(z)$  be two meromorphic functions and  $a$  be a complex number then  $f(z)$  and  $g(z)$  share  $a$  CM(IM), if  $f(z) - a$  and  $g(z) - a$  have the same zeros counting multiplicities (ignoring multiplicities). In 1929, Nevanlinna [15] proved the Five Value Uniqueness Theorem which states that, if two non-constant meromorphic functions share five distinct values then they must be identically equal. This foundational result initiated extensive research into the concept of value sharing, particularly for counting multiplicities (CM) and ignoring multiplicities (IM). Rubel and Yang [12], considered sharing of two values between a non-constant entire function and its derivative, making a significant development in the field. Subsequently, Li and Yang [6] extended these results to higher-order derivatives. Analogous uniqueness theorems have also been explored in the context of shift, difference and differential-difference operators as discussed in [4,1,8,16,14,13].

### 2. Preliminaries

In 2017, Li et al., [7] proved uniqueness theorem on sharing of 0 CM and  $a$  IM.

**Theorem 2.1** [7] *Let  $c(\neq 0) \in \mathbb{C}$ ,  $n \in \mathbb{N}$ , and  $f(z)$  be a non constant entire function of finite order. If  $f(z)$  and  $\Delta_c^n f(z)$  share 0 CM and a non-zero complex constant  $a$  IM, then  $f(z) \equiv \Delta_c^n f(z)$ .*

In 2018, Qi et al., [9] proved

**Theorem 2.2** [9] *Let  $f(z)$  be a transcendental entire function of finite order, and  $a$  be a non-zero finite value. If  $f(z)$  and  $f(z+c)$  share 0, a CM, then  $f(z) = f(z+c)$ .*

Qi and Yang [10] proved the following

**Theorem 2.3** [10] *Let  $f(z)$  be a nonconstant entire function of finite order, and let  $a, \eta$  be two nonzero finite complex values. If  $f'(z)$  and  $f(z+\eta)$  share 0 CM and a IM, then  $f'(z) \equiv f(z+\eta)$ .*

X. Huang [5] proved the result.

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**Theorem 2.4** [5] Let  $f(z)$  be a transcendental entire function of finite order, let  $\eta \neq 0$  be a finite complex number,  $n \geq 1, k \geq 0$  two integers and let  $a, b$  be two distinct finite complex values. If  $f(z)$  and  $(\Delta_\eta^n f(z))^{(k)}$  share a CM and share  $b$  IM, then  $f(z) \equiv (\Delta_\eta^n f(z))^{(k)}$ .

In 2022, B. Narasimha Rao and Shilpa N. [11] proved

**Theorem 2.5** [11] Let  $f(z)$  be a transcendental entire function of finite order and  $\psi(f) = \sum_{i_1 \in I_1} \alpha_{i_1}(z) f^{(k_{i_1})}(z) + \sum_{i_2 \in I_2} \alpha_{i_2}(z) f^{(k_{i_2})}(z + b_{i_2}) + \sum_{i_3 \in I_3} \alpha_{i_3}(z) f^{(k_{i_3})}(z + c_{i_3})$  be differential polynomial with  $\sum_{i_3 \in I_3} \alpha_{i_3}(z) \neq 0$ . If  $f(z)$  and  $\psi(f)$  share finite value  $a$  CM and  $f(z)$  has exceptional value  $\beta (\neq a)$ ,

(i) If  $a \neq 0$  and  $\beta$  is a Nevanlinna exceptional value of  $f(z)$  then

$$\frac{\psi(f) - a}{f - a} = \tau (\neq 0).$$

(ii) If  $\beta$  is a Borel exceptional value of  $f(z)$  then

$$\frac{\psi(f) - a}{f - a} = \frac{a}{a - \beta}.$$

**Lemma 2.1** [2,5] Let  $f$  be a nonconstant meromorphic function of finite order, and let  $\eta$  be a non-zero complex number. Then

$$m\left(r, \frac{f(z + \eta)}{f(z)}\right) = S(r, f)$$

for all  $r$  outside of a possible exceptional set  $E$  with finite logarithmic measure.

**Lemma 2.2** [15,5] Suppose  $f_1, f_2$  are two nonconstant meromorphic functions in the complex plane, then

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right).$$

**Lemma 2.3** [15,5] Let  $f$  be a nonconstant meromorphic function, and let  $P(f) = a_0 f^p + a_1 f^{p-1} + \dots + a_p (a_0 \neq 0)$  be a polynomial of degree  $p$  with constant coefficients  $a_j (j = 0, 1, \dots, p)$ . Suppose that  $b_j (j = 0, 1, \dots, q) (q > p)$ . Then

$$m\left(r, \frac{P(f) f'}{(f - b_1)(f - b_2) \dots (f - b_q)}\right) = S(r, f).$$

**Lemma 2.4** [5] Let  $f$  be a transcendental entire function of finite order, let  $\eta \neq 0$  be a finite complex number,  $n \geq 1, k \geq 0$  two integers, and let  $a$  be a nonzero complex value. If  $f$  and  $(\Delta_\eta^n)^{(k)}$  share a CM, and  $N\left(r, \frac{1}{(\Delta_\eta^n)^{(k)}}\right) = S(r, f)$ , then there is a polynomial  $p$  such that either  $T(r, e^p) = S(r, f)$ , or  $(\Delta_\eta^n)^{(k)} = H e^p$ , where  $H \neq 0$  is a small function of  $e^p$ .

**Lemma 2.5** [5,3,15] Suppose that  $f(z)$  is a meromorphic function in the complex plane and  $p(f) = a_0 f^n(z) + a_1 f^{n-1}(z) + \dots + a_n$ , where  $a_0 (\neq 0), a_1, \dots, a_n$  are small functions of  $f(z)$ . Then

$$T(r, p(f)) = nT(r, f(z)) + S(r, f).$$

**Lemma 2.6** [15] Let  $f$  be a transcendental meromorphic function of finite order, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$$

**Lemma 2.7** [5] Let  $f$  and  $g$  be two nonconstant entire functions, and  $a, b$  be two finite distinct complex values. If

$$H = \frac{f'}{(f - a)(f - b)} - \frac{g'}{(g - a)(g - b)} \equiv 0,$$

and  $f$  and  $g$  share a CM and share  $b$  IM, then either  $2T(r, f) \leq \bar{N}\left(r, \frac{1}{f - a}\right) + \bar{N}\left(r, \frac{1}{f - b}\right) + S(r, f)$  or  $f \equiv g$ .

### 3. Main Results

**Theorem 3.1** *Let  $f(z)$  be a transcendental entire function of finite order, let  $a, b$  be two distinct finite complex numbers and  $L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \dots + a_0 f$  be a linear differential polynomial of a function  $f$ , where  $a_0, a_1, \dots, a_k (\neq 0)$  are complex constants. If  $f$  and  $L(f)$  share a CM and  $b$  IM then  $f \equiv L(f)$ .*

**Proof:** If  $f \equiv L(f)$  then there is nothing to prove. Suppose  $f \not\equiv L(f)$ . Since  $f$  is a transcendental entire function of finite order,  $f$  and  $L(f)$  share a CM then we get

$$\frac{L(f) - a}{f - a} = e^h \quad (3.1)$$

where  $h$  is a polynomial. Since  $f$  and  $L(f)$  share a CM and  $b$  IM then by the second fundamental theorem and Lemma (2.1) we have

$$\begin{aligned} T(r, f) &\leq \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f-b}\right) + S(r, f) \\ &= \bar{N}\left(r, \frac{1}{L(f)-a}\right) + \bar{N}\left(r, \frac{1}{L(f)-b}\right) \leq N\left(r, \frac{1}{f-L(f)}\right) + S(r, f) \\ &\leq T(r, f - L(f)) + S(r, f) \leq m(r, f - L(f)) + S(r, f) \\ &\leq m(r, f) + m\left(r, 1 - \frac{L(f)}{f}\right) + S(r, f) \leq T(r, f) + S(r, f). \end{aligned}$$

That is

$$T(r, f) = \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f-b}\right) + S(r, f). \quad (3.2)$$

According to Lemma (2.1), (3.1) and (3.2) we have

$$T(r, f) = T(r, f - L(f)) + S(r, f) = N\left(r, \frac{1}{f-L(f)}\right) + S(r, f). \quad (3.3)$$

and

$$T(r, e^h) = m(r, e^h) = m\left(r, \frac{L(f) - a}{f - a}\right) \leq m\left(r, \frac{1}{f - a}\right) + S(r, f). \quad (3.4)$$

Then it follows from (3.1) and (3.3) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &= m\left(r, \frac{1 - e^h}{f - L(f)}\right) \leq m\left(r, \frac{1}{f - L(f)}\right) + m(r, 1 - e^h) \\ &\leq T(r, e^h) + S(r, f). \end{aligned} \quad (3.5)$$

Then by (3.4) and (3.5)

$$T(r, e^h) = m\left(r, \frac{1}{f-a}\right) + S(r, f). \quad (3.6)$$

On the other hand we rewrite (3.1) as

$$1 - e^h = \frac{f - L(f)}{f - a}, \quad (3.7)$$

which implies

$$\bar{N}\left(r, \frac{1}{f-a}\right) \leq \bar{N}\left(r, \frac{1}{1 - e^h}\right) = T(r, e^h) + S(r, f). \quad (3.8)$$

Then by (3.2), (3.6) and (3.8)

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) &= \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f-b}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{1-e^h}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-a}\right) + S(r, f), \end{aligned}$$

that is

$$N\left(r, \frac{1}{f-a}\right) = \bar{N}\left(r, \frac{1}{f-a}\right) + S(r, f). \quad (3.9)$$

And

$$\bar{N}\left(r, \frac{1}{f-b}\right) = T(r, e^h) + S(r, f). \quad (3.10)$$

Set

$$\omega = \frac{f'(L(f) - f)}{(f-a)(f-b)}. \quad (3.11)$$

And

$$\chi = \frac{L'(f)(L(f) - f)}{(L(f) - a)(L(f) - b)}. \quad (3.12)$$

Easy to know that  $\omega \neq 0$  because of  $f \neq L(f)$  and  $\omega$  is an entire function. By Lemmas (2.1), (2.3) we have

$$\begin{aligned} T(r, \omega) = m(r, \omega) &= m\left(r, \frac{f'(L(f) - f)}{(f-a)(f-b)}\right) + S(r, f) \\ &\leq m\left(r, \frac{f'f'}{(f-a)(f-b)}\right) + m\left(r, \frac{L(f)}{f} - 1\right) + S(r, f) \\ &\leq S(r, f), \end{aligned}$$

that is

$$T(r, \omega) = S(r, f). \quad (3.13)$$

Let  $d = a - j(a-b)$  ( $j \neq 0, 1$ ). Obviously by Lemmas (2.1), (2.3) we obtain

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &= m\left(r, \frac{1}{(a-b)\omega} \left[ \frac{f'}{(f-a)} - \frac{f'}{(f-b)} \right] \left( \frac{L(f)}{f} - 1 \right)\right) \\ &\leq m\left(r, \frac{1}{\omega}\right) + m\left(r, \frac{f'}{f-a} - \frac{f'}{f-b}\right) + m\left(r, \frac{L(f)}{f} - 1\right) + S(r, f) \\ &= S(r, f) \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} m\left(r, \frac{1}{f-d}\right) &= m\left(r, \frac{f'(L(f) - f)}{\omega(f-a)(f-b)(f-d)}\right) \\ &\leq m\left(r, \frac{L(f)}{f} - 1\right) + m\left(r, \frac{f'f}{(f-a)(f-b)(f-d)}\right) + S(r, f) \\ &= S(r, f). \end{aligned} \quad (3.15)$$

Set

$$\phi = \frac{L'(f)}{(L(f) - a)(L(f) - b)} - \frac{f'}{(f-a)(f-b)}. \quad (3.16)$$

Now, we have two cases.

**Case 1.**  $\phi = 0$ . By (3.16) we have

$$\frac{L(f) - a}{L(f) - b} = c \frac{f - a}{f - b} \quad (3.17)$$

where  $c$  is a non-zero constant. Since  $f \neq L(f)$  then by Lemma (2.7) we get

$$2T(r, f) \leq \bar{N}\left(r, \frac{1}{f - a}\right) + \bar{N}\left(r, \frac{1}{f - b}\right) + S(r, f) \quad (3.18)$$

which contradicts with (3.2).

**Case 2.**  $\phi \neq 0$ , by (3.3), (3.13) and (3.16) we can obtain

$$\begin{aligned} m(r, f) &= m(r, f - L(f)) + S(r, f) \\ &= m\left(r, \frac{\phi[f - L(f)]}{\phi}\right) + S(r, f) \\ &= m\left(r, \frac{\chi - \omega}{\phi}\right) + S(r, f) \\ &\leq T\left(r, \frac{\phi}{\chi - \omega}\right) + S(r, f) \\ &\leq T(r, \chi - \omega) + T(r, \phi) + S(r, f) \\ &\leq T(r, \chi) + T(r, \phi) + S(r, f) \\ &\leq T(r, \chi) + \bar{N}\left(r, \frac{1}{f - b}\right) + S(r, f). \end{aligned} \quad (3.19)$$

On the other hand

$$\begin{aligned} T(r, \chi) &= T\left(r, \frac{L'(f)[L(f) - f]}{(L(f) - a)(L(f) - b)}\right) \\ &= m\left(r, \frac{L'(f)(L(f) - f)}{(L(f) - a)(L(f) - b)}\right) + S(r, f) \\ &\leq m\left(r, \frac{L'(f)}{(L(f) - a)}\right) + m\left(r, \frac{(L(f) - f)}{(L(f) - b)}\right) + S(r, f) \\ &\leq m\left(r, \frac{1}{f - a}\right) + S(r, f) = \bar{N}\left(r, \frac{1}{f - b}\right) + S(r, f). \end{aligned} \quad (3.20)$$

Hence combining (3.19) and (3.20) we obtain

$$T(r, f) \leq 2\bar{N}\left(r, \frac{1}{f - b}\right) + S(r, f). \quad (3.21)$$

Next we have three subcases.

**Subcase 2.1**  $a = 0$ . Then by (3.1) and Lemma (2.1) we get

$$m(r, e^h) = m\left(r, \frac{L(f)}{f}\right) = S(r, f). \quad (3.22)$$

Then by (3.10), (3.21), (3.22) we can have  $T(r, f) = S(r, f)$  a contradiction.

**Subcase 2.2**  $b = 0$ , then by (3.6), (3.10), (3.21) and Lemma (2.1), we get

$$\begin{aligned} T(r, f) &\leq m\left(r, \frac{1}{f - a}\right) + \bar{N}\left(r, \frac{1}{L(f)}\right) + S(r, f) \\ &\leq m\left(r, \frac{1}{L(f)}\right) + \bar{N}\left(r, \frac{1}{L(f)}\right) + S(r, f) \\ &\leq T(r, L(f)) + S(r, f). \end{aligned} \quad (3.23)$$

From the fact that

$$T(r, L(f)) \leq T(r, f) + S(r, f), \quad (3.24)$$

which follows from (3.23) that

$$T(r, f) = T(r, L(f)) + S(r, f). \quad (3.25)$$

By the second fundamental theorem, Lemma (2.1), (3.2) and (3.25) we have

$$\begin{aligned} 2T(r, f) &\leq 2T(r, L(f)) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{L(f)-a}\right) + \bar{N}\left(r, \frac{1}{L(f)}\right) + \bar{N}\left(r, \frac{1}{L(f)-d}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f-a}\right) + \bar{N}\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{L(f)-d}\right) - m\left(r, \frac{1}{L(f)-d}\right) + S(r, f) \\ &\leq T(r, f) + T(r, L(f)) - m\left(r, \frac{1}{L(f)-d}\right) + S(r, f) \\ &\leq 2T(r, f) - m\left(r, \frac{1}{L(f)-d}\right) + S(r, f). \end{aligned}$$

Thus

$$m\left(r, \frac{1}{L(f)-d}\right) = S(r, f). \quad (3.26)$$

From the first fundamental theorem, Lemmas (2.1), (2.2), (3.14), (3.15), (3.25), (3.26) and that  $f$  is a transcendental entire function of finite order, we obtain

$$\begin{aligned} m\left(r, \frac{f-d}{L(f)-d}\right) &\leq m\left(r, \frac{f}{L(f)-d}\right) + m\left(r, \frac{d}{L(f)-d}\right) + S(r, f) \\ &\leq T\left(r, \frac{f}{L(f)-d}\right) - N\left(r, \frac{f}{L(f)-d}\right) + S(r, f) \\ &= m\left(r, \frac{L(f)-d}{f}\right) + N\left(r, \frac{L(f)-d}{f}\right) - N\left(r, \frac{1}{L(f)-d}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{L(f)-d}\right) + S(r, f) \\ &= T\left(r, \frac{1}{f}\right) - T\left(r, \frac{1}{L(f)-d}\right) + S(r, f) \\ &= T(r, f) - T(r, L(f)) + S(r, f) = S(r, f). \end{aligned}$$

Thus we get

$$m\left(r, \frac{f-d}{L(f)-d}\right) = S(r, f). \quad (3.27)$$

It's easy to see that  $N(r, \chi) = S(r, f)$  and (3.12) can be rewritten as

$$\chi = \left[ \frac{a-d}{a} \frac{L'(f)}{L(f)-a} + \frac{d}{a} \frac{L'(f)}{L(f)} \right] \left[ 1 - \frac{f-d}{L(f)-d} \right]. \quad (3.28)$$

Then by (3.27) and (3.28) we can get

$$T(r, \chi) = m(r, \chi) + N(r, \chi) = S(r, f). \quad (3.29)$$

By (3.2), (3.19) and (3.29) we get

$$\bar{N}\left(r, \frac{1}{f-a}\right) = S(r, f). \quad (3.30)$$

Moreover, by (3.2), (3.25) and (3.30) we have

$$m\left(r, \frac{1}{L(f)}\right) = S(r, f). \quad (3.31)$$

Which implies

$$\bar{N}\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{f-a}\right) \leq m\left(r, \frac{1}{L(f)}\right) = S(r, f). \quad (3.32)$$

Then by (3.2), (3.30) and (3.32) we obtain  $T(r, f) = S(r, f)$ , a contradiction.

**Subcase 2.3**  $ab \neq 0$ . So by (3.6), (3.10), (3.21) and second fundamental theorem, we get

$$\begin{aligned} T(r, f) &\leq 2m\left(r, \frac{1}{f-a}\right) + S(r, f) \\ &\leq 2m\left(r, \frac{1}{L(f)}\right) + S(r, f) \\ &= 2T(r, L(f)) - 2N\left(r, \frac{1}{L(f)}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{L(f)-a}\right) + \bar{N}\left(r, \frac{1}{L(f)-b}\right) + \bar{N}\left(r, \frac{1}{L(f)}\right) - 2N\left(r, \frac{1}{L(f)}\right) + S(r, f) \\ &\leq T(r, f) - N\left(r, \frac{1}{L(f)}\right) + S(r, f), \end{aligned}$$

which deduces that

$$N\left(r, \frac{1}{L(f)}\right) = S(r, f). \quad (3.33)$$

It follows from the second fundamental theorem of Nevanlinna that

$$\begin{aligned} T(r, L(f)) &\leq \bar{N}\left(r, \frac{1}{L(f)}\right) + \bar{N}\left(r, \frac{1}{L(f)-a}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{L(f)-a}\right) + S(r, f) \\ &\leq T(r, L(f)) + S(r, f), \end{aligned}$$

which implies that

$$T(r, L(f)) = \bar{N}\left(r, \frac{1}{L(f)-a}\right) + S(r, f). \quad (3.34)$$

Similarly

$$T(r, L(f)) = \bar{N}\left(r, \frac{1}{L(f)-b}\right) + S(r, f). \quad (3.35)$$

Then by (3.2), (3.34), (3.35) and the fact that  $f$  and  $L(f)$  share  $a$  CM and  $b$  IM we get

$$T(r, f) = 2T(r, L(f)) + S(r, f). \quad (3.36)$$

Easy to see from (3.16) that

$$T(r, \phi) = N(r, \phi) + S(r, f) \leq \bar{N}\left(r, \frac{1}{L(f)-b}\right) + S(r, f). \quad (3.37)$$

We claim that

$$T(r, \phi) = \bar{N}\left(r, \frac{1}{L(f)-b}\right) + S(r, f). \quad (3.38)$$

Otherwise

$$T(r, \phi) < \overline{N} \left( r, \frac{1}{L(f) - b} \right) + S(r, f). \quad (3.39)$$

We can deduce from (3.2), (3.12) and Lemma (2.2) that

$$\begin{aligned} T(r, \chi) &= T \left( r, \frac{L'(f)(L(f) - f)}{(L(f) - a)(L(f) - b)} \right) \\ &= m \left( r, \frac{L'(f)(L(f) - f)}{(L(f) - a)(L(f) - b)} \right) + S(r, f) \\ &\leq m \left( r, \frac{L'(f)}{L(f) - a} \right) + m \left( r, 1 - \frac{f - b}{L(f) - b} \right) \\ &\leq m \left( r, \frac{L(f) - b}{f - b} \right) + N \left( r, \frac{L(f) - b}{f - b} \right) - N \left( r, \frac{f - b}{L(f) - b} \right) + S(r, f) \\ &\leq m \left( r, \frac{1}{f - b} \right) + N \left( r, \frac{1}{f - b} \right) - N \left( r, \frac{1}{L(f) - b} \right) + S(r, f) \\ &\leq T(r, f) - \overline{N} \left( r, \frac{1}{L(f) - b} \right) + S(r, f) \\ &\leq \overline{N} \left( r, \frac{1}{f - a} \right) + S(r, f), \end{aligned}$$

which is

$$T(r, \chi) \leq \overline{N} \left( r, \frac{1}{f - a} \right) + S(r, f). \quad (3.40)$$

Then combining (3.2), (3.39), (3.40) and (3.19) we obtain

$$\begin{aligned} \overline{N} \left( r, \frac{1}{f - a} \right) + \overline{N} \left( r, \frac{1}{f - b} \right) &= T(r, f) + S(r, f) \\ &\leq \overline{N} \left( r, \frac{1}{f - a} \right) + T(r, \phi) + S(r, f) \end{aligned}$$

that is

$$\overline{N} \left( r, \frac{1}{L(f) - b} \right) \leq T(r, \phi) + S(r, f) \quad (3.41)$$

a contradiction. Similarly, we can also obtain

$$T(r, \phi) = \overline{N} \left( r, \frac{1}{L(f) - a} \right) + S(r, f). \quad (3.42)$$

By Lemma (2.4), if  $T(r, e^p) = S(r, f)$  then we can obtain  $T(r, f) = S(r, f)$  from (3.10) and (3.21) a contradiction.

Hence  $L(f) = He^p$ , where  $H \neq 0$  is a small function of  $e^p$ .  $\square$

**Theorem 3.2** *Let  $f$  be a nonconstant meromorphic function with  $\rho_2(f) < 1$ , let  $a, \alpha$  be two distinct small functions related to  $f$  and  $W(f) = \sum_{j \in J} A_j(z)(\Delta_c f)^{(k_j)}$  be a differential-difference polynomial with  $a \neq W(\alpha)$ . If  $f$  and  $W(f)$  share  $a, \infty$  CM and  $\alpha$  is a Nevanlinna exceptional small function of  $f$  then*

$$\frac{W(f) - a}{f - a} = \tau (\neq 0).$$

**Proof:** Let

$$\Phi = \frac{W(f) - a}{f - a}, \quad (3.43)$$

where  $\Phi$  is a meromorphic function. Since  $f$  and  $W(f)$  share  $a, \infty$  CM, we have

$$N(r, \Phi) = S(r, f), \quad N\left(r, \frac{1}{\Phi}\right) = S(r, f).$$

From (3.43) it follows that

$$\frac{\Phi}{a(f-\alpha)} - \frac{W(f-\alpha)}{a(f-\alpha)} \frac{\Phi-a}{W(f-\alpha)} = \frac{1}{f-\alpha}. \quad (3.44)$$

By Lemmas (2.1), (2.6) and Nevanlinna first fundamental theorem, we have

$$\begin{aligned} T(r, \Phi) &= m(r, \Phi) + N(r, \Phi) \\ &= m\left(r, \frac{W(f-\alpha)}{f-a}\right) + S(r, f) \\ &\leq m\left(r, \frac{\sum_{j \in J} A_j(z)(\Delta_c f)^{(k_j)} - a}{f-a}\right) + S(r, f) \\ &\leq m\left(r, \frac{\sum_{j \in J} A_j(z)(f(z+c) - f(z))^{(k_j)} - a}{f-a}\right) + S(r, f) \\ &\leq m\left(r, \frac{\sum_{j \in J} A_j(z)[f^{(k_j)}(z+c) - f^{(k_j)}(z) - a^{(k_j)}(z)]}{f-a}\right) \\ &\quad + m\left(r, \frac{\sum_{j \in J} A_j(z)a^{(k_j)}(z) - a}{f-a}\right) + S(r, f) \\ &\leq \sum_{j \in J} \left[ m(r, A_j(z)) + m\left(r, \frac{f^{(k_j)}(z+c)}{f-a}\right) + m\left(r, \frac{f^{(k_j)}(z) - a^{(k_j)}(z)}{f-a}\right) \right] \\ &\quad + m\left(r, \frac{1}{f-a}\right) + S(r, f) \\ &\leq m\left(r, \frac{1}{f-a}\right) + S(r, f) \\ &\leq T(r, f) + S(r, f). \end{aligned}$$

It follows that

$$S(r, \Phi) = S(r, f). \quad (3.45)$$

Since  $\alpha$  is a Nevanlinna exceptional small function of  $f$ , we deduce that  $m\left(r, \frac{1}{f-\alpha}\right) \geq \gamma T(r, f)$  for sufficiently large  $r$ , where  $\gamma$  is some positive constant. Then by (3.44) we have

$$\begin{aligned} T(r, f) &\leq \frac{1}{\gamma} m\left(r, \frac{1}{f-\alpha}\right) \\ &= \frac{1}{\gamma} \left[ m\left(r, \frac{\Phi}{a(f-\alpha)}\right) + m\left(r, \frac{W(f-\alpha)}{a(f-\alpha)}\right) + m\left(r, \frac{\Phi-a}{W(f-\alpha)}\right) \right] + S(r, f) \\ &\leq \frac{2}{\gamma} T(r, \Phi) + S(r, f). \end{aligned}$$

It follows that

$$S(r, f) = S(r, \Phi). \quad (3.46)$$

From (3.45), (3.46),  $a \neq W(\alpha)$  and Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T(r, \Phi) &\leq \bar{N}(r, \Phi) + \bar{N}\left(r, \frac{1}{\Phi}\right) + \bar{N}\left(r, \frac{1}{\Phi - \frac{a-W(\alpha)}{a-\alpha}}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{a-\alpha}{(a-\alpha)\Phi - a + W(\alpha)}\right) + S(r, f) \\ &\leq \bar{N}(r, a-\alpha) + \bar{N}\left(r, \frac{1}{(a-\alpha)\Phi - a + W(\alpha)}\right) \\ &\leq T(r, \Phi) + S(r, f). \end{aligned}$$

Thus, we have

$$\bar{N}\left(r, \frac{1}{(a-\alpha)\Phi - a + W(\alpha)}\right) = T(r, \Phi) + S(r, f)$$

Now

$$\begin{aligned} m\left(r, \frac{\Phi}{a(f-\alpha)}\right) &= m\left(r, \frac{\Phi}{a} \frac{1}{f-\alpha}\right) = m\left(r, \frac{W(f)-a}{a(f-a)} \frac{1}{f-\alpha}\right) \\ &\leq m\left(r, \frac{W(f)-a}{a(f-a)}\right) + m\left(r, \frac{1}{f-\alpha}\right) + S(r, f) \\ &= S(r, f) \end{aligned} \tag{3.47}$$

$$\begin{aligned} m\left(r, \frac{\Phi-a}{W(f-\alpha)}\right) &= m\left(r, \frac{\Phi}{W(f-\alpha)}\right) + m\left(r, \frac{a}{W(f-\alpha)}\right) + S(r, f) \\ &\leq m\left(r, \frac{W(f)-a}{(f-a)W(f-\alpha)}\right) + m\left(r, \frac{a}{W(f-\alpha)}\right) + S(r, f) \\ &\leq m\left(r, \frac{W(f)-a}{f-a}\right) + m\left(r, \frac{1}{W(f-\alpha)}\right) + m\left(r, \frac{a}{W(f-\alpha)}\right) + S(r, f) \\ &= m\left(r, \frac{\Phi-a}{W(f-\alpha)}\right) = S(r, f). \end{aligned} \tag{3.48}$$

It follows from (3.44), (3.47), (3.48) and Lemma (2.1), (2.6) that

$$\begin{aligned} m\left(r, \frac{1}{f-\alpha}\right) &= m\left(r, \frac{\Phi}{a(f-\alpha)} - \frac{W(f-\alpha)}{a(f-\alpha)} \frac{\Phi-a}{W(f-\alpha)}\right) \\ &\leq m\left(r, \frac{\Phi}{a(f-\alpha)}\right) + m\left(r, \frac{W(f-\alpha)}{a(f-\alpha)}\right) + m\left(r, \frac{\Phi-a}{W(f-\alpha)}\right) \\ &= S(r, f). \end{aligned}$$

Which contradicts with  $\alpha$  is a Nevanlinna small function of  $f$ . Hence  $\Phi$  is a constant. That is  $\frac{W(f)-a}{f-a} = \tau$ .  $\square$

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*Jayashri Pattar,*  
*Department of Mathematics,*  
*Maharani's Science College for Women,*  
*Maharani Cluster University,*  
*Bengaluru,*  
*India.*  
*E-mail address: jspmcu.2019@gmail.com*

*and*

*Shilpa N.,*  
*Department of Mathematics,*  
*School of Engineering,*  
*Presidency University,*  
*Bengaluru,*  
*India.*  
*E-mail address: shilpajaikumar@gmail.com*

*and*

*Smita R. Kabbur,*  
*Department of Mathematics,*  
*KLS Gogte Institute of Technology,*  
*Belagavi,*  
*India.*  
*E-mail address: srkabbur@gmail.com*