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A Fan-KKM Approach to Vector Hemivariational-Like Inequalities with $\zeta - \Xi$ Monotonicity

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ABSTRACT: This study introduces a new class of vector hemivariational-like inequalities in Hausdorff topological vector spaces (HTVS) governed by $\zeta - \Xi$-monotone operators. The Fan-KKM principle is used to establish new existence results for this extended inequality system, with suitable assumptions made about the nonlinear mappings involved. This formulation broadens the classical theory of variational and hemivariational inequalities by include more complex and non-smooth operator behaviors. Our findings not only broaden the theoretical landscape, but also offer practical answers to non-smooth equilibrium problems in Contact Mechanics.

Keywords: Existence outcomes, non-smooth analysis, generalized monotonicity, vector hemivariational inequality, Hausdorff spaces.

Contents

1. Introduction

Inequalities of the vector hemi-variational kind include non-coercive multivalued relations in mathematical physics, optimization, and mechanics. Compared to classic variational inequalities that necessitate monotonicity and single-valued mappings, vector hemivariational inequalities permit a greater number of issues with non-unique solutions. Optimization, non-smooth analysis, and variational formulations are closely related, as has been shown by recent developments in hemivariational inequality theory. These connections help to solve complicated problems in contact mechanics, nonlinear elasticity, and other engineering fields that need to look at a lot of different approaches or states. Consequently, vector hemi-variational inequalities not only contribute to the field of mathematics, but they also aid engineers and scientists in comprehending multi-criteria, interactive systems (see [5,12,13,3,1,14,2,18,21,20]).

Ahmad and Khan were able to establish weak linear variational-like inequalities with the use of an hgeneralized convex function, which enabled them to approach the existence problem in novel ways [4]. In response to the unresolved question posed by Hou and Chen [7], I would like to propose another relevant topic. In their publication [6], Fang and Huang discovered novel existence results for resilient linear variational inequalities in Banach spaces. They presented these results specifically within the framework of Banach spaces. Xie and Gong reported research on fuzzy mapping variational variance and Zadeh's decomposition theory in 2019. Their discoveries brought these aspects to light. Additionally, they formalized the precise sets that directly correspond to fuzzy mapping cut sets [24]. This work elucidates the complex relationship between fuzzy mapping and variational inequalities, suggesting potential avenues for further investigation. Furthermore, their findings could improve our understanding of resilience in mathematical models, leading to advancements in both theoretical and practical settings. Ultimately,

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this improved comprehension may lead to more robust solutions in a variety of disciplines, including engineering, economics, and optimization, by enhancing methodologies for managing uncertainty in various systems. It is anticipated that the integration of fuzzy logic and variational approaches will reveal novel methods for addressing complex real-world issues as researchers continue to expand upon these fundamental concepts. Khisbag, Hashoosh, and their colleagues developed a new definition of invexity known as $N-\zeta-\mho-$ invex in their research. They also explored various concepts related to complementarity problems that involve variational inequalities and topologically ordered vector spaces [19]. As a component of their works, this was carried out. In the year 2024, Noor suggested and analyzed a number of iterative strategies for higherorder hemivariational inequalities by making use of the auxiliary principal methodology [22]. The goal of this study is to show that there is a new family of vector inequalities that are similar to hemivariational inequalities. This will help build on and expand on what has already been found. The study is broken up into three main parts. After the introduction, we talk about the basic ideas and theorems that you need to know in order to understand the main results that are talked about in the next part. We get some new results in the third part by using the FKKM-mapping method. Any type of convex set is looked at in the study, whether it is compact or not. This study, according to several authors, both bolsters and generalizes the conclusions of prior research (see [15,?,16,10]). These conclusions highlight the significance of having a solid understanding of the geometric features of convex sets in various mathematical settings through their application. By exploring both compact and noncompact cases, this research aims to provide a more comprehensive framework that can be applied to diverse theoretical and practical problems.

2. Preliminaries

I suggest that we explore the possibility that Υ and M be distinct sets that are not included in the HTVS. Furthermore, let us presume that the set ϖ, a subset of Υ, is a closed, convex, pointed cone with its apex located at the origin. If D is non-empty, it is a convex and closed set within M. An ordered HTVS is represented by the letter Υ and is denoted by the notation (Υ, ϖ). The information contained in Υ suggests that the following is a design of an ordering link which includes:

$$\forall \lambda, v \in \Upsilon, \lambda \leq v \text{ iff } v-\lambda \in \varpi.$$

Weak ordering relations in Υ are more commonly referred to when int ϖ is non-zero.

$$\forall \lambda, \psi \in \Upsilon, \lambda \not< \psi \text{ iff } \psi-\lambda \notin \text{ int } \varpi.$$

For example, let $\mathrm{T}: \mathrm{M} \rightarrow \Theta(\mathrm{M}, \Upsilon)$ and $\Theta(\mathrm{M}, \Upsilon)$ is the collection of every continuous vector-valued mapping from the set M to the space Υ. We refer to the evaluation of $1 \in \Theta(\mathrm{M}, \Upsilon)$ on $\lambda \in \mathrm{M}$ via $(1, \lambda)$.

This investigation, we examine a family for cones $\varpi(\lambda) \subseteq \Upsilon$, which are indexed by elements $\lambda \in D$. Each cone is presumed to be closed, convex, and pointed. We will establish two mappings:

$$\Xi: D \times D \rightarrow M \text{ and } g: D \times D \rightarrow \Upsilon. \text{ For all } \lambda \in D, \operatorname{int} \varpi(\lambda) \neq \varnothing.$$

Definition 2.1 *[9] Consider a function $J: L \rightarrow R$ that is a locally Lipschitz. $J^0(u;\psi)$ represents the generalized derivative of $\psi \in L$.*

$$J^0(u;\psi)=\lim_{\substack{w \rightarrow u \\ \alpha \downarrow 0}} \sup \frac{J(w+\alpha\psi)-J(w)}{\alpha}$$

Lemma 2.1 *[9] Considering a function $J: L \rightarrow \mathbb{R}$ is Locally Lipschitz of rank L_u near the point $u \in L, J^0(u;\psi)$ be the generalized directional derivative as defined in Definition [?]. Then the following properties hold:*

i. A function $\psi \rightarrow J^0(u;\psi)$ is a sub additive that is finite, homogenous, positively, and contents that $\left|J^0(u;\psi)\right| \leq L_u\|\psi\|_B$.

ii. $J^0(u;\psi)$ is upper as a function of (u,ψ).

iii. $J^0(u;-\psi),=(-J)^0(u;\psi)$.

Lemma 2.2 *[8]. Let* D *be a non-empty convex and closed set in a H.T.S. Consider a multi-valued mapping* $\Lambda:\mathrm{D}\rightarrow 2^M$. *For any finite subset* $\{u_1,u_2,\ldots,u_n\}\subset\mathrm{D}$, *it holds that* $\operatorname{conv}\{u_1,u_2,\ldots,u_n\}\subset\cup_{i=1}^{n}\Lambda(u_i)$. *It shows that* Λ *is a KKM-mapping, and* $\Lambda(u)$ *is closed for all* u *in* M *and compact for some* u *. The convex hull is "conv". Then,* $\bigcap_{u\in M}\Lambda(u)$ *is not empty.*

Theorem 2.1 *[11]. Assume that* D *is a non-empty set of an HTVS M. Considering the set-valued mapping* $\Phi:D\rightarrow 2^M$ *is a KKM map, in which*

i. $\Phi(\varrho)$ *is closed set, for every* $\varrho\in\mathrm{D}$.

ii. $\Phi(\varrho)$ *is a compact set for at least single* $\varrho\in\mathrm{D}$.

Thus, $\cap\{\Phi(\varrho):\varrho\in\mathrm{D}\}\neq\varnothing$.
The notations for study findings that are listed below will be utilized by us in order to demonstrate the findings that we have obtained.

Remark 2.1 [23]. The following terms is used for clarity and consistency:

i. $\psi\notin-\operatorname{int}\mathrm{w}$ iff $\psi\geq_{\varpi}0$.

ii. $\psi\in$-int w iff $\psi<_{\varpi}0$.

iii. $\psi-w\notin-\operatorname{int}\varpi$ iff $\psi-w\geq_{\varpi}0$.

iv. $\psi\notin-\operatorname{int}\mathrm{w}$ and $w\notin-\operatorname{int}$ imply $\psi+w\notin-\operatorname{int}\mathrm{w}$.

Definition 2.2 *[17] Assume that* $T:M\rightarrow L(M,\Upsilon),\zeta:M\times M\rightarrow\Upsilon$ *and* $\Xi:\mathrm{D}\times\mathrm{D}\rightarrow\mathrm{D}$ *are three mappings in which* $\varpi=\bigcap_{u\in D}\varpi(u)$. *At that time,* T *is held to be* $\zeta-E-$ *monotone in* ϖ *iff*

$$<T(\psi)-T(u),\Xi(\psi,u)>+\zeta(\psi,u)\in\varpi \textit{ for all } u,\psi\in\mathrm{D}.$$

Definition 2.3 *[9] Assume that* $T:D\rightarrow\Theta(M,\Upsilon)$ *and* $\Xi:D\times D\rightarrow D$. *We emphasize that* T *is* Ξ*-hemi continuous if the mapping* $\mathrm{t}\mapsto\langle\mathrm{T}(\psi+\mathrm{t}(\mathrm{u}-\psi)),\mathrm{E}(\psi,u)\rangle$ *is continuous at* 0^+, *given for all* $u,\psi\in\mathrm{D}$, *and* $\mathrm{t}\in(0,1)$, *that is represented for* $\Xi-$ *H.C.*

Generalized vector hemivariational-like (GVHLI) are a class of linear variational inequalities that we shall study in the following: Find $\sigma\in\mathrm{D}$ *such that for every* $\psi\in\mathrm{D}$,

$$<\mathrm{T}(\lambda),\Xi(\psi,\lambda)>+J^0(\lambda,\Xi(\psi,\lambda))+\mathrm{g}(\psi,\lambda)\notin-\operatorname{int}\varpi(\lambda). \tag{2.1}$$

besides the resolution of problem (2.1), Given the following hypotheses, we are going to presume that they have been satisfied:

$\mathrm{H}_1:\lim_{t\rightarrow 0^+}\frac{\zeta(\mathrm{t}\psi+(1-\mathrm{t})\lambda,\lambda)}{\mathrm{t}}=0$;
$\mathrm{H}_2:\mathrm{T}:\mathrm{D}\rightarrow\Theta(\mathrm{M},\Upsilon)$ *is an* $\Xi-$ *H.C mapping;*
$\mathrm{H}_3:\varpi=\bigcap_{\lambda\in\mathrm{D}}\varpi(\lambda)\neq\varnothing$ *and the operator* T *is* $\zeta-\Xi$*-monotone in* ϖ.
H_4 *: The mapping* $\Xi(\cdot;\cdot):X\times X\rightarrow X$ *satisfies the next circumstances*

i. $\Xi(u;u)=0$ *for any* $u\in X$*;*

ii. $\Xi(\cdot;u)$ *is linear operator for all* $u\in X$*;*

iii. $\Xi(v;u_m)\rightharpoonup\theta(v;u)$, *whenever* $u_m\rightharpoonup u$.

3. Key Findings

Within the scope of this section, we investigate the newly discovered category of vector hemivariational-like inequalities that are defined within (HTVS). Using the properties of the be $\zeta-\Xi$-monotone operator, we construct a structure that extends the domain of conventional variational inequality theories to include more complex functional linkages. This framework is a result of our utilization of these qualities.

Theorem 3.1 *Supposing M to be (HTVS) with D be a closed, nonempty, convex subset of M. For every λ in D let $(Y,\varpi(u))$ be an OTLS in which int $\varpi(\mho)\neq\emptyset$. Let $\zeta:\mathrm{M}\times\mathrm{M}\to\Upsilon$ be a bifunction that fulfills the conditions $(\mathrm{H}_1-\mathrm{H}_4)$. Considering that $\mathrm{E}:\mathrm{D}\times\mathrm{D}\to\mathrm{M},\mathrm{g}:\mathrm{D}\times\mathrm{D}\to\mathrm{M}$. Then, The issues are equivalent. Let $\lambda\in D,\forall\psi\in D$ in which*

$$\langle \mathrm{T}(\lambda),\Xi(\psi,\lambda)\rangle+J^0(\lambda,\Xi(\psi,\lambda))+\mathrm{g}(\psi,\lambda)\notin-\mathrm{int}\,\varpi(\lambda).$$

(A) Let $\lambda\in\mathrm{D},\forall\psi\in\mathrm{D}$ in which

$$\langle \mathrm{T}(\lambda),\Xi(\psi,\lambda)\rangle+\zeta(\psi,\lambda)+J^0(\lambda,\Xi(\psi,\lambda))+\mathrm{g}(\psi,\lambda)\notin-\mathrm{int}\,\varpi(\lambda).$$

Proof: If (A) is valid, $\lambda\in\mathrm{D}$, can exist.

$$\langle \mathrm{T}(\lambda),\Xi(\psi,\lambda)\rangle+J^0(\lambda,\Xi(\psi,\lambda))+\mathrm{g}(\psi,\lambda)\notin-\mathrm{int}\,\varpi(\lambda),\forall\psi\in\mathrm{D}.$$

The fact that T is $\zeta-\Xi$-monotone allows one to obtain the result.

$$\langle \mathrm{T}(\psi)-\mathrm{T}(\lambda),\Xi(\psi,\lambda)\rangle+\zeta(\psi,\lambda)\in\varpi\quad\forall\lambda,\psi\in\mathrm{D}.$$

Indeed, it is accurate.

$$\langle \mathrm{T}(\lambda),\Xi(\psi,\lambda)\rangle-\langle \mathrm{T}(\psi),\Xi(\psi,\lambda)\rangle-\zeta(\psi,\lambda)\in-\varpi,\forall\psi\in\mathrm{D}.$$

From, $\varpi=\bigcap_{\lambda\in\mathrm{D}}\varpi(\lambda)$, then $\forall\psi\in\mathrm{D}$

$$\begin{aligned}\big[\langle \mathrm{T}(\lambda),\Xi(\psi,\lambda)\rangle &+J^0(\lambda,\Xi(\psi,\lambda))+\mathrm{g}(\psi,\lambda)\big]-\big[\langle \mathrm{T}(\psi),\Xi(\psi,\lambda)\rangle+\\ &J^0(\lambda,\Xi(\psi,\lambda))+\zeta(\psi,\lambda)+\mathrm{g}(\psi,\lambda)\big]\in-\varpi\subset-\varpi(\lambda)\end{aligned}$$

After applying Remark (2.1), then $\forall\psi\in\mathrm{D}$,

$$\langle \mathrm{T}(\psi)+\Xi(\psi,\lambda)\rangle+J^0(\lambda,\Xi(\psi,\lambda))+\mathrm{g}(\psi,\lambda)+\zeta(\psi,\lambda)\notin-\mathrm{int}\,\varpi(\lambda).$$

Given the fact that this is the situation, the answer to formula (B) is denoted by the symbol λ. In contrast, if (B) turns out to be true, then it does. all values of $\psi\in D$, where $\lambda_0\in D$. After that,

$$\langle \mathrm{T}(\psi),\Xi(\psi,\lambda_0)\rangle+J^0(\lambda_0,\Xi(\psi,\lambda_0))+\zeta(\psi,\lambda_0)+\mathrm{g}(\psi,\lambda_0)\notin-\mathrm{int}\,\varpi(\lambda_0),$$

for each $\psi\in\mathrm{D},\mathrm{t}\in(0,1)$, let $\psi_{\mathrm{t}}=\mathrm{t}\psi+(1-\mathrm{t})\lambda_{\mathrm{o}}$. It appears to be obvious that $\psi_{\mathrm{t}}\in\mathrm{D}$, then

$$\langle \mathrm{T}(\psi_{\mathrm{t}}),\Xi(\psi_{\mathrm{t}},\lambda_0)\rangle+J^0(\lambda_0,\Xi(\psi_{\mathrm{t}},\lambda_0))+\zeta(\psi_{\mathrm{t}},\lambda_0)+\mathrm{g}(\psi_{\mathrm{t}},\lambda_0)\notin-\mathrm{int}\,\varpi(\lambda_0).$$

With the affine $\mathrm{g},\mathrm{Xi},\mathrm{H}_1$ and H_4 at one's disposal, it is possible to

$$\begin{aligned}&\langle \mathrm{T}(\mathrm{t}\psi+(1-\mathrm{t})\lambda_0),\mathrm{t}\Xi(\psi,\lambda_0)\rangle+J^0(\lambda_0,\mathrm{t}\Xi(\psi,\lambda_0))+\mathrm{tg}(\psi,\lambda_0)\\ &=\mathrm{t}\langle\ \mathrm{T}(\mathrm{t}\psi+(1-\mathrm{t})\lambda_0),\Xi(\psi,\lambda_0)\rangle+\mathrm{t}J^0(\lambda_0,\Xi(\psi,\lambda_0))+\zeta(\mathrm{t}\psi+(1-\mathrm{t})\lambda_0,\lambda_0)+\mathrm{tg}(\psi,\lambda_0)\notin-\mathrm{int}\,\varpi(\lambda_0)\\ &=\langle \mathrm{T}(\mathrm{t}\psi+(1-\mathrm{t})\lambda_0),\Xi(\psi,\lambda_0)\rangle+J^0(\lambda_0,\Xi(\psi,\lambda_0))+\frac{\zeta(\mathrm{t}\psi+(1-\mathrm{t})\lambda_0,\lambda_0)}{t}+\mathrm{g}(\psi,\lambda_0)\notin-\mathrm{int}\,\varpi(\lambda_0)\end{aligned}$$

Based on the assumptions that $\mathrm{t}\to0^+$, and the $\Xi-$ H.C of T and H_1, we derive the following:

$$\langle \mathrm{T}(\lambda_0), \Xi(\psi, \lambda_0)\rangle + J^0(\lambda_0, \Xi(\psi, \lambda_0)) + \mathrm{g}(\psi, \lambda_0) \notin -\operatorname{int} \varpi(\lambda_0), \forall \psi \in D.$$

The proof is completed. □

Consider the topological vector space M to be a convex and closed set, and let us call it D . Additionally, we would like to define $\{\varpi(\sigma) : \sigma \in \mathrm{D}\}$ as a collection of cones that are convex, closed, and directed within the topological space Υ. This ensures that $\operatorname{int} \sigma(\sigma) \neq \emptyset$ for any $\sigma \in \mathrm{D}$. A setvalued map is employed in this assignment. For all $\sigma \in \mathrm{D}, \varpi : \mathrm{D} \to 2^{\mathrm{E}}$ as demonstrated $\varpi(\lambda) = \Upsilon \backslash \{-\operatorname{int} \varpi(\lambda)\}$, for each $\lambda \in D\}$.

Theorem 3.2 *Assume that M is an HTVS with a non-empty set D is compact, closed, and convex of M. Following that, and and $(\Upsilon, \varpi(\lambda))$ is an OTLS, as int $\varpi(\lambda) \neq \varnothing$ for every $\lambda \in D$. Let $\Theta : D \times D \to M$ and $g : D \times D \to M$ are affine bifunctions in which $\Xi(\lambda, \lambda) = 0 = g(\lambda, \lambda)$, for any $\lambda \in K$: The conditions $(\mathrm{H}_1 - \mathrm{H}_4)$ are satisfied, and $\zeta : \mathrm{M} \times \mathrm{M} \to \Upsilon$ are continuous bifunctions. If a function $\varpi : D \to 2^{\Upsilon}$ is an upper semicontinuous. Consequently, there is $\lambda_0 \in D$ in which*

$$< \mathrm{T}(\lambda_0), \Xi(\psi, \lambda_0) > + J^0(\lambda, \Xi(\psi, \lambda_0)) + \mathrm{g}(\psi, \lambda_0) \notin -\operatorname{int} \varpi(\lambda_0) \quad \forall \psi \in \mathrm{D}.$$

Proof: For $\psi \in \mathrm{D}$, we define

$$\mathrm{L}_1(\psi) = \left\{\lambda \in \mathrm{D} :< \mathrm{T}(\lambda), \Xi(\psi, \lambda) > + J^0(\lambda, \Xi(\psi, \lambda)) + \mathrm{g}(\psi, \lambda) \notin -\operatorname{int} \varpi(\lambda)\right\};$$
$$\mathrm{L}_2(\psi) = \left\{\lambda \in \mathrm{D} :< \mathrm{T}(\psi), \Xi(\psi, \lambda) > + J^0(\lambda, \Xi(\psi, \lambda)) + \zeta(\psi, \lambda) + \mathrm{g}(\psi, \lambda) \notin -\operatorname{int} \varpi(\lambda)\right\}.$$

Considering this, L_1 and L_2 are not empty. For your convenience, we have broken up our proof into three separate pieces.
The first step: We will start by introducing L_1 is a KKM mapping. Now, if L_1 is not a KKM so there are $v_1, v_2, \ldots, v_{\mathrm{n}} \in D$ and $\Lambda_1 \geq 0, \Lambda_2 \geq 0, \ldots, \Lambda_{\mathrm{m}} \geq 0$ through $\sum_{\mathrm{i}=1}^{\mathrm{m}} \Lambda_{\mathrm{i}} = 1$ and $\mho = \sum_{\mathrm{i}=1}^{\mathrm{m}} \Lambda_{\mathrm{i}} v_{\mathrm{i}}$ in which $\mho \notin \bigcup_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{L}_1(v_{\mathrm{i}})$. This indicates that, for every $\mathrm{i} = 1, 2, \ldots,$ m.

$$< \mathrm{T}(\mho), \Xi(v_{\mathrm{i}}, \mho) > + J^0(\mu, \Xi(v_{\mathrm{i}}, \mho)) + \mathrm{g}(v_{\mathrm{i}}, \mho) \in -\operatorname{int} \varpi(\mho).$$

Given affine Ξ, g , and semicontinuous $\mathcal{H}$, for all $\mho \in \mathrm{D}$, we have

$$\begin{aligned} &< \mathrm{T}(\mho), \Xi(\mho, \mho) > + J^0(\mho, \Xi(\mho, \mho)) + \mathrm{g}(\mho, \mho) \\ &\quad = < \mathrm{T}(\mho), \Xi\left(\sum_{\mathrm{i}=1}^{\mathrm{m}} \Lambda_{\mathrm{i}} v_{\mathrm{i}}, \mho\right) > + J^0\left(\mho, \Xi\left(\sum_{\mathrm{i}=1}^{\mathrm{m}} \Lambda_{\mathrm{i}} v_{\mathrm{i}}, \mho\right)\right) + \mathrm{g}\left(\sum_{\mathrm{i}=1}^{\mathrm{m}} \Lambda_{\mathrm{i}} v_{\mathrm{i}}, \mho\right) \\ &\quad = \sum_{\mathrm{i}=1}^{\mathrm{m}} \Lambda_{\mathrm{i}} \left[< \mathrm{T}(\mho), \Xi(v_{\mathrm{i}}, \mho) > + J^0(\mho, \Xi(v_{\mathrm{i}}, \mho)) + \mathrm{g}(v_{\mathrm{i}}, \mho)\right] \in -\operatorname{int} \varpi(\mho) \end{aligned}$$

When we look at it from the other side, we notice that $\Theta(\mho, \mho) = g(\mho, \mho) = 0$, and we find

$$0 =< \mathrm{T}(\mho), \Xi(\mho, \mho) > + J^0(\mho, \Xi(\mho, \mho)) + \mathrm{g}(\mho, \mho) \in -\operatorname{int} \varpi(\mho).$$

The mapping $L_1 : D \to 2^D$ is classified as a KMM due to its impossibility of execution.
Step two requires the articulation of

$$\bigcap_{\psi \in \mathrm{D}} \mathrm{L}_1(\psi) = \bigcap_{\psi \in \mathrm{D}} \mathrm{L}_2(\psi)$$

Actually, if $\lambda \in L_1(\psi)$ then

$$< \mathrm{T}(\lambda), \Xi(\psi, \lambda) > + J^0(\lambda, \Xi(\psi, \lambda)) + \mathrm{g}(\psi, \lambda) \notin -\operatorname{int} \varpi(\lambda), \text{ from } \mathrm{H}_3 \text{ we get}$$
$$< \mathrm{T}(\psi) - \mathrm{T}(\lambda), \Xi(\psi, \lambda) > + \zeta(\psi, \lambda) \in \varpi,$$

So,

$$\begin{aligned}&\left[< \mathrm{T}(\lambda), \Xi(\psi,\lambda) > + J^0(\lambda, \Xi(\psi,\lambda)) + \mathrm{g}(\psi,\lambda)\right] \\ &\quad - \left[< \mathrm{T}(\psi), \Xi(\psi,\lambda) > - J^0(\lambda, \Xi(\psi,\lambda)) + \zeta(\psi,\lambda) + \mathrm{g}(\psi,\lambda)\right] \in -\varpi \subset -\varpi(\lambda).\end{aligned}$$

This becomes possible via the Remark (2.1).

$$< \mathrm{T}(\psi), \Xi(\psi,\lambda) > + J^0(\lambda, \Xi(\psi,\lambda)) + \zeta(\psi,\lambda) + \mathrm{g}(\psi,\lambda) \notin -\operatorname{int}\varpi(\lambda).$$

On the other hand, we are able to $\lambda \in \mathrm{L}_2(\psi)$ for each $\psi \in \mathrm{D}$. That means, $\mathrm{L}_1(\psi) \subset \mathrm{L}_2(\psi)$. Thus, $\bigcap_{\psi\in \mathrm{D}} \mathrm{L}_1(\psi) \subset \bigcap_{\psi\in \mathrm{D}} \mathrm{L}_2(\psi)$.
Conversely, one can logically deduce that $\lambda \in \bigcap_{\psi\in D} L_2(\psi)$. Consequently,

$$< \mathrm{T}(\psi), \Xi(\psi,\lambda) > + J^0(\lambda, \Xi(\psi,\lambda)) + \zeta(\psi,\lambda) + \mathrm{g}(\psi,\lambda) \notin -\operatorname{int}\varpi(\lambda), \forall \psi \in \mathrm{D}.$$

The subsequent steps derived from Theorem 3.1 are as follows:

$$< \mathrm{T}(\lambda), \Xi(\psi,\lambda) > + J^0(\lambda, \Xi(\psi,\lambda)) + \mathrm{g}(\psi,\lambda) \notin -\operatorname{int}\varpi(\lambda), \forall \psi \in \mathrm{D}.$$

So,
$\lambda \in \bigcap_{\psi\in \mathrm{D}} \mathrm{L}_1(\psi)$ and $\bigcap_{\psi\in \mathrm{D}} \mathrm{L}_2(\psi) \subset \bigcap_{\psi\in \mathrm{D}} \mathrm{L}_1(\psi)$. It shows that $\bigcap_{\psi\in \mathrm{D}} \mathrm{L}_1(\psi) = \bigcap_{\psi\in \mathrm{D}} \mathrm{L}_2(\psi)$.
In the third stage, we state that $\bigcap_{\psi\in \mathrm{D}} \mathrm{L}_2(\psi) \neq \emptyset$.
In considering knowing that L_1 is a KKM mapping, we are aware of the element that, for each finite set $\{\psi_1, \psi_2, \ldots, \psi_{\mathrm{n}}\} \in \mathrm{D}$, it is required for one to possess certain characteristics.

$$\operatorname{conv}\{\psi_1, \psi_2, \ldots, \psi_{\mathrm{n}}\} \subset \bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{L}_1(\psi_{\mathrm{i}}) \subset \bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{L}_2(\psi_{\mathrm{i}})$$

So, L_2 is a KKM map is clear. Our goal to show $\mathrm{L}_2(\psi)$ is closed for any $\psi \in \mathrm{D}$. Assume that there is a net $\{\lambda_\alpha\} \subset L_2(\psi)$ by $\lambda_\alpha \to \lambda \in D$, one can get

$$< \mathrm{T}(\psi), \Xi(\psi,\lambda_\alpha) > + J^0(\lambda_\alpha, \Xi(\psi,\lambda_\alpha)) + \zeta(\psi,\lambda_\alpha) + \mathrm{g}(\psi,\lambda_\alpha) \notin -\operatorname{int}\varpi(\lambda_\alpha).$$

With the utilization of the definition of wit is possible to acquire

$$< \mathrm{T}(\psi), \Xi(\psi,\lambda_\alpha) > + J^0(\lambda_\alpha, \Xi(\psi,\lambda_\alpha)) + \zeta(\psi,\lambda_\alpha) + \mathrm{g}(\psi,\lambda_\alpha) \notin \varpi(\lambda_\alpha).$$

It may be deduced that considering the continuous quality that is pertinent to both Θ and $\mathbf{g}$, is approach to

$$< \mathrm{T}(\psi), \Xi(\psi,\lambda_\alpha) > + J^0(\lambda_\alpha, \Xi(\psi,\lambda_\alpha)) + \zeta(\psi,\lambda_\alpha) + \mathrm{g}(\psi,\lambda_\alpha)$$

$$< \mathrm{T}(\psi), \Xi(\psi,\lambda) > + J^0(\lambda, \Xi(\psi,\lambda)) + \zeta(\psi,\lambda) + \mathrm{g}(\psi,\lambda).$$

This is due to the element that ϖ is a lsc by nearby values. We may prove that ϖ is closed using Lemma 2.2, we can determine that ϖ is closed; consequently,

$$< \mathrm{T}(\psi), \Xi(\psi,\lambda) > + J^0(\lambda, \Xi(\psi,\lambda)) + \zeta(\psi,\lambda) + \mathrm{g}(\psi,\lambda) \in \varpi(\lambda).$$

It is clear from this that

$$< \mathrm{T}(\psi), \Xi(\psi,\lambda) > + J^0(\lambda, \Xi(\psi,\lambda)) + \zeta(\psi,\lambda) + \mathrm{g}(\psi,\lambda) \notin \operatorname{int}\varpi(\lambda),$$

The conclusion is that $\mathrm{L}_2(\psi)$ is closed. Because of the compactness of D and the proximity of $\mathrm{L}_2(\psi) \subset \mathrm{D}$, the following is the consequence. The compactness of $\mathrm{L}_2(\psi)$ can be observed by everyone. Through the utilization of Theorem 2.1, it is possible to acquire it.

$$\bigcap_{\psi \in \mathrm{D}} \mathrm{L}_2(\psi) \neq \emptyset$$

This leads to the conclusion that

$$\bigcap_{\psi \in \mathrm{D}} \mathrm{L}_1(\psi) \neq \emptyset.$$

This implies, there is $\lambda_0 \in \mathrm{D}$ in which

$$< \mathrm{T}(\lambda_0), \Xi(\psi, \lambda_0) > + J^0(\lambda_0, \Xi(\psi, \lambda_0)) + \mathrm{g}(\psi, \lambda_0) \notin -\operatorname{int} \varpi(\lambda_0), \forall \psi \in \mathrm{D}.$$

The result is that (GVHLI) is resolved. The evidence is complete. □

Example 3.1 Through the consideration of a straightforward mechanical system that is driven by non-smooth energy components, we demonstrate the applicability of the generalized vector hemivariational-like inequalities (GVHLI) model.

Take into consideration a particle that is moving along a rough surface because of dry friction. This particle is characterized within a Hausdorff topological linear space $E = \mathbb{R}$. In order to define the positive cone, let us assume that the feasible set is $\mathrm{D} = [0, 1]$.
we'll define the positive cone as $\varpi = \mathbb{R}+$. Specify the mappings that are as follows:
$T(\lambda) = k \cdot u$ is a linear operator that represents the surface's stiffness when $k > 0$ (for example, $k = 2$). In the local Lipschitz function $J(\lambda) = \mho \cdot |\lambda|$, the coefficient of dry friction is denoted by $\mho > 0$ (e.g., $\mho = 0.5$).

The displacement difference is represented by the bifunction $\Xi(\psi, \lambda) = \psi - \lambda, g(\psi, \lambda) = \alpha \cdot (\psi - \lambda)^2$ is the perturbation function, where α exceeds 0 (e.g., $\alpha = 0.1$).

Next, the GVHLI issue looks for $\lambda \in \mathrm{D}$ so that for every $\psi \in \mathrm{D}$:

$$< \mathrm{T}(\lambda), \Xi(\psi, \lambda) > + J^0(\lambda, \Xi(\psi, \lambda)) + \mathrm{g}(\psi, \lambda) \notin -\operatorname{int} \varpi(\lambda).$$

Changing the definitions results in:

If ψ is in the interval $[0, 1]$, then $k \cdot \lambda \cdot (\psi - \lambda) + \mho \cdot \operatorname{sgn}(\lambda) \cdot (\psi - \lambda) + \alpha \cdot (\psi - \lambda)^2 \geq 0.$ where the sign function is indicated by $\operatorname{sgn}(\lambda)$. Let us choose $\lambda = 0.3$ and $\psi = 0.4$:

$$\begin{aligned} & 2 \cdot 0.3 \cdot (0.4 - 0.3) + 0.5 \cdot 1 \cdot (0.4 - 0.3) + 0.1 \cdot (0.4 - 0.3)^2 \\ & = 0.06 + 0.05 + 0.001 = 0.111 > 0 \end{aligned}$$

Consequently, the inequality is satisfied, indicating that the system is in equilibrium under the specified hypotheses. This example demonstrates the potential of the GVHLI framework to model mechanical systems that entail non-smooth phenomena, such as friction or unilateral constraints.

4. Conclusion and Upcoming Projects

4.1. Conclusion

The $\boldsymbol{\zeta} - \boldsymbol{\Xi}$ monotone operator is employed to create a novel category of inequalities in Hausdorff topological linear spaces (HTVS) that are analogous to vector hemivariational inequalities. The study demonstrates the similarity of this novel class of inequalities to vector hemivariational inequalities by employing the FKKM method and applying appropriate restrictions on the nonlinear mappings. This research contributes to the solutions proposed by other authors by enhancing our theoretical and practical comprehension of specific discrepancies.

4.2. Upcoming Projects

The current analysis suggests several research avenues. One possibility is applying the proved results to larger topological vector spaces, ones without local convexity. In addition, introducing uncertainty or stochastic components to the hemivariational-like framework may make it more suitable for noisy or insufficient data systems. Another possibility is designing and testing numerical iterative methods to address inequality concerns. Finally, applications in contact mechanics, economics, and control theory, which naturally entail non-smooth and multi-criteria equilibrium issues, will validate this work's theoretical contributions.

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