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A Note on Frobenius–Tangent–Fibonacci Polynomials

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ABSTRACT: In this study, we develop a new family of two-variable Frobenius–Tangent–Fibonacci polynomials and their corresponding numerical sequences within the framework of the Golden F–Calculus. By employing generating functions, we establish several essential algebraic and analytic properties, including recurrence relations, summation formulas, symmetry identities, and F–derivative representations. Moreover, we reveal explicit connections between these polynomials and the Stirling–Fibonacci numbers of the second kind, deriving multiple summation and convolution-type identities. The framework is further extended through the introduction of parametric generalizations featuring trigonometric generating mechanisms, which are explored via F–differential operator methods and functional equations. The results presented herein broaden the scope of Fibonacci-based special polynomial theory and provide potential tools for future applications in combinatorics, number theory, approximation theory, and matrix analysis.

Keywords: Golden calculus, tangent polynomials, Frobenius-Tangent-Fibonacci polynomials, symmetric identities, stirling numbers of the second kind.

Contents

1. Introduction

In recent years, there has been a growing interest among mathematicians [1,6,11,7,8,17] in constructing and analyzing generating functions associated with new classes of Fibonacci-based special polynomials, including the F–Bernoulli, F–Euler, and F–Genocchi families. Through the frameworks of Frobenius–Euler–Fibonacci and Frobenius–Euler–Genocchi–Fibonacci structures, these studies have unveiled rich algebraic and analytic properties, encompassing summation theorems, recurrence relations, symmetry identities, and intricate connections with classical polynomial systems. Furthermore, such explorations have led to elegant interrelations among different polynomial families, providing unified approaches to derive their properties and extensions. Building upon these advancements, the present work introduces a new class of Frobenius–Tangent–Fibonacci polynomials and investigates their analytical behavior. We establish several fundamental results, including addition and recurrence formulas, implicit summation identities, and relationships with previously defined special polynomials [18,19,20,21,22,23,24], thereby expanding the theoretical foundation of F–based polynomial structures.

The Apostol–type Frobenius–Euler polynomials $\mathcal{H}_{\omega}^{(\alpha)}(\xi;\mathfrak{U};\lambda)$ of order $\alpha \in \mathbb{C}$ are defined by [7,8,13]:

$$\left(\frac{1-\mathfrak{U}}{\lambda e^{d}-\mathfrak{U}}\right)^{\alpha} e^{\xi d} = \sum_{\omega=0}^{\infty} \mathcal{H}_{\omega}^{(\alpha)}(\xi;\mathfrak{U};\lambda)\frac{d^{\omega}}{\omega!}, \tag{1.1}$$

where $\mathfrak{U} \in \mathcal{C} \setminus \{1\}$, $\xi \in \mathcal{R}$ and $|d| < \left|\log\left(\frac{\lambda}{\mathfrak{U}}\right)\right|$.

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At the point $\xi=0$, $\mathcal{H}_{\omega}^{(\alpha)}(\mathfrak{U};\lambda)=\mathcal{H}_{\omega}^{(\alpha)}(0;\mathfrak{U};\lambda)$ are called the Apostol–type Frobenius–Euler numbers of order α. From (1.1), we find

$$\mathcal{H}_{\omega}^{(\alpha)}(\xi;\mathfrak{U};\lambda)=\sum_{\nu=0}^{\omega}\left(\begin{array}{c}\omega\\ \nu\end{array}\right)\mathcal{H}_{\nu}^{(\alpha)}(\mathfrak{U};\lambda)\xi^{\omega-\nu}. \tag{1.2}$$

Ryoo defined the Frobenius-Tangent polynomials through the following generating function [29] :

$$\sum_{j=0}^{\infty}\mathbb{T}_{j,F}(\xi;\mathfrak{U})\frac{d^j}{j!}=\frac{1-\mathfrak{U}}{e^{2d}-\mathfrak{U}}e^{\xi d}. \tag{1.3}$$

Note that, for $\xi=0$, the Fibonacci Tangent polynomials become the Fibonacci Tangent numbers. The author then gave several identities for these types of polynomials and numbers. For more details related to Fibonacci-based special polynomials, please refer to [7,8,9,10,11,12,13]. For $j\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, the Stirling numbers of the first kind are defined by [25,26,27]

$$(\xi)_j=\sum_{p=0}^{j}S_1(j,p)\xi^p, \tag{1.4}$$

where $(\xi)_0=1$, and $(\xi)_j=\xi(\xi-1)\cdots(\xi-j+1),(j\geq 1)$. From (1.4), we get

$$\frac{1}{r!}(\log(1+d))^r=\sum_{j=r}^{\infty}S_1(j,r)\frac{d^j}{j!},\quad (r\geq 0). \tag{1.5}$$

For $j\in\mathbb{N}_0$, the Stirling numbers of the second kind are defined by [28,15,16]

$$\xi^j=\sum_{q=0}^{j}S_2(j,q)(\xi)_q. \tag{1.6}$$

From (1.6), we see that

$$\frac{1}{k!}(e^d-1)^k=\sum_{j=k}^{\infty}S_2(j,k)\frac{d^j}{j!}. \tag{1.7}$$

For any non-negative integer r, the r–Stirling numbers $S_{2,r}(j,k)$ of the second kind are defined by [7]

$$\frac{1}{k!}e^{rd}(e^d-1)^k=\sum_{j=k}^{\infty}S_{2,r}(j+r,k+r)\frac{d^j}{j!}. \tag{1.8}$$

The Fibonacci sequence is defined by the following recurrence relation:

$$F_n=F_{n-1}+F_{n-2},\quad n\geq 2$$

with $F_0=0$, $F_1=1$. Fibonacci numbers can be expressed explicitly as

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},\quad n\in\mathbb{N}_0$$

where $\alpha=\frac{1+\sqrt{5}}{2}$ (Golden ratio) and $\beta=\frac{1-\sqrt{5}}{2}$. The use of the golden ratio in various branches of science and mathematics is well-documented. Additionally, this mystical number also makes appearances in the fields of architecture and art. Recently, the properties of F–calculus have been exhaustively defined and studied by Pashaev and Nalci [2]. Please refer to the following papers for further reading: Krot [4], Özvatan [5], Pashaev [3], and Kuş et al. [6]. These sources provide additional information and insights on the topic at hand.

For $w \in \mathbb{N}$, the F–factorial, derived from the Fibonacci sequence, was formally characterized as follows:

$$F_1F_2F_3\dots F_w = F_w!, \tag{1.9}$$

where $F_0! = 1$.

Presented here is the formal representation of the binomial theorem as it applies to F–analogues, often referred to as the Golden binomial theorem:

$$(a+b)_F^w = \sum_{s=0}^{w} (-1)^{\binom{s}{2}} \binom{w}{s}_F a^{w-s}b^s, \tag{1.10}$$

or

$$(a +_F b)^w = \sum_{s=0}^{w} \binom{w}{s}_F a^{w-s}b^s, \tag{1.11}$$

in terms of the Golden binomial coefficients, called as Fibonomials

$$\binom{u}{s}_F = \frac{F_u!}{F_{u-s}!F_s!},$$

with u and s being non-negative integers, $u \geq s$. The Golden derivative (F–Derivative Formula) defined as follows:

$$\frac{d_F}{d_F\xi}\left(g(\xi)\right) = \frac{g\left(\alpha\xi\right) - g\left(-\frac{\xi}{\alpha}\right)}{\left(\alpha - \left(-\frac{1}{\alpha}\right)\right)\xi} = \frac{g\left(\alpha\xi\right) - g\left(\beta\xi\right)}{\left(\alpha-\beta\right)\xi}. \tag{1.12}$$

The first and second types of Golden exponential functions are represented as

$$e_F^{\xi} = \sum_{s=0}^{\infty} \frac{(\xi)_F^s}{F_s!}, \tag{1.13}$$

and

$$E_F^{\xi} = \sum_{s=0}^{\infty} (-1)^{\binom{s}{2}} \frac{(\xi)_F^s}{F_s!}, \tag{1.14}$$

respectively. Briefly, we use this notation throughout the article

$$e_F^{\xi} = \sum_{s=0}^{\infty} \frac{\xi^s}{F_s!}, \tag{1.15}$$

and

$$E_F^{\xi} = \sum_{s=0}^{\infty} (-1)^{\binom{s}{2}} \frac{\xi^s}{F_s!}. \tag{1.16}$$

Using these functions (see [4,2]), the following equations can be given:

$$e_F^{\xi}e_F^{\eta} = e_F^{(\xi +_F \eta)}, \tag{1.17}$$

$$e_F^{\xi}E_F^{\eta} = e_F^{(\xi+\eta)_F}. \tag{1.18}$$

The Fibonacci cosine and sine functions, commonly known as the Golden trigonometric functions, are expressed through the power series

$$\cos_F\left(\xi\right) = \sum_{s=0}^{\infty} (-1)^s \frac{\xi^{2s}}{F_{2s}!}, \tag{1.19}$$

and

$$\sin_F\left(\xi\right) = \sum_{s=0}^{\infty} (-1)^s \frac{\xi^{2s+1}}{F_{2s+1}!}. \tag{1.20}$$

For arbitrary number ϕ, Golden derivatives of $e_F^{\phi\xi}$, $E_F^{\phi\xi}$, $\cos_F(\phi\xi)$, and $\sin_F(\phi\xi)$ functions are given as follows:

$$\frac{d_F}{d_F\xi}\left(e_F^{\phi\xi}\right)=\phi e_F^{\phi\xi}, \tag{1.21}$$

$$\frac{d_F}{d_F\xi}\left(E_F^{\phi\xi}\right)=\phi E_F^{-\phi\xi}, \tag{1.22}$$

$$\frac{d_F}{d_F\xi}\left(\cos_F(\phi\xi)\right)=-\phi\sin_F(\phi\xi), \tag{1.23}$$

and

$$\frac{d_F}{d_F\xi}\left(\sin_F(\phi\xi)\right)=\phi\cos_F(\phi\xi). \tag{1.24}$$

In 2024, Alatawi et al. [7] introduced the generalized Apostol-type Frobenius-Euler-Fibonacci polynomials as:

$$\left(\frac{1-\mathfrak{U}}{e_F^d-\mathfrak{U}}\right)^\alpha e_F^{\zeta d}E_F^{\eta d}=\sum_{w=0}^{\infty}\mathcal{H}_{w,F}^{(\alpha)}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!},\quad \mid d\mid\neq\pi, \tag{1.25}$$

where $\mathfrak{U}\in\mathcal{C}$ with $\mathfrak{U}\neq 1$, and $w\in\mathcal{N}_0$.

Note that, for $\xi=0$, the generalized Apostol-type Frobenius-Euler-Fibonacci polynomials become the generalized Apostol-type Frobenius-Euler-Fibonacci numbers. The author then gave several identities for these types of polynomials and numbers. For more details related to Fibonacci-based special polynomials, please refer to [7,8,9,10,11,12,13].

The Stirling–Fibonacci numbers of the second kind are defined by [14,7]

$$\sum_{k=m}^{\infty}S_2^F(k,m)\frac{d^k}{F_k!}=\frac{(e_F^d-1)^m}{F_m!}. \tag{1.26}$$

The overall organization of this paper is structured as follows. Section 2 introduces a new class of two-variable Frobenius–Tangent–Fibonacci polynomials and their corresponding numbers, formulated within the framework of Golden Calculus through Equation (2.1). In this section, we develop their generating functions and utilize functional equations to establish several core properties, including recurrence relations, summation formulas, and F–derivative identities. Section 3 extends these foundational results by exploring the parametric forms of the Frobenius–Tangent–Fibonacci polynomials in the bivariate setting. Within this framework, we investigate their structural characteristics and derive a variety of related identities using advanced techniques from Golden Calculus. This systematic approach provides deeper insights into the algebraic and analytic behavior of these polynomial families.

2. Properties of Frobenius–Tangent–Fibonacci Polynomials

In this section, we define Frobenius–Sigmoid–Fibonacci polynomials and numbers. Then, we derive the basic properties such as additivity and symmetrically regarding related to it. Lastly, we introduce certain properties of these numbers and polynomials.

Definition 2.1 Let $\mathfrak{U}\in\mathcal{C}$ with $\mathfrak{U}\neq 1$, and $w\in\mathcal{N}_0$. Then, we can define the Frobenius–Tangent–Fibonacci polynomials $\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})$:

$$\frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}e_F^{\zeta d}E_F^{\eta d}=\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!},\quad \mid d\mid\neq\pi. \tag{2.1}$$

Letting $\mathfrak{U}=-1$ in (2.1), we get

$$\frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}e_F^{\zeta d}E_F^{\eta d}=\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!},$$

where, $\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})$ are called the w^{th}–Tangent–Fibonacci polynomials.

Again, put $\zeta=\eta=0$ in (2.1), we note that

$$\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(0;\mathfrak{U})\frac{d^w}{F_w!}=\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\mathfrak{U})\frac{d^w}{F_w!}=\frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}},$$

where, $\mathcal{T}_{w,F}(\mathfrak{U})$ are called the w^{th}- Frobenius–Sigmoid–Fibonacci numbers.

To derive structural properties of these polynomials, we now prove a series of summation, recurrence, and convolution–type identities.

Theorem 2.1 *The following summation formulas for the polynomials $\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})$ hold true:*

$$\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})=\sum_{j=0}^{w}\left(\begin{array}{c}w\\ j\end{array}\right)_F\mathcal{T}_{j,F}(\mathfrak{U})(\zeta+\eta)_F^{w-j},\tag{2.2}$$

$$\mathcal{T}_{j,F}(\zeta,\eta;\mathfrak{U})=\sum_{j=0}^{w}\left(\begin{array}{c}w\\ j\end{array}\right)_F\mathcal{T}_{j,F}(0,\eta;\mathfrak{U})\zeta^{w-j},\tag{2.3}$$

$$\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})=\sum_{j=0}^{w}\left(\begin{array}{c}w\\ j\end{array}\right)_F\mathcal{T}_{w-j,F}(\zeta;\mathfrak{U})(-1)^{\frac{j(j-1)}{2}}\eta^j.\tag{2.4}$$

Proof: By appropriately applying equations (1.2) and (1.8) within the generating function (2.1), three distinct forms are derived. Subsequently, the F–Cauchy product rule is utilized in the resulting expressions, and by comparing the corresponding powers of d on both sides of the resultant equation, we obtain formulas (2.2)-(2.4). □

Theorem 2.2 *The following recursive formulas for the two variable Frobenius–Tangent–Fibonacci polynomials $\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})$ hold true:*

$$\frac{\partial_F}{\partial_F\zeta}\{\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})\}=F_w\mathcal{T}_{w-1,F}(\zeta,\eta;\mathfrak{U}),\tag{2.5}$$

and

$$\frac{\partial_F}{\partial_F\eta}\{\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})\}=F_w\mathcal{T}_{w-1,F}(\zeta,-\eta;\mathfrak{U}),\tag{2.6}$$

where $\frac{\partial_F}{\partial_F\zeta}$ is F–partial derivative operator.

Proof: Differentiating generating function (2.1) with respect to ζ and η, utilizing equations (1.3) and (2.3), and subsequently simplifying through the application of the F–Cauchy product rule formulas yields the results given in (2.5) and (2.6). □

Theorem 2.3 *Let $w\geq 0$ be an integer. Then we have*

$$\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})=\sum_{k=0}^{w}\sum_{m=0}^{k}\binom{w}{k}\mathcal{T}_{w-k,F}(\eta;\mathfrak{U})(\zeta)_m S_{2,F}(k,m).\tag{2.7}$$

Proof: By virtue (1.26) and (2.1), we find

$$\begin{aligned}\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!} &= \frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}(e_F^d-1+1)^{\zeta}E_F^{\eta d}\\ &= \frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}E_F^{\eta d}\sum_{m=0}^{\infty}(\zeta)_m\frac{(e_F^d-1)^m}{F_m!}\\ &= \frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}E_F^{\eta d}\sum_{m=0}^{\infty}(\zeta)_m\sum_{k=m}^{\infty}S_{2,F}(k,m)\frac{d^k}{F_k!}\\ &= \sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\eta;u)\frac{d^w}{F_w!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}(\zeta)_m S_{2,F}(k,m)\frac{d^k}{F_k!}\\ &= \sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\sum_{m=0}^{k}\binom{w}{k}\mathcal{T}_{w-k,F}(\eta;u)(\zeta)_m S_{2,F}(k,m)\right)\frac{d^w}{F_w!}. \qquad (2.8)\end{aligned}$$

To derive the result, we equate the coefficients of $\frac{d^w}{F_w!}$ on both sides of the last equation. □

Theorem 2.4 *Let $w \geq 0$ be an integer. Then we have*

$$\mathcal{T}_{w,F}(\zeta+r,\eta;\mathfrak{U}) = \sum_{k=0}^{w}\sum_{m=0}^{k}\binom{w}{k}_F\mathcal{T}_{w-k,F}(\eta;\mathfrak{U})(\zeta)_m S_{2,F}(k+r,m+r). \qquad (2.9)$$

Proof: By virtue of (1.26), and (2.1), we find

$$\begin{aligned}\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\zeta+r,\eta;\mathfrak{U})\frac{d^w}{F_w!} &= \frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}e_F^{rd}E_F^{\eta d}(e^d-1+1)^{\zeta}\\ &= \frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}e_F^{rd}E_F^{\eta d}\sum_{m=0}^{\infty}(\zeta)_m\frac{(e^d-1)^m}{F_m!}\\ &= \frac{1-\mathfrak{U}}{e_F^{-d}-\mathfrak{U}}E_F^{\eta d}\sum_{m=0}^{\infty}(\zeta)_m\sum_{k=m}^{\infty}S_{2,F}(k+r,m+r)\frac{d^k}{F_k!}\\ &= \sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\eta;\mathfrak{U})\frac{d^w}{F_w!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}(\zeta)_m S_{2,F}(k+r,m+r)\frac{d^k}{F_k!}\\ &= \sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\sum_{m=0}^{k}\binom{w}{k}_F\mathcal{T}_{w-k,F}(\eta;\mathfrak{U})(\zeta)_m S_{2,F}(k+r,m+r)\right)\frac{d^w}{F_w!}. \qquad (2.10)\end{aligned}$$

To derive the result, we equate the coefficients of $\frac{d^w}{F_w!}$ on both sides of the last equation. □

Theorem 2.5 *Let $w \geq 0$ be an integer. then we have*

$$\sum_{k=0}^{w}\binom{n}{k}_F\mathcal{T}_{k,F}(\mathfrak{U})(2)^{w-k}-\mathfrak{U}\mathcal{T}_{w,F}(\mathfrak{U}) = \begin{cases}1-\mathfrak{U}, & \text{if } w=0,\\ 0, & \text{if } w \geq 0.\end{cases}. \qquad (2.11)$$

Proof: If $e_F^{2d} \neq \mathfrak{U}$ in the generating function of the Frobenius–Tangent–Fibonacci number, then we find

$$\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\mathfrak{U})\frac{d^w}{F_w!}\left(\sum_{w=0}^{\infty}(-1)^w\frac{d^w}{F_w!}-\mathfrak{U}\right) = 1-\mathfrak{U}$$

$$\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F \mathcal{T}_{k,F}(\mathfrak{U})(-1)^{w-k}-\mathfrak{U}\mathcal{T}_{w,F}(\mathfrak{U})\right)\frac{d^w}{F_w!}=1-\mathfrak{U}.$$

Hence, complete proof of the theorem.

□

Corollary 2.1 *Using the similar method of proof from Theorem 2.5, we have*

$$\sum_{k=0}^{w}\binom{w}{k}_F \mathcal{T}_{k,F}(\zeta,\eta;\mathfrak{U})(2)^{w-k}-\mathfrak{U}\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})=(\zeta+\eta)_F^w. \tag{2.12}$$

Theorem 2.6 *Let $w \geq 0$ be an integer. Then we have*

$$\sum_{k=0}^{w}\binom{w}{k}_F \alpha^{w-k}\beta^k\mathcal{T}_{w-k,F}(\alpha^{-1}\zeta,\eta;\mathfrak{U})\mathcal{T}_{k,F}(\beta^{-1}\phi,\xi;\mathfrak{U})$$
$$=\sum_{k=0}^{w}\binom{w}{k}_F \beta^{w-k}\alpha^k\mathcal{T}_{w-k,F}(\beta^{-1}\zeta,\eta;\mathfrak{U})\mathcal{T}_{k,F}(\alpha^{-1}\phi,\xi;\mathfrak{U}). \tag{2.13}$$

Proof: Let

$$A=\frac{(1-\mathfrak{U})^2}{(e^{2\alpha d}-\mathfrak{U})(e^{2\beta d}-\mathfrak{U})}e_F^{(\zeta+\phi)d}E_F^{(\eta+\xi)d}. \tag{2.14}$$

Then, we have

$$A=\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\alpha^{-1}\zeta,\eta;\mathfrak{U})\frac{(\alpha d)^w}{F_w!}\sum_{k=0}^{\infty}\mathcal{T}_{k,F}(\beta^{-1}\phi,\xi;\mathfrak{U})\frac{(\beta d)^k}{F_k!}$$
$$=\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F \alpha^{w-k}\beta^k\mathcal{T}_{w-k,F}(\alpha^{-1}\zeta,\eta;\mathfrak{U})\mathcal{T}_{k,F}(\beta^{-1}\phi,\xi;\mathfrak{U})\right)\frac{d^w}{F_w!}. \tag{2.15}$$

Similarly, we have

$$A=\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F \beta^{w-k}\alpha^k\mathcal{T}_{w-k,F}(\beta^{-1}\zeta,\eta;\mathfrak{U})\mathcal{T}_{k,F}(\alpha^{-1}\phi,\xi;\mathfrak{U})\right)\frac{d^w}{F_w!}. \tag{2.16}$$

By virtue of (2.15) and (2.16), we attain the result. □

Theorem 2.7 *Let $w \geq 0$ be an integer. Then we have*

$$\mathfrak{U}\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})=\mathcal{T}_{w,F}(\zeta-1,\eta;\mathfrak{U})-(1-\mathfrak{U})(\zeta+\eta)_F^w. \tag{2.17}$$

Proof: Suppose

$$\frac{\mathfrak{U}}{(e_F^{2d}-\mathfrak{U})e_F^{2d}}=\frac{1}{e_F^{2d}-\mathfrak{U}}-\frac{1}{e_F^{2d}}.$$

Evaluating the following fraction using the above identity, we find

$$\frac{\mathfrak{U}(1-\mathfrak{U})e_F^{\zeta d}E_F^{\eta d}}{(e_F^{2d}-\mathfrak{U})e_F^{2d}}=\frac{(1-\mathfrak{U})e_F^{\zeta d}E_F^{\eta d}}{(e_F^{2d}-\mathfrak{U})}-\frac{(1-\mathfrak{U})e_F^{\zeta d}E_F^{\eta d}}{2e_F^{2d}}$$

$$\mathfrak{U}\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{w!}=\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\zeta-1,\eta;\mathfrak{U})\frac{d^w}{F_w!}-(1-\mathfrak{U})\sum_{w=0}^{\infty}(\zeta+\eta)_F^w\frac{d^w}{F_w!}.$$

Hence, the proof of the theorem is complete. □

Corollary 2.2 *On setting $\mathfrak{U} = -1$ in Theorem 2.7, we have*

$$(\zeta+\eta)_F^w = \mathcal{T}_{w,F}(\zeta,\eta;-1) + \mathcal{T}_{w,F}(\zeta-1,\eta;-1).$$

Theorem 2.8 *Let $w \geq 0$ be an integer. Then we have*

$$\sum_{k=0}^{w}\binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(a\zeta,\eta;\mathfrak{U})\mathcal{H}_{k,F}(b\phi,\xi;\mathfrak{U})}{a^{w-k}b^k} = \sum_{k=0}^{w}\binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(b\zeta,\eta;\mathfrak{U})\mathcal{H}_{k,F}(a\phi,\xi;\mathfrak{U})}{b^{w-k}a^k}, \tag{2.18}$$

where $\mathcal{H}_{w,F}(\zeta,\eta;\mathfrak{U})$ is the Frobenius–Euler–Fibonacci polynomials.

Proof: Let

$$B(d) = \frac{(1-\mathfrak{U})^2 e_F^{(\zeta+\phi)d} E_F^{(\eta+\xi)d}}{(e^{\frac{2d}{a}}-\mathfrak{U})(e^{\frac{d}{b}}-\mathfrak{U})}.$$

Now, we can write

$$B(d) = \sum_{w=0}^{\infty}\mathcal{T}_{w,F}(a\zeta,\eta;\mathfrak{U})\frac{d^w}{a^w F_w!}\sum_{k=0}^{\infty}\mathcal{H}_{k,F}(b\phi,\xi;\mathfrak{U})\frac{d^k}{b^k F_k!}$$

$$= \sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(a\zeta,\eta;\mathfrak{U})\mathcal{H}_{k,F}(b\phi,\xi;\mathfrak{U})}{a^{w-k}b^k}\right)\frac{d^w}{F_w!}. \tag{2.19}$$

Also, we can transform the form $B(d)$ as follows:

$$B(d) = 2\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(b\zeta,\eta;\mathfrak{U})\mathcal{H}_{k,F}(a\phi,\xi;\mathfrak{U})}{b^{w-k}a^k}\right)\frac{d^w}{F_w!}. \tag{2.20}$$

By virtue of (2.19) and (2.20), we obtain the result. □

Corollary 2.3 *Letting $a = 1$ in Theorem 2.8, we get*

$$\sum_{k=0}^{w}\binom{w}{k}\frac{\mathcal{T}_{w-k}(\xi;\mathfrak{U})\mathcal{H}_k(b\eta;\mathfrak{U})}{b^k} = \sum_{k=0}^{w}\binom{w}{k}\frac{\mathcal{T}_{w-k}(b\xi;\mathfrak{U})\mathcal{H}_k(\eta;\mathfrak{U})}{b^{w-k}}.$$

Theorem 2.9 *Let $a, b > 0$ and $a \neq b$. Then*

$$\sum_{k=0}^{w}\binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(a\zeta,\eta;\mathfrak{U})\mathcal{T}_{k,F}(b\phi,\xi;\mathfrak{U})}{a^{w-k}b^k} = \sum_{k=0}^{w}\binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(b\zeta,\eta;\mathfrak{U})\mathcal{T}_{k,F}(a\phi,\xi;\mathfrak{U})}{b^{w-k}a^k}. \tag{2.21}$$

Proof: Let

$$C(d) = \frac{(1-\mathfrak{U})^2 e_F^{(\zeta+\phi)d} E_F^{(\eta+\xi)d}}{(e^{\frac{2d}{a}}-\mathfrak{U})(e^{\frac{2d}{b}}-\mathfrak{U})}.$$

Now, we can write

$$C(d) = 4\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(a\zeta,\eta;\mathfrak{U})\frac{d^w}{a^w F_w!}\sum_{k=0}^{\infty}\mathcal{T}_{k,F}(b\phi,\xi;\mathfrak{U})\frac{d^k}{b^k F_k!}$$

$$= \sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(a\zeta,\eta;\mathfrak{U})\mathcal{T}_{k,F}(b\phi,\xi;\mathfrak{U})}{a^{w-k}b^k}\right)\frac{d^w}{F_w!}. \tag{2.22}$$

Also, we can transform the form C(d) as follows:

$$C(d) = \sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{n}{k}_F \frac{\mathcal{T}_{w-k,F}(b\zeta,\eta;\mathfrak{U})\mathcal{T}_{k,F}(a\phi,\xi;\mathfrak{U})}{b^{w-k}a^k}\right)\frac{d^w}{F_w!}. \tag{2.23}$$

In virtue of (2.22) and (2.23), we obtain the result. □

Theorem 2.10 *Let $a, b > 0$ and $a \neq b$. Then*

$$\sum_{k=0}^{w} \binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(-a\zeta,\eta;\mathfrak{U})\mathcal{H}_k(b\phi,\xi;\mathfrak{U})}{(-a)^{w-k}b^k} = \sum_{k=0}^{w} \binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(b\zeta,\eta;\mathfrak{U})\mathcal{H}_{k,F}(\phi,\xi;\mathfrak{U})}{b^{w-k}a^k}. \tag{2.24}$$

Proof: Let

$$E(d) = \frac{(1-\mathfrak{U})^2 e_F^{(\zeta+\phi)d} E_F^{(\eta+\xi)d}}{(e^{\frac{2d}{a}}+\mathfrak{U})(e^{\frac{d}{b}}-\mathfrak{U})}.$$

Now, we can write

$$\begin{aligned} E(d) &= \sum_{w=0}^{\infty} \mathcal{T}_{w,F}(-a\zeta,\eta;\mathfrak{U}) \frac{d^w}{(-a)^w F_w!} \sum_{k=0}^{\infty} \mathcal{H}_{k,F}(b\phi,\xi;\mathfrak{U}) \frac{d^k}{b^k F_k!} \\ &= 2\sum_{w=0}^{\infty} \left(\sum_{k=0}^{w} \binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(-a\zeta,\eta;\mathfrak{U})\mathcal{H}_k(b\phi,\xi;\mathfrak{U})}{(-a)^{w-k}b^k} \right) \frac{d^w}{F_w!}. \end{aligned} \tag{2.25}$$

Also, we can transform the form B as follows:

$$E(d) = \sum_{w=0}^{\infty} \left(\sum_{k=0}^{w} \binom{w}{k}_F \frac{\mathcal{T}_{w-k,F}(-b\zeta,\eta;\mathfrak{U})\mathcal{H}_{k,F}(a\phi,\xi;\mathfrak{U})}{b^{w-k}a^k} \right) \frac{d^w}{F_w!}. \tag{2.26}$$

By (2.25) and (2.26), we obtain the result. □

3. Parametric type of Frobenius–Tangent–Fibonacci Numbers and Polynomials

This section extends the two-variable Tangent–Fibonacci polynomials by integrating parametric trigonometric elements through the application of the golden cosine and sine functions. Let $p, q \in \mathbb{R}$. The Taylor series of the functions $e_F^{pd}\cos_F(qd)$ and $e_F^{pd}\sin_F(qd)$ are given as follows [8]:

$$e_F^{pd}\cos_F qd = \sum_{w=0}^{\infty} \mathcal{C}_{w,F}(p,q)\frac{d^w}{F_w!}, \tag{3.1}$$

and

$$e_F^{pd}\sin_F qd = \sum_{w=0}^{\infty} \mathcal{S}_{w,F}(p,q)\frac{d^w}{F_w!}, \tag{3.2}$$

where

$$\mathcal{C}_{w,F}(p,q) = \sum_{k=0}^{\lfloor \frac{w}{2} \rfloor} (-1)^k \binom{w}{2k}_F p^{w-2k}q^{2k}, \tag{3.3}$$

$$\mathcal{S}_{w,F}(p,q) = \sum_{k=0}^{\lfloor \frac{w-1}{2} \rfloor} (-1)^k \binom{w}{2k+1}_F p^{w-2k-1}q^{2k+1}. \tag{3.4}$$

With the help of Equation (2.1), we define Frobenius–Tangent–Fibonacci polynomials represented by the following generating function:

$$\frac{1-\mathfrak{U}}{e^{2d}-\mathfrak{U}} e_F^{\zeta d} = \sum_{w=0}^{\infty} \mathcal{T}_{w,F}(\zeta;\mathfrak{U})\frac{d^w}{F_w!}. \tag{3.5}$$

If we substitute $\zeta = 0$ into Equation (3.5), we obtain Frobenius–Tangent–Fibonacci numbers as follows:

$$\frac{1-\mathfrak{U}}{e^{2d}-\mathfrak{U}} = \sum_{w=0}^{\infty} \mathcal{T}_{w,F}(\mathfrak{U})\frac{d^w}{F_w!}. \tag{3.6}$$

Based on the aforementioned definitions of $\mathcal{C}_{w,F}(p,q)$ and $\mathcal{S}_{w,F}(p,q)$, as well as the number of a $\mathcal{S}_{w,F}$ we can specify two parameters for Frobenius–Tangent–Fibonacci polynomials as follows:

Definition 3.1 Two parametric families of the Frobenius–Tangent–Fibonacci polynomials are hereby defined via the following generating functions:

$$\frac{1-\mathfrak{U}}{e^{2d}-\mathfrak{U}}e_F^{\zeta d}\cos_F \eta d=\sum_{w=0}^{\infty}\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!}, \tag{3.7}$$

and

$$\frac{1-\mathfrak{U}}{e^{2d}-\mathfrak{U}}e_F^{\zeta d}\sin_F \eta d=\sum_{w=0}^{\infty}\mathcal{T}^{(s)}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!}. \tag{3.8}$$

Based on the aforementioned definitions, we have arrived at the following principal results.

Theorem 3.1 *Let $w \geqslant 0$ be an integer. Then*

$$\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})=\sum_{l=0}^{w}\binom{w}{l}_F\mathcal{S}_{l,F}(\mathfrak{U})\ \mathcal{C}_{w-l,F}(\zeta,\eta), \tag{3.9}$$

and

$$\mathcal{T}^{(s)}_{w,F}(\zeta,\eta;\mathfrak{U})=\sum_{l=0}^{w}\binom{w}{l}_F\mathcal{S}_{l,F}(\mathfrak{U})\ \mathcal{S}_{w-l,F}(\zeta,\eta). \tag{3.10}$$

Proof: Using (3.1) and (3.5), we have

$$\begin{aligned}\frac{1-\mathfrak{U}}{e^{2d}-\mathfrak{U}}e_F^{\zeta d}\cos_F(\eta d)&=\sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\mathfrak{U})\frac{d^w}{F_w!}\sum_{w=0}^{\infty}\mathcal{C}_{w}(\zeta,\eta)\frac{d^w}{F_w!}\\&=\sum_{w=0}^{\infty}\left(\sum_{l=0}^{w}\binom{w}{l}_F\mathcal{T}_{l,F}(\mathfrak{U})\ \mathcal{C}_{w-l}(\zeta,\eta)\right)\frac{d^w}{F_w!}\\ \sum_{w=0}^{\infty}\mathcal{T}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!}&=\sum_{w=0}^{\infty}\left(\sum_{l=0}^{w}\binom{w}{l}_F\mathcal{T}_{l,F}(\mathfrak{U})\ \mathcal{C}_{w-l,F}(\zeta,\eta)\right)\frac{d^w}{F_w!}.\end{aligned} \tag{3.11}$$

By equating the coefficients of $\frac{d^w}{F_w!}$ on both sides of the above equation, we derive the intended result (3.9). Equation (3.10) can be obtain in a similar manner. □

Theorem 3.2 *The following identities hold for:*

$$\mathcal{T}^{(c)}_{w,F}(\zeta+\theta,\eta;\mathfrak{U})=\sum_{l=0}^{w}\binom{w}{l}_F(-1)^{\binom{l}{2}}\mathcal{T}^{(c)}_{w-l,F}(\zeta,\eta;\mathfrak{U})\theta^k, \tag{3.12}$$

and

$$\mathcal{T}^{(s)}_{w,F}(\zeta+\theta,\eta;\mathfrak{U})=\sum_{l=0}^{w}\binom{w}{l}_F(-1)^{\binom{l}{2}}\mathcal{T}^{(s)}_{w-l,F}(\zeta,\eta;\mathfrak{U})\theta^k. \tag{3.13}$$

Proof: Utilizing (3.5), we obtain the following functional equation through derivation:

$$\begin{aligned}\sum_{w=0}^{\infty}\mathcal{T}^{(c)}_{w,F}(\zeta+\theta,\eta;\mathfrak{U})\frac{d^w}{F_w!}&=\frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}e_F^{\zeta d}E_F^{\theta d}\cos_F \eta d\\&=\sum_{w=0}^{\infty}\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!}\sum_{w=0}^{\infty}(-1)^{\binom{w}{2}}\ \theta^w\frac{d^w}{F_w!}\\&=\sum_{w=0}^{\infty}\left(\sum_{l=0}^{w}\binom{w}{l}_F(-1)^{\binom{l}{2}}\mathcal{T}^{(c)}_{w-l,F}(\zeta,\eta;\mathfrak{U})\theta^k\right)\frac{d^w}{F_w!}.\end{aligned}$$

By equating the coefficients of $\frac{d^w}{F_w!}$ on both sides of the aforementioned equation, it can be observed that

$$\mathcal{T}^{(c)}_{w,F}(\zeta+\theta,\eta;\mathfrak{U})=\sum_{l=0}^{w}\binom{w}{l}_F(-1)^{\binom{l}{2}}\mathcal{T}^{(c)}_{w-l,F}(\xi,\eta;\mathfrak{U})\theta^k.$$

The proof of the result (3.12) is provided. The assertion (3.13) can also be established in a similar manner. □

Theorem 3.3 *(F–Derivative Formulas) It is known that the following identities hold for:*

$$\frac{\partial_F}{\partial_F\zeta}\left(\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\right)=F_w\mathcal{T}^{(c)}_{w-1,F}(\zeta,\eta;\mathfrak{U}), \tag{3.14}$$

$$\frac{\partial_F}{\partial_F\zeta}\left(\mathcal{T}^{(s)}_{w,F}(\zeta,\eta;\mathfrak{U})\right)=F_w\mathcal{T}^{(s)}_{w-1,F}(\zeta,\eta;\mathfrak{U}), \tag{3.15}$$

$$\frac{\partial_F}{\partial_F\eta}\left(\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\right)=-F_w\mathcal{T}^{(s)}_{w-1,F}(\zeta,\eta;\mathfrak{U}), \tag{3.16}$$

$$\frac{\partial_F}{\partial_F\eta}\left(\mathcal{T}^{(s)}_{w,F}(\zeta,\eta;\mathfrak{U})\right)=F_w\mathcal{T}^{(c)}_{w-1,F}(\zeta,\eta;\mathfrak{U}). \tag{3.17}$$

Proof: Utilizing the Golden derivative operator $\frac{\partial_F}{\partial_F\zeta}$, and using (1.9) and (3.5), we have

$$\begin{aligned}\sum_{w=0}^{\infty}\frac{\partial_F}{\partial_F\zeta}\left(\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\right)\frac{d^w}{F_w!}=\frac{\partial_F}{\partial_F\zeta}\left\{\frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}e_F^{\zeta d}\cos_F\eta d\right\}&=\sum_{w=0}^{\infty}\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^{w+1}}{F_w!}\\&=\sum_{w=1}^{\infty}\mathcal{T}^{(c)}_{w-1,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_{w-1}!}.\end{aligned}$$

Since $\frac{\partial_F}{\partial_F\zeta}\left(\mathcal{T}^{(c)}_{0,F}(\zeta,\eta;\mathfrak{U})\right)=0$; for $w\geq 1$, by analyzing the coefficients of $\frac{d^w}{F_w!}$ on both sides of the final equation, the desired result is achieved in (3.14). The remaining results can then be easily deduced. □

Theorem 3.4 *The following identities holds true:*

$$C_{w,F}(\zeta,\eta)=\frac{1}{1-\mathfrak{U}}\left[\sum_{k=0}^{w}\binom{w}{k}_F(-1)^{w-k}\mathcal{T}^{(c)}_{k,F}(\zeta,\eta;\mathfrak{U})-\mathfrak{U}\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\right], \tag{3.18}$$

and

$$S_{w,F}(\zeta,\eta)=\frac{1}{1-\mathfrak{U}}\left[\sum_{k=0}^{w}\binom{w}{k}_F(-1)^{w-k}\mathcal{T}^{(s)}_{k,F}(\zeta,\eta;\mathfrak{U})-\mathfrak{U}\mathcal{T}^{(s)}_{w,F}(\zeta,\eta;\mathfrak{U})\right]. \tag{3.19}$$

Proof: By using (3.5), we have

$$\frac{1-\mathfrak{U}}{e_F^{2d}-\mathfrak{U}}e_F^{\zeta d}\cos_F(\eta d)=\sum_{w=0}^{\infty}\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!},$$

and

$$\begin{aligned}(1-\mathfrak{U})e_F^{\zeta d}\cos_F(\eta d)&=\left(e_F^{2d}-\mathfrak{U}\right)\sum_{k=0}^{\infty}\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!}\\&=\sum_{w=0}^{\infty}(-1)^w\frac{d^w}{F_w!}\sum_{k=0}^{\infty}\mathcal{T}^{(c)}_{k,F}(\zeta,\eta;\mathfrak{U})\frac{d^k}{F_k!}-\mathfrak{U}\sum_{w=0}^{\infty}\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\frac{d^w}{F_w!}\\&=2\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F(-1)^{w-k}\mathcal{T}^{(c)}_{k,F}(\zeta,\eta;\mathfrak{U})-\mathfrak{U}\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\right)\frac{d^w}{F_w!}.\end{aligned}$$

From (3.1), we get

$$\sum_{w=0}^{\infty} C_{w,F}(\zeta,\eta)\frac{d^w}{F_w!} = \frac{1}{1-\mathfrak{U}}\sum_{w=0}^{\infty}\left(\sum_{k=0}^{w}\binom{w}{k}_F(-1)^{w-k}\mathcal{T}^{(c)}_{k,F}(\zeta,\eta;\mathfrak{U}) - \mathfrak{U}\mathcal{T}^{(c)}_{w,F}(\zeta,\eta;\mathfrak{U})\right)\frac{d^w}{F_w!}.$$

By equating the coefficients of $\frac{d^w}{F_w!}$, we obtain the desired result. Equation (3.19) can be derived in a similar manner. □

4. Conclusion

In this study, we introduced and analyzed the Frobenius–Tangent–Fibonacci polynomials as the F–extensions of the classical Frobenius–Tangent polynomials, formulated in a two-variable setting within the framework of Golden Calculus. Extending and enhancing the results of earlier investigations (see [7]), we established their generating functions and derived several key properties, including recurrence relations, summation formulas, and derivative identities. In addition, we defined the associated Frobenius–Sigmoid–Fibonacci numbers and explored their intrinsic characteristics and connections with related polynomial families.

The results obtained in this work offer a unified and generalized approach to the interaction between Frobenius–Tangent–type structures and Fibonacci-based systems. This framework not only enriches the theoretical foundation of F–based special polynomials but also provides promising avenues for future research and applications in combinatorics, number theory, approximation theory, and various branches of applied mathematics.

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