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Complex Valued Perturbed Metric Space and its Fixed Points

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ABSTRACT: In this paper, we introduce complex valued perturbed metric spaces which generalize traditional complex valued metric spaces by permitting the variations in complex valued distances. This framework has enabled the formulation of fixed point results analogous to Banach's classical theorem for complex valued perturbed contraction mappings. In addition, the results of Azam et. al. [1] also generalized.

Key Words: Complex valued perturbed metric space, contractive condition, unique common fixed point.

Contents

1. Introduction

The literature on Fixed Point Theory encompasses numerous extensions and generalizations of Banach's fixed point theorem. The concept of a metric space has been employed not only within Mathematics but also across various qualitative sciences. For instance, one of the notable generalizations of a metric—the so-called partial metric [10]—was introduced to address certain problems in Domain Theory within Computer Science. Subsequently, the notion of a partial metric was further refined through the introduction of the dislocated metric [7]. Beyond these abstract constructions, the metric concept has been generalized and extended in several distinct directions. Among them, some well-established and particularly intriguing generalizations deserve attention. An early example is the quasi-metric, which arises when the axiom of symmetry is omitted. Another classical variation is the semi-metric [11], defined without imposing the triangular inequality. In contrast, Branciari [4] proposed a modified distance function obtained by replacing the triangular inequality with a quadrilateral inequality. Furthermore, an additional generalization was achieved by multiplying the right-hand side of the triangular inequality by a fixed constant $s \geq 1$, leading to the notion of a b-metric [6]. We also recall the concept of cone metric [8], formulated by replacing the domain in the definition of a metric space with a cone in a Banach space satisfying particular properties. This idea was subsequently extended to the complex-valued metric [1].

Azam et. al. [1] defined complex valued metric spaces which have attracted significant interest due to their richer structures and wide applications. These spaces generalize classical metric spaces by allowing distances to take values in the complex plane.

These developments have allowed for fixed point results regarding mappings that satisfy rational inequalities within complex valued metric spaces. Furthermore, the framework extends naturally to complex valued normed spaces and complex valued Hilbert spaces, providing fertile ground for further advances in mathematical analysis.

To proceed, we first recall some notations and definitions that will be used throughout this paper. Let $\mathbb{C}$ represent the set of all complex numbers and take any two elements $z_1, z_2 \in \mathbb{C}$.

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Define a partial order relation $\precsim$ on $\mathbb{C}$ by:

$$z_1 \precsim z_2 \quad \text{if and only if} \quad \Re(z_1) \leq \Re(z_2) \quad \text{and} \quad \Im(z_1) \leq \Im(z_2),$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z, respectively.

From this definition, the partial order between z_1 and z_2 means at least one of the following holds:

(i) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$,

(ii) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$,

(iii) $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$,

(iv) $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$.

Specifically, the notation $z_1 \lnsim z_2$ indicates that $z_1 \neq z_2$ and that one of the conditions (i), (ii), or (iii) holds. Further, $z_1 \prec z_2$ means that only condition (iii) is fulfilled.

Observe that for any $z_1, z_2 \in \mathbb{C}$, if $0 \precsim z_1 \lnsim z_2$, then it follows that $|z_1| < |z_2|$. Additionally, given $z_1 \precsim z_2$ and $z_2 \prec z_3$, we conclude $z_1 \prec z_3$.

These definitions follow the framework established by Azam et al. [1].

Definition 1.1 *[1] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:*

1. *$0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
3. *$d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then d is called a complex valued metric *on X, and (X, d) is called a* complex valued metric space.

Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$d(x_n, x) \prec c,$$

then $\{x_n\}$ is said to be convergent, x_n converges to x, and x is called the limit of $\{x_n\}$. We denote this as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

If for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$d(x_n, x_{n+m}) \prec c,$$

then $\{x_n\}$ is called a Cauchy sequence in (X, d).

If every Cauchy sequence in (X, d) converges, then (X, d) is called a complete complex valued metric space.

Lemma 1.2 *[1] Let (X, d) be a complex valued metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to x if and only if*

$$|d(x_n, x)| \to 0 \quad \textit{as} \quad n \to \infty.$$

Lemma 1.3 *[1] Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if*

$$|d(x_n, x_{n+m})| \to 0 \quad \textit{as} \quad n \to \infty.$$

Banach's fixed point theorem is a classical result stating that if (X, d) is a complete metric space, then every contraction mapping has a unique fixed point in X.

However, considering possible errors in measuring distances, suppose instead of the classical metric d, an experimental measurement D satisfies contraction mapping, where D is not necessarily a metric, raises the question of how the Banach fixed point theorem behaves under such perturbations.

Recent work has generalized Banach's theorem by introducing the notion of perturbed metric spaces (X, D, P), where D is the perturbed metric related to an exact metric via a perturbation mapping P. Within this framework, fixed point results analogous to the classical Banach theorem have been established for perturbed contraction mappings [9].

Definition 1.4 *[9] Let $D, P : X \times X \to [0, \infty)$ be two given mappings. We say that D is a perturbed metric on X with respect to P, if*

$$D - P : X \times X \to \mathbb{R}, \quad (x, y) \mapsto D(x, y) - P(x, y)$$

is a metric on X; that is, for all $x, y, z \in X$,

(i) $(D - P)(x, y) \geq 0$;

(ii) $(D - P)(x, y) = 0$ *if and only if* $x = y$;

(iii) $(D - P)(x, y) = (D - P)(y, x)$;

(iv) $(D - P)(x, y) \leq (D - P)(x, z) + (D - P)(z, y)$.

We call P a perturbation mapping*, $d = D - P$ an* exact metric*, and (X, D, P) a* perturbed metric space.

Notice that a perturbed metric on X is not necessarily a metric on X. Some examples are provided to illustrate this fact [9].

2. Complex Valued Perturbed Metric Spaces

We introduce below the notion of a complex valued perturbed metric space. Throughout this paper, X denotes an arbitrary non-empty set.

Definition 2.1 *Let $D, P : X \times X \to \mathbb{C}$ be two given mappings. We say that D is a complex valued perturbed metric on X with respect to P if*

$$D - P : X \times X \to \mathbb{C},$$

$$(x, y) \mapsto D(x, y) - P(x, y)$$

is a complex valued metric on X, i.e., for all $x, y, z \in X$:

(i) $(D - P)(x, y) \succcurlyeq 0$;

(ii) $(D - P)(x, y) = 0$ *if and only if* $x = y$;

(iii) $(D - P)(x, y) = (D - P)(y, x)$;

(iv) $(D - P)(x, y) \preccurlyeq (D - P)(x, z) + (D - P)(z, y)$.

We call P a complex valued perturbed mapping, $d = D - P$ an exact complex valued metric, and (X, D, P) a complex valued perturbed metric space.

Notice that a complex valued perturbed metric on X is not necessarily a complex valued metric on X.

Example 2.2 *Let $X = \mathbb{C}$ and $D : X \times X \to \mathbb{C}$ be the mapping defined by*

$$D(x, y) = |x - y| + xy, \quad x, y \in X.$$

Then D is a complex valued perturbed metric on X with respect to the perturbed mapping $P : X \times X \to \mathbb{C}$ given by

$$P(x, y) = xy, \quad x, y \in X.$$

In this case, the exact complex metric is the mapping $d : X \times X \to \mathbb{C}$ defined by

$$d(x, y) = |x - y|, \quad x, y \in X.$$

Remark that D is not a complex valued metric on X. This can be easily seen observing that $D(1, 1) = 1 \neq 0$.

Example 2.3 *Let $X = \mathbb{C}$ and $D : X \times X \to \mathbb{C}$ be the mapping defined by*

$$D(x, y) = (x + y)^2, \quad x, y \in X.$$

Then D is a complex valued perturbed metric on X, where the perturbed mapping

$$P : X \times X \to \mathbb{C}$$

is given by

$$P(x, y) = (x + y)^2 - |x - y|, \quad x, y \in X,$$

and the exact complex metric $d : X \times X \to \mathbb{C}$ is given by

$$d(x, y) = |x - y|, \quad x, y \in X.$$

Remark that D is not a complex valued metric on X. This can be easily seen observing that $D(1, 1) = 4 \neq 0$.

Example 2.4 *Consider $X = \mathbb{C}$, the set of complex numbers, and define a complex valued metric $d : X \times X \to \mathbb{C}$ by*

$$d(z_1, z_2) = (1 + i)\, |z_1 - z_2|$$

for all $z_1, z_2 \in X$. This defines a complex valued metric space (X, d).

Now define a perturbation of this metric by defining a mapping

$$D(z_1, z_2) = d(z_1, z_2) + i\frac{|z_1| + |z_2|}{2}$$

for all $z_1, z_2 \in X$.

This D can be shown to be a complex valued perturbed metric, which is a perturbed form of the original metric d. It is not a complex valued metric space since

$$D(z, z) = i|z| \neq 0,$$

representing a perturbation of the original metric structure.

In analogy of Jleli and Samet [9], we state some elementary properties of complex valued perturbed metric spaces.

Definition 2.5 *Let (X, D, P) be a complex valued perturbed metric space, $\{z_n\}$ a sequence in X, and $\mathcal{T} : X \to X$.*

(i) We say that $\{z_n\}$ is a perturbed convergent sequence in (X, D, P) if $\{z_n\}$ is a convergent sequence in the complex valued metric space (X, d), where d is complex valued metric (i.e., $d = D - P$).

(ii) *We say that $\{z_n\}$ is a perturbed Cauchy sequence in (X, D, P) if $\{z_n\}$ is a Cauchy sequence in the complex valued metric space (X, d).*

(iii) *We say that (X, D, P) is a complete perturbed complex valued metric space if (X, d) is a complete complex-valued metric space, or equivalently, if every perturbed Cauchy sequence in (X, D, P) is a perturbed convergent sequence in (X, D, P).*

(iv) *We say that $\mathcal{T}$ is a perturbed continuous mapping if $\mathcal{T}$ is continuous with respect to the complex valued metric d.*

3. A Generalization of Banach's Fixed Point Theorem

The aim of this section is to generalize Banach fixed point theorem from complex valued metric spaces to complex valued perturbed metric spaces.

Theorem 3.1 *Let (X, D, P) be a complete complex valued perturbed metric space and $\mathcal{T} : X \to X$ be a given perturbed continuous mapping. If there exists $\lambda \in (0, 1)$ such that*

$$D(\mathcal{T}x, \mathcal{T}y) \precsim \lambda D(x, y) \tag{3.1}$$

for all $x, y \in X$. Then $\mathcal{T}$ has a unique fixed point.

Proof: Let $z_0 \in X$ be fixed. Consider the Picard sequence $\{z_n\} \subset X$ defined by

$$z_{n+1} = \mathcal{T}z_n, \quad n \in \mathbb{N}.$$

Taking $(x, y) = (z_0, z_1)$ in (3.1), we obtain

$$D(\mathcal{T}z_0, \mathcal{T}z_1) \precsim \lambda D(z_0, z_1), \tag{3.2}$$

that is,

$$D(z_1, z_2) \precsim \lambda D(z_0, z_1). \tag{3.3}$$

Similarly, taking $(x, y) = (z_1, z_2)$ in (3.1), we obtain

$$D(z_2, z_3) \precsim \lambda D(z_1, z_2),$$

which implies by (3.3) that

$$D(z_2, z_3) \precsim \lambda^2 D(z_0, z_1).$$

Inductively, we get

$$D(z_n, z_{n+1}) \precsim \lambda^n \tau, \quad n \in \mathbb{N}, \tag{3.4}$$

where $\tau = D(z_0, z_1)$.

Let $d = D - P$ be the exact complex valued metric. We have from (3.4)

$$d(z_n, z_{n+1}) + P(z_n, z_{n+1}) \precsim \lambda^n \tau, \quad n \in \mathbb{N}.$$

Since $d(z_n, z_{n+1}) \precsim d(z_n, z_{n+1}) + P(z_n, z_{n+1})$, it holds that

$$|d(z_n, z_{n+1})| \leq |d(z_n, z_{n+1})| + |P(z_n, z_{n+1})|,$$

and so

$$|d(z_n, z_{n+1})| \leq \lambda^n \tau, \quad n \in \mathbb{N}.$$

The use of above inequality and Lemma 1.3 implies that $\{z_n\}$ is a Cauchy sequence in the complex valued perturbed metric space (X, d), that is, $\{z_n\}$ is a perturbed Cauchy sequence in the perturbed metric space (X, D, P), by the completeness of (X, D, P) that there exists $z^* \in X$ such that

$$\lim_{n\to\infty} d(z_n, z^*) = 0. \tag{3.5}$$

Claim: z^* is a fixed point of $\mathcal{T}$. Since $\mathcal{T}$ is a perturbed continuous mapping, then (3.5) yields

$$\lim_{n\to\infty} d(\mathcal{T}z_n, \mathcal{T}z^*) = 0,$$

that is,

$$\lim_{n\to\infty} d(z_{n+1}, \mathcal{T}z^*) = 0. \tag{3.6}$$

Since $d = D - P$ is a complex valued metric on X, the uniqueness of the limit implies that

$$z^* = \mathcal{T}z^*,$$

that is, z^* is a fixed point of $\mathcal{T}$.

For uniqueness of fixed point of $\mathcal{T}$, suppose on the contrary that $u, v \in X$ are two distinct fixed points of $\mathcal{T}$. By (3.1), we have

$$D(u, v) = D(\mathcal{T}u, \mathcal{T}v) \precsim \lambda D(u, v),$$

and so

$$d(u, v) + P(u, v) \precsim \lambda(d(u, v) + P(u, v)).$$

Since $u \neq v$, then $d(u, v) + P(u, v) \neq 0$, and the above inequality yields $\lambda \geq 1$, which contradicts the hypothesis. Consequently, z^* is the unique fixed point of $\mathcal{T}$. □

As an application of Theorem 3.1, we now prove the following corollary regarding Banach's fixed point theorem in complete complex valued metric space.

Corollary 3.2 *Let (X, d) be a complete complex valued metric space and $\mathcal{T} : X \to X$ be a given mapping. Assume that there exists $\lambda \in (0, 1)$ such that*

$$d(\mathcal{T}x, \mathcal{T}y) \precsim \lambda d(x, y) \tag{3.7}$$

for all $x, y \in X$. Then $\mathcal{T}$ has a unique fixed point.

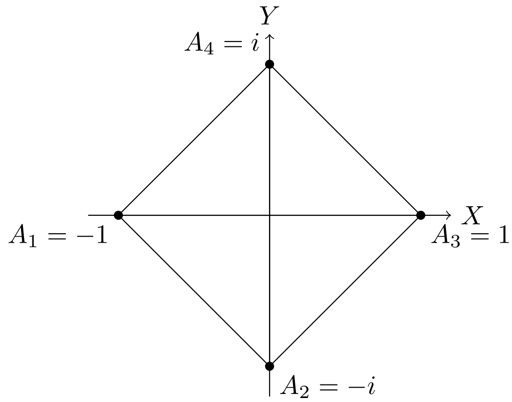
Proof: Let $D = d$ and $P \equiv 0$ (i.e., $P(x, y) = 0$ for all $x, y \in X$). Then (X, D, P) is a complete complex valued perturbed metric space. Furthermore, by (3.7), $\mathcal{T}$ is continuous with respect to the exact complex valued metric d, and (3.1) holds. The corollary now follows by the Theorem 3.1. □

In support of our Theorem 3.1, we provide an example.

Example 3.3 *Let $X = \{1, -1, i, -i\}$ be the set of fourth roots of unity in the complex numbers. This set is fundamental in complex number theory, signal processing, and many uses in mathematical engineering. These elements represent the complex solutions to the equation*

$$z^4 = 1.$$

Let these numbers be denoted in the xy-complex plane as points:



The norm of adjacent sides is $|A_1 - A_2| = |A_2 - A_3| = |A_3 - A_4| = |A_4 - A_1| = \sqrt{2}$ *where as diagonal sides have norm* $|A_1 - A_3| = |A_2 - A_4| = 2$.

Define a mapping $\mathcal{T} : X \to X$ *by*

$$\mathcal{T}A_1 = A_1, \quad \mathcal{T}A_2 = A_3, \quad \mathcal{T}A_3 = A_4, \quad \mathcal{T}A_4 = A_1,$$

and introduce the perturbation $P : X \times X \to \mathbb{C}$ *by*

$$P(A_1, A_3) = P(A_3, A_1) = 0 = P(A_1, A_4) = P(A_4, A_1),$$

$$P(A_3, A_4) = P(A_4, A_3) = i,$$

$$P(A_1, A_2) = P(A_2, A_1) = 2i = P(A_2, A_3) = P(A_3, A_2),$$

$$P(A_2, A_4) = P(A_4, A_2) = 4i,$$

and

$$P(A_r, A_r) = 0, \quad r = 1, 2, 3, 4.$$

Let the mapping $D : X \times X \to \mathbb{C}$ *be defined by*

$$D(A_r, A_s) = |A_r - A_s| + P(A_r, A_s), \qquad r, s \in \{1, 2, 3, 4\}.$$

Clearly (X, D, P) *is a complex valued perturbed metric space where the exact metric* $d : X \times X \to \mathbb{C}$ *is defined by*

$$d(A_r, A_s) = |A_r - A_s|, \qquad r, s \in \{1, 2, 3, 4\}.$$

Since

$$|D(A_2, A_4)| = |2 + P(A_2, A_4)| = |2 + 4i| = \sqrt{20},$$
$$|D(A_2, A_3)| = |\sqrt{2} + P(A_2, A_3)| = |\sqrt{2} + 2i| = \sqrt{6},$$
$$|D(A_3, A_4)| = |\sqrt{2} + P(A_3, A_4)| = |\sqrt{2} + i| = \sqrt{3},$$

which shows that $|D(A_2, A_4)| > |D(A_2, A_3)| + |D(A_3, A_4)|$ *and hence* D *is not a complex metric.*

It is clear that $\mathcal{T}$ *is a perturbed continuous mapping.*

We can write (3.1) as

$$|\mathcal{T}A_r - \mathcal{T}A_s| + P(\mathcal{T}A_r, \mathcal{T}A_s) \leq \lambda \left(|A_r - A_s| + P(A_r, A_s)\right) \tag{3.8}$$

for every $r, s \in \{1, 2, 3, 4\}$.

Clearly if $r = s$, *(3.8) is satisfied for all* $\lambda > 0$. *So let* $r \neq s$. *Table 1 provides the values of* $D(A_r, A_s) = |\mathcal{T}A_r - \mathcal{T}A_s| + P(\mathcal{T}A_r, \mathcal{T}A_s)$ *and* $|A_r - A_s| + P(A_r, A_s) = D^*(A_r, A_s)$ *(say). From Table 1, we deduce that*

$$\max_{1 \leq r < s \leq 4} \frac{|D(A_r, A_s)|}{|D^*(A_r, A_s)|} \approx .816 \leq .82.$$

Table 1: The values of $D(A_r, A_s)$ & $D^*(A_r, A_s)$.

(r, s)	$\lvert D(A_r, A_s)\rvert$	$\lvert D^*(A_r, A_s)\rvert$	$\frac{\lvert D(A_r,A_s)\rvert}{\lvert D^*(A_r,A_s)\rvert}$
(1,2)	2	$\sqrt{6}$	$\approx .816$
(1,3)	$\sqrt{2}$	2	$\approx .707$
(1,4)	0	$\sqrt{2}$	0
(2,3)	$\sqrt{3}$	$\sqrt{6}$	$\approx .707$
(2,4)	2	$\sqrt{20}$	$\approx .447$
(3,4)	$\sqrt{2}$	$\sqrt{3}$	$\approx .816$

Then, by symmetry (notice that P is a symmetric mapping), we deduce that (3.8) holds for all $\lambda \in (.82, 1)$, which shows that condition (3.1) of Theorem 3.1 is satisfied. Further the only fixed point of $\mathcal{T}$ is the point A_1.

It is to be pointed that the Banach fixed point theorem (see Corollary 3.2) is not applicable in the complex valued metric space (X, d), where d is the exact complex valued metric. This can be easily seen observing, for instance, that

$$\frac{d(\mathcal{T}A_1, \mathcal{T}A_2)}{d(A_1, A_2)} = \frac{d(A_1, A_3)}{d(A_1, A_2)} = \sqrt{2}.$$

4. An Extension of Azam et. al. Fixed Point Theorem

Theorem 4.1 *Let (X, D, P) be a complete complex valued perturbed metric space, and suppose the mappings $S, \mathcal{T} : X \to X$ satisfy, for all $x, y \in X$,*

$$D(Sx, \mathcal{T}y) \precsim \alpha\, D(x, y) + \beta\, \frac{D(x, Sx)D(y, \mathcal{T}y)}{1 + D(x, y)}, \tag{4.1}$$

where $\alpha, \beta \geq 0$ are real numbers such that $\alpha + \beta < 1$.

Then S and $\mathcal{T}$ have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X and define the sequence:

$$x_{2k+1} = Sx_{2k}, \quad x_{2k+2} = \mathcal{T}x_{2k+1}, \quad k = 0, 1, 2, \ldots$$

Then,

$$\begin{aligned} D(x_{2k+1}, x_{2k+2}) &= D(Sx_{2k}, \mathcal{T}x_{2k+1}) \\ &\precsim \alpha D(x_{2k}, x_{2k+1}) + \beta \frac{D(x_{2k+1}, \mathcal{T}x_{2k+1})D(x_{2k}, Sx_{2k})}{1 + D(x_{2k}, x_{2k+1})} \\ &\precsim \alpha D(x_{2k}, x_{2k+1}) + \beta \frac{D(x_{2k+1}, x_{2k+2})D(x_{2k}, x_{2k+1})}{1 + D(x_{2k}, x_{2k+1})} \\ &\precsim \alpha D(x_{2k}, x_{2k+1}) + \beta D(x_{2k+1}, x_{2k+2}), \\ &\quad \textit{since}\ D(x_{2k}, x_{2k+1}) \precsim 1 + D(x_{2k+1}, x_{2k+2}) \\ &\precsim \frac{\alpha}{1-\beta} D(x_{2k}, x_{2k+1}). \end{aligned}$$

Similarly,

$$\begin{aligned}
D(x_{2k+2}, x_{2k+3}) &= D(Sx_{2k+2}, Tx_{2k+1}) \\
&\preccurlyeq \alpha D(x_{2k+2}, x_{2k+1}) + \beta \frac{D(x_{2k+1}, Tx_{2k+1}) D(x_{2k+2}, Sx_{2k+2})}{1 + D(x_{2k+2}, x_{2k+1})} \\
&\preccurlyeq \alpha D(x_{2k+2}, x_{2k+1}) + \beta \frac{D(x_{2k+1}, x_{2k+2}) D(x_{2k+2}, x_{2k+3})}{1 + D(x_{2k+1}, x_{2k+2})} \\
&\preccurlyeq \alpha D(x_{2k+2}, x_{2k+1}) + \beta D(x_{2k+2}, x_{2k+3}), \\
&\preccurlyeq \frac{\alpha}{1-\beta} D(x_{2k}, x_{2k+1}).
\end{aligned}$$

Now, with $h = \frac{\alpha}{1-\beta}$, we have

$$D(x_{n+1}, x_{n+2}) \preccurlyeq h\, D(x_n, x_{n+1}) \preccurlyeq \cdots \preccurlyeq h^{n+1} D(x_0, x_1).$$

So for any $m > n$,

$$\begin{aligned}
D(x_n, x_m) &\preccurlyeq D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) + \cdots + D(x_{m-1}, x_m) \\
&\preccurlyeq \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) D(x_0, x_1).
\end{aligned}$$

Using the sum of a geometric series,

$$\sum_{k=n}^{m-1} h^k = h^n \frac{1 - h^{m-n}}{1-h} = \frac{h^n}{1-h}.$$

Consequently,

$$D(x_n, x_m) \preccurlyeq \frac{h^n}{1-h} D(x_0, x_1),$$

and so

$$|D(x_m, x_n)| \leq \frac{h^n}{1-h} |D(x_0, x_1)| \to 0, \quad \text{as } m, n \to \infty,$$

and this implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$.

It follows that $u = Su$, otherwise let

$$D(u, Su) = z > 0,$$

and we would have

$$\begin{aligned}
z &= D(u, Su) \\
&\preccurlyeq D(u, x_{2k+2}) + D(x_{2k+2}, Su) \\
&\preccurlyeq D(u, x_{2k+2}) + D(Tx_{2k+1}, Su) \\
&\preccurlyeq D(u, x_{2k+2}) + \alpha D(x_{2k+1}, u) + \beta \frac{D(x_{2k+1}, Tx_{2k+1}) D(u, Su)}{1 + D(u, x_{2k+1})} \\
&\preccurlyeq D(u, x_{2k+2}) + \alpha D(x_{2k+1}, u) + \beta \frac{D(x_{2k+1}, x_{2k+2}) z}{1 + D(u, x_{2k+1})}.
\end{aligned}$$

Taking absolute values yields

$$|z| \leq |D(u, x_{2k+2})| + \alpha |D(x_{2k+1}, u)| + \beta \frac{|D(x_{2k+1}, x_{2k+2})| \cdot |z|}{1 + |D(u, x_{2k+1})|}.$$

Since the right-hand side tends to zero as $k \to \infty$, we have $|z| = 0$, which is a contradiction. Hence, $u = Su$.

Similarly, one can show that $u = \mathcal{T}u$.

Now, suppose $u^* \in X$ is a second common fixed point of S and $\mathcal{T}$. Then,

$$\begin{aligned} d(u, u^*) &= d(Su, \mathcal{T}u^*) \\ &\precsim \alpha d(u, u^*) + \beta \frac{d(u, Su) d(u^*, \mathcal{T}u^*)}{1 + d(u, u^*)} \\ &\precsim \alpha d(u, u^*). \end{aligned}$$

This implies $u^* = u$, proving the uniqueness of fixed point. □

Taking $\mathcal{T} = S$ in Theorem 4.1, we have

Corollary 4.2 *Let (X, D, P) be a complete complex valued perturbed metric space and suppose the mapping $\mathcal{T} : X \to X$ satisfies, for all $x, y \in X$,*

$$D(\mathcal{T}x, \mathcal{T}y) \precsim \alpha\, D(x, y) + \beta\, \frac{D(x, \mathcal{T}x)\, D(y, \mathcal{T}y)}{1 + D(x, y)}, \tag{4.2}$$

where $\alpha, \beta \geq 0$ are real numbers such that $\alpha + \beta < 1$.

Then $\mathcal{T}$ has a unique fixed point in X.

Corollary 4.3 *[1, Theorem 4] Let (X, d) be a complete complex valued metric space and suppose the mapping $\mathcal{T} : X \to X$ satisfies, for all $x, y \in X$,*

$$d(\mathcal{T}x, \mathcal{T}y) \precsim \alpha\, d(x, y) + \beta\, \frac{d(x, \mathcal{T}x)\, d(y, \mathcal{T}y)}{1 + d(x, y)}, \tag{4.3}$$

where $\alpha, \beta \geq 0$ are real numbers such that $\alpha + \beta < 1$.

Then $\mathcal{T}$ has a unique fixed point in X.

Proof: Let $D = d$ and $P \equiv 0$ (i.e., $P(x, y) = 0$ for all $x, y \in X$) in (4.2). Then (X, D, P) is a complex valued perturbed complete metric space. Furthermore, by (4.3), $\mathcal{T}$ is continuous with respect to the exact complex valued metric d, and (4.2) holds, the corollary follows. □

Corollary 4.4 *Let (X, D, P) be a complete complex valued perturbed metric space and suppose the mapping $\mathcal{T} : X \to X$ satisfies, for all $x, y \in X$,*

$$D(\mathcal{T}^n x, \mathcal{T}^n y) \precsim \alpha\, D(x, y) + \beta\, \frac{D(x, \mathcal{T}^n x)\, D(y, \mathcal{T}^n y)}{1 + D(x, y)}, \tag{4.4}$$

where $\alpha, \beta \geq 0$ are real numbers such that $\alpha + \beta < 1$.

Then $\mathcal{T}^n$ has a unique fixed point in X.

Proof: By Corollary 4.2, there exists $v \in X$ such that

$$\mathcal{T}^n v = v.$$

The result then follows from the fact that

$$\begin{aligned} D(\mathcal{T}(v, v) &= D(\mathcal{T}\mathcal{T}^n v, \mathcal{T}^n v) = D(\mathcal{T}^n \mathcal{T}v, \mathcal{T}^n v) \\ &\precsim \alpha D(\mathcal{T}v, v) + \beta \frac{D(\mathcal{T}v, \mathcal{T}^n \mathcal{T}v) D(v, \mathcal{T}^n v)}{1 + D(\mathcal{T}v, v)} \\ &\precsim \alpha D(\mathcal{T}^n v, v), \end{aligned}$$

since

$$D(v,v)=0,$$

and therefore

$$D(\mathcal{T}v,v)\precsim \alpha D(\mathcal{T}v,v).$$

Since $\alpha<1$, this implies

$$|D(\mathcal{T}v,v)|=0,$$

which means $Tv=v$. □

Corollary 4.5 *[1, Corollary 6] Let (X,d) be a complete complex valued metric space and let the mapping $\mathcal{T}:X\to X$ satisfy, for all $x,y\in X$,*

$$d(\mathcal{T}^n x,\mathcal{T}^n y)\precsim \alpha d(x,y)+\beta\frac{d(x,\mathcal{T}^n x)d(y,\mathcal{T}^n y)}{1+d(x,y)}, \tag{4.5}$$

where $\alpha,\beta\geq 0$ and $\alpha+\beta<1$. Then $\mathcal{T}$ has a unique fixed point in X.

Proof: Let $D=d$ and $P\equiv 0$ (i.e., $P(x,y)=0$ for all $x,y\in X$) in (4.4). Then (X,D,P) is a complex valued perturbed complete metric space. Furthermore, by (4.5), $\mathcal{T}$ is continuous with respect to the exact metric d, and (4.4) holds, the corollary follows. □

5. Conclusion

This paper introduces the concept of complex valued perturbed metric spaces, thereby generalizing traditional complex valued metric spaces by allowing variations in complex valued distances. This novel framework successfully extends classical fixed point results, specifically Banach's contraction principle, to complex valued perturbed contraction mappings. Furthermore, the paper generalizes existing results by Azam et al. [1], showing the broader applicability and deeper insight provided by this new structure. The introduction of complex valued perturbed metric spaces opens up new directions for future research in fixed point theory within complex metric frameworks.

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