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Uni-Labeling of Graphs Over Finite Commutative Rings

Pranjali*, Seema Swami

ABSTRACT: This paper introduces the notion of uni-labeling of a graph and its empirical study demonstrates that every finite graph admits a uni-labeling with respect to some finite commutative ring. In the course of the investigation, it is found that unit graph turns out to be maximal with respect to the optimal uni-labeling index. Furthermore, the paper determines the optimal uni-labeling index for several well-known families of graphs. Several new directions for further research are also indicated through problems.

Key Words: Labeling, ideal, unit graph, finite commutative ring.

Contents

1. Introduction

For undefined graph theoretical and abstract algebra notations or terminologies that are not described here, we refer to the reader to standard textbooks [5] and [6], respectively. Throughout this paper, all graphs are assumed to be simple and finite, and all rings are assumed to be finite, commutative, and possess unity, unless stated otherwise.

Graph labeling is a significant concept in graph theory aimed at assigning numbers or labels to the vertices or edges of a graph according to specific rules. Since the mid-1960s, when Rosa [2] introduced the foundational notions of graph labeling (such as α-labeling, β-labeling, and ρ-labeling), numerous variants and generalizations have been developed. Graph labeling has applications in areas including coding theory, x-ray crystallography, communication networks, and more. In these studies, the structural properties of graphs and algebraic structures like rings and fields have proven central. In a β-labeling of a graph G with q edges, a function f assigns each vertex a unique number from 0 to q, such that each edge (xy) is assigned the label $|f(x) - f(y)|$, and all edge labels are distinct. This labeling was later referred as *graceful* labeling by Golomb [4]. Over time, a large number of studies have focused on different aspects of graph labelings. Besides advancing the theory, researchers have also been interested in exploring their practical applications. For a detailed overview of various labeling techniques and their applications, reader is referred to [1].

The concept of β-labeling, also known as graceful labeling, introduced a way to assign integers to vertices so that the labels induced on edges meet certain distinctness criteria. This area has received much attention due to its blend of combinatorics and algebra and its diverse applications. Motivated by the depth and versatility of these labeling problems, this work seeks to further explore how algebraic structures, specifically finite commutative rings, can be leveraged to define and study new types of labelings for various graph families. With this motivation here our aim is to construct a general framework that unifies known results and opens paths to new findings about labelings over rings.

Substantial research has addressed diverse graph labeling techniques beyond those of Rosa [2] and Golomb [4], whose work on β-labeling and related problems has been foundational. Surveys such as Gallian's [1] extensive range of results, reflecting the richness of the field. Various methods have explored both theoretical properties and applications, often focusing on graphs labeled over finite fields or with additive group structures (see [8,9]).

Despite the substantial advances, few studies have systematically examined graph labeling via injective functions from the vertex set into finite commutative rings, with the additional constraint that the sum

* Corresponding author.
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of labels of adjacent vertices yields a unit in the ring. The notion of a "uni-labeling" as introduced in this work fills this gap by generalizing classical labeling schemes to a broader algebraic context. The present research aims to determine existence conditions for uni-labelings, compute optimal indices for fundamental graph families, and initiate the broader study of uni-labeling indices across graphs and rings, establishing both foundational results and open questions.

The formal definition of the new notion is as follows:

Definition 1.1 Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let R be a commutative ring with unity. An injective function $f : V(G) \to R$ is called a *uni-labeling* of G if for all $(u, v) \in E(G)$, $f(u) + f(v) \in U(R)$, where $U(R)$ denotes the set of units of R.

Example 1 Consider the graph of order 4. To ensure the existence of a uni-labeling, the ring must contain at least 4 elements. Up to isomorphism, there are four such rings, namely, $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/\langle x^2\rangle$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{F}_4$. Among these, the graph admits uni-labeling using elements of $\mathbb{F}_4$ only.

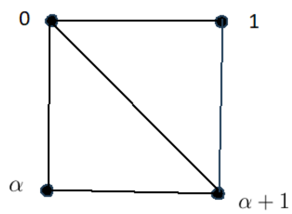

Figure 1: The uni-labeling of graph of order 4

2. Main results

In this section, we present the main results concerning uni-labelings of graphs over finite commutative rings. Building upon the foundational definitions and preliminary observations discussed earlier, we first establish necessary conditions for the existence of uni-labeling on various classes of graphs in the following theorem:

Theorem 2.1 *Let G be a graph of order n. If $|R| < n$, then G does not admit a uni-labeling from R.*

Proof: Suppose to the contrary that G admits a uni-labeling $f : V(G) \to R$. By definition, f is injective; hence $|f(V(G))| = |V(G)| = n$. But $f(V(G)) \subseteq R$ implies $n = |f(V(G))| \leq |R|$, contradicting $|R| < n$. Therefore no such labeling exists. □

Therefore, a necessary condition for a graph of order n to admit a uni-labeling is that $|R| \geq n$.

Theorem 2.2 *Let G be a finite graph and let R be a commutative ring with unity. Suppose that G contains an odd cycle and G admits a uni-labeling from R. Then the following conditions necessarily hold:*

i) $|R| \geq |V(G)|$.

ii) for every maximal ideal $\mathfrak{m} \subset R$, the quotient ring $R/\mathfrak{m}$ is not isomorphic to $\mathbb{Z}_2$.

iii) there exist $a, b \in U(R)$ with $a \neq b$ such that $a + b \in U(R)$.

Proof: (i) Since $f : V(G) \to R$ is injective and G is finite, the set R must contain at least $|V(G)|$ distinct elements. Hence $|R| \geq |V(G)|$.

(ii) Suppose, for contradiction, that there exists a maximal ideal $\mathfrak{m} \subset R$ with residue field $R/\mathfrak{m} \cong \mathbb{Z}_2$. Let $\pi : R \to R/\mathfrak{m} \cong \mathbb{Z}_2$ be the quotient map and consider the labeling $\bar{f} := \pi \circ f : V(G) \to \mathbb{Z}_2$. For each edge $(u,v) \in E(G)$ we have $f(u)+f(v) \in U(R)$, hence $\pi(f(u)+f(v)) \neq 0$ in $R/\mathfrak{m}$; but in $\mathbb{Z}_2$ the unique unit is 1, so

$$\bar{f}(u)+\bar{f}(v)=1 \qquad \text{for every } (u,v)\in E(G).$$

Thus $\bar{f}$ is a function assume only value 0 and 1. Let C be an odd cycle $C = v_1v_2\cdots v_{2m+1}v_1$ we obtain the alternating pattern $0,1,0,1,\ldots$, that gives $\bar{f}(v_1)=\bar{f}(v_{2m+1})$ and simultaneously $\bar{f}(v_1)+\bar{f}(v_{2m+1})=1$, which is a contradiction. Therefore no maximal ideal $\mathfrak{m}$ has $R/\mathfrak{m}\cong\mathbb{Z}_2$.

(iii) Assume further that R is a local ring with maximal ideal $\mathfrak{m}$; then $U(R)=R\setminus\mathfrak{m}$ and the sum of two elements of $\mathfrak{m}$ lies in $\mathfrak{m}$. Let $C=v_1v_2\cdots v_{2m+1}v_1$ be an odd cycle in G. Let us partition the vertex set into two sets

$$U:=\{v_i:\ f(v_i)\in U(R)\},\qquad N:=\{v_i:\ f(v_i)\in\mathfrak{m}\}.$$

Because $\mathfrak{m}$ is an ideal, no edge of C can join two vertices from N otherwise $f(u)+f(v)\in\mathfrak{m}$. If, on the other hand, no edge of C joined two vertices from U, then the cycle C would alternate between U and N, which is impossible for an odd cycle. Hence there exists an edge v_iv_{i+1} in C with $f(v_i)+f(v_{i+1})\in U(R)$. Set $a:=f(v_i)$ and $b:=f(v_{i+1})$. By injectivity, $a\neq b$, and by the uni-labeling property for the edge v_iv_{i+1} we have $a+b\in U(R)$. Thus there exist $a,b\in U(R)$, $a\neq b$, with $a+b\in U(R)$. □

Remark 2.1 The argument establishing (iii) uses the locality of R to rule out edges between two non-units. In a general (non-local) ring the sum of two non-units may be a unit, and the existence of a unit–unit edge on an odd cycle need not follow from the uni-labeling condition alone. Part (ii) is independent of locality.

Corollary 2.1 *Let R be a finite commutative ring where the sum of two units is not a unit of R. If graph G admits a uni-labeling over R, then G must be bipartite.*

From the foregoing analysis, a natural question arises: does every graph admit a uni-labeling? In this context, we have the following result:

Theorem 2.3 *Every finite graph admits unit-labeling.*

Proof: Let G be a finite graph of order n and consider a ring which is field with characteristic 2. Define a function $f: V(G)\to\mathbb{F}_{2^k}$ such that

$$f(u_i)=x_i \quad \forall u_i\in V(G),\ \text{where } k\geq\lceil\log_2 n\rceil\,.$$

Now, for each $(u_i,u_j)\in E(G)$, the labels $f(u_i)$ and $f(u_j)$ are distinct non-zero vectors in $\mathbb{F}_{2^k}$. And, if $f(u_i)\neq f(u_j)$, then

$$f(u_i)+f(u_j)\neq 0.$$

This implies that, for every edge (u_i,u_j), the sum $f(u_i)+f(u_j)\in U(\mathbb{F}_{2^k})$.
Hence, f is a uni-labeling. □

At this stage one may naturally ask about optimal index. In this regard we have the following:

Definition 2.1 Let G be a graph and let $\mathcal{R}$ denote the class of all finite commutative rings with unity. The *uni-labeling index* of G, denoted by $\theta^0(G)$, is defined as

$$\begin{aligned}\theta^0(G)=\min\{|R|\mid R\in\mathcal{R},\ &\text{there exists an injective } f:V(G)\to R \text{ such that}\\ &(u,v)\in E(G)\implies f(u)+f(v)\in U(R)\}\end{aligned}$$

where $U(R)$ is the set of units of R. Any uni-labeling f of G is optimal if it uses $\theta^0(G)$ labels.

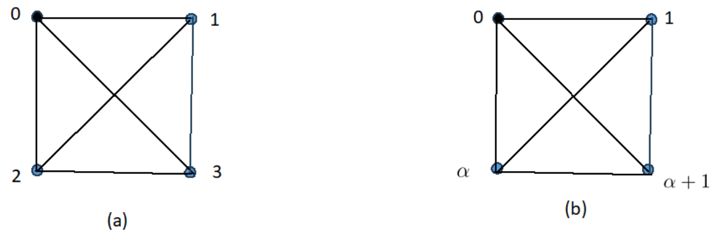

Figure 2: The uni-labeling of K_4 by $\mathbb{Z}_7$ and $\mathbb{F}_4$, respectively

Example 2 The uni-labeling of K_4 with respect to $\mathbb{Z}_7$ is shown in Figure 2(a) and 2(b), respectively. Based on Definition 2.1, it is clearly seen that

$$\theta^0(K_4) = \min\{4, 7\}.$$

Therefore, the optimal index $\theta^0(K_4) = 4$.

Now consider the wheel graph W_6. The uni-labeling with respect to $\mathbb{Z}_7$ and $\mathbb{Z}_{11}$ are shown in Figure 3. Again, in view of Definition 2.1, it can be seen that

$$\theta^0(W_6) = \min\{7, 11\}.$$

Therefore, the optimal index $\theta^0(W_6) = 7$.

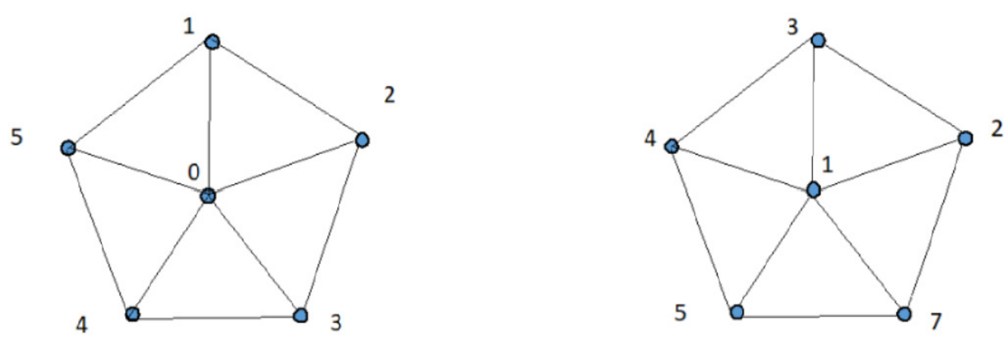

Figure 3: The Uni-labeling of W_6 by $\mathbb{Z}_7$ and $\mathbb{Z}_{11}$, respectively

In view of Theorem 2.3 and Definition 2.1, for a finite graph G of order n, we have the following inequality.

$$n \leq \theta^0(G) \leq 2^k, \tag{2.1}$$

where k is the ceiling of $\log_2 n$.

The bounds in (2.1) indicate the following problem of fundamental importance.

Problem 1 Characterize graphs for which the bounds in (2.1) are attained. Also, determine all the commutative rings, which provide an optimal uni-labeling for these graphs.

Invoking the above results, we obtain a partial solution to Problem 1 for certain special classes of graphs.

Theorem 2.4 *For the complete graph* K_n

$$\theta^0(K_n) = n \textit{ if and only if } n \textit{ is power of 2.}$$

Proof: **(⇒)** Assume $\theta^0(K_n) = n$. By definition of optimal index, there exists a finite commutative ring with unity R such that $|R| = n$ which provides a uni-labeling $f : V(K_n) \to R$. Since $|V(K_n)| = n = |R|$ and f is injective, which gives f is bijective. Hence, for *every* pair of distinct elements $r, s \in R$ (take $r = f(u), s = f(v)$ for distinct vertices $u \neq v$), the uni-labeling condition for the edge (u, v) yields

$$r + s \in U(R).$$

Note that r and s are arbitrary, thus for all distinct $r, s \in R$, $r + s \in U(R)$.

Next we claim that the only possible ring R is field of characteristic 2.

Let if possible R is not field. Then for any $x \in R \setminus \{0\}$ and $x \notin U(R)$ for which $x + 0 \notin U(R)$. It is not possible as all pair of vertices in K_n are adjacent, so R must be field.

Next, if characteristic of field is $p, (p > 2)$, then there exist x and y which are additive inverse of each other. If we label any pair of adjacent vertices by x and y, then $x + y \notin U(R)$, a contradiction to uni-labeling condition. Hence characteristic of field must be 2. Since R is a finite field of characteristic 2, so $|R| = 2^k$ for some $k \in \mathbb{N}$. Thus, we conclude that $n = 2^k$.

(⇐) Conversely, suppose $n = 2^k$. Take $R = \mathbb{F}_{2^k}$, the finite field of order 2^k (characteristic 2). Label the vertices of K_n injectively by distinct elements of R. For any distinct vertices labeled $a, b \in R$ we have $a + b \neq 0$ and each non-zero element of field is a unit so $a + b \in U(R)$, so the uni-condition holds on every edge of K_n. Therefore, we get $\theta^0(K_n) = n$. □

Theorem 2.5 *Let* $K_{m,n}$ *denotes the complete bipartite graph on* $m + n$ *vertices. Then for all* $m, n \in \mathbb{N}$, *one has*

$$m + n \;\leq\; \theta^0(K_{m,n}) \;\leq\; 2^{\lceil \log_2(m+n) \rceil}.$$

Moreover, if $m + n$ *is a power of prime, then* $\theta^0(K_{m,n}) \;=\; m + n$.

Proof: It can be noticed that $\theta^0(K_{m,n}) \geq m + n$ is immediate from injectivity condition of uni-labeling. For the upper bound, take R be field of characteristic 2. Let $R = \mathbb{F}_{2^r}$ with $2^r \geq m + n$ and $r = \lceil \log_2(m + n) \rceil$. In view of Theorem 2.3 and Definition 2.1, one have $\theta^0(K_{m,n}) \leq 2^r$.

Let $m + n = p^k$ for some prime. We shall show the existence of uni-labeling by using exactly $m + n$ labels, i.e., $f : V(K_{m,n}) \to \mathbb{F}_{p^k}$. Let us partition the elements of field into two subsets S and T with $|S| = m$ and set $T := \mathbb{F}_{p^k} \setminus (-S)$. Then $|T| = p^k - m = n$. Label the m vertices in one part by the elements of S, and the n vertices in the other part by the elements of T. This labeling is injective because $S \cap T = \varnothing$ and $|S| + |T| = p^k$.

For any edge between $u \in S$ and $t \in T$, we have $u \neq -v$ by construction, hence $u + v \neq 0$ in $\mathbb{F}_{p^k}$. Since all nonzero elements are units in a field, $u + v \in U(\mathbb{F}_{p^k})$. Thus the uni-condition holds on every edge of $K_{m,n}$.

Thus, we obtain $\theta^0(K_{m,n}) = p^k = m + n$.

Hence the result. □

An *n-dimensional hypercube* $Q_n =: (V_n, E_n)$ is n-regular bipartite graph with 2^n nodes and $n2^{n-1}$ edges, which is obtained by taking two copies of Q_{n-1} and making their corresponding vertices adjacent, where Q_1 is just a path P_2.

Theorem 2.6 *For the d-dimensional hypercube* Q_d

$$\theta^0(Q_d) = 2^d.$$

Proof: By Definition 2.1, $\theta^0(Q_d)$ is the minimum of $|R|$ over all finite commutative rings R with unity that admit an injective labeling $f : V(Q_d) \to R$ such that for every edge $(u, v) \in E(Q_d)$ one has $f(u) + f(v) \in U(R)$.

Clearly, in view of Definition 2.1, we have $\theta^0(Q_d) \geq 2^d$. To show the upper bound let us consider R be a field of characteristic 2 has 2^d elements, and choose any bijection $f : V(Q_d) \xrightarrow{\sim} \mathbb{F}_{2^d}$. For any edge $(u, v) \in E(Q_d)$ the endpoints are distinct vertices, so $f(u) \neq f(v)$. In a field of characteristic 2, the equality $x + y = 0$ holds if and only if $x = y$. Thus $f(u) + f(v) \neq 0$ for every edge. Moreover each non-zero elements of $\mathbb{F}_{2^d}$ is unit so $f(u) + f(v) \in U(\mathbb{F}_{2^d})$. Hence f is a uni-labeling of Q_d over a ring of size 2^d. Thus, $\theta^0(Q_d) \leq 2^d$. □

Theorem 2.7 *For the cycle graph C_n on n vertices,*

$$\theta^0(C_n) = \begin{cases} 4, & n = 3, \\ n, & n \neq 3. \end{cases}$$

Proof: In light of Inequality (2.1), we found that $\theta^0(C_n) \geq n$.

For $n = 3$, there exist precisely one ring, namely, $\mathbb{Z}_3$ from which 1 and 2 can not label end vertex of an edge, so $\theta^0(C_3) > 3$. Next we shall look the ring of order 4, there are 4 rings $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/\langle x^2\rangle$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{F}_4$. Among them only $\mathbb{F}_4$ provides label to C_3 such that it is uni-labeling and $\theta^0(C_3) = 4$.

For $n \neq 3$, take $R = \mathbb{Z}_n$. Consider the unitary addition Cayley graph on the vertex set $\mathbb{Z}_n$ in which x, y are adjacent iff $x + y \in U(\mathbb{Z}_n)$. The map $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$, $\varphi(x) = -x$, is a graph isomorphism from unitary addition Cayley graph to the unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, U(\mathbb{Z}_n))$, whose edges are the pairs x, y with $x - y \in U(\mathbb{Z}_n)$. It is known that $\mathrm{Cay}(\mathbb{Z}_n, U(\mathbb{Z}_n))$ is Hamiltonian for all $n \geq 3$. Hence unitary addition Cayley graph has a Hamilton cycle $a_0, a_1, \ldots, a_{n-1}$ with each $a_i + a_{i+1} \in U(\mathbb{Z}_n)$. Label the i-th vertex of C_n by a_i. Then every edge-sum is a unit in $\mathbb{Z}_n$, so this is a uni-labeling over $\mathbb{Z}_n$, which implies that $\theta^0(C_n) \leq n$. Combined with the lower bound we get $\theta^0(C_n) = n$ for $n \neq 3$. □

Theorem 2.8 *Let P_n be the path on n vertices. Then for every $n \geq 2$,*

$$\theta^0(P_n) = n.$$

Proof: Clearly, if $n = 2$ and 3, the labels are taken from $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively. For $n > 3$, by Theorem 2.7, we have $\theta^0(C_n) = n$, and consequently $\theta^0(P_n) = n$. □

Theorem 2.9 *Let T be a tree on $n \geq 2$ vertices. Then*

$$n \;\leq\; \theta^0(T) \;\leq\; 2^{\lceil \log_2 n \rceil}.$$

Moreover, if n is a power of a prime, then

$$\theta^0(T) = n.$$

Proof: The lower bound $\theta^0(T) \geq n$ follows directly from the definition of $\theta^0(\cdot)$ along with inequality (2.1). For the moreover statement assume that n is a prime power, i.e., $n = p^k$ and consider the field $R = \mathbb{F}_{p^k}$, which consists of p^k elements. It is known that tree is bipartite so its vertex set can be partitioned into two disjoint sets A and B such that every edge joins a vertex in A to a vertex in B. Now there are two possibilities for the characteristic of field:

i) If $p = 2$, then due to Theorem 2.3 any injective function $f : V(T) \to \mathbb{F}_{2^k}$ is uni-labeling. This indicate that $\theta^0(T) = n$

ii) If $p \neq 2$, then partition all the elements of field into $\{0\}$ and $(p^k - 1)/2$ pairs consisting of $\{\alpha, -\alpha\}$, $\alpha \in \mathbb{F}_{p^k}$. Now we shall label each element of A and B. Assign elements of each pair $\{\alpha, -\alpha\}$ to either A or B and 0 on other set according to the requirement of $|A|$ and $|B|$. Since additive inverse lie only in one partition, which ensure that no pair of adjacent vertices receive label α, $-\alpha$ under uni-labeling.

Therefore, for any edge (u, v) with $u \in A$ and $v \in B$, the sum $f(u) + f(v)$ is a unit. This satisfies the condition of uni-labeling using exactly p^k elements from the field $\mathbb{F}_{p^k}$. Hence, $\theta^0(T) \leq n$, and together with the lower bound, we conclude that $\theta^0(T) = n$ when n is a prime power. □

Corollary 2.2 *If H is a subgraph of a graph G, then $\theta^0(H) \leq \theta^0(G)$.*

Proof: Any uni-labeling of G restricts to a uni-labeling of H using the same ring. □

Theorem 2.10 *Let $G = \bigcup_{i=1}^{k} G_i$ be a graph with connected components G_i. Let R be a commutative ring with unity. Suppose that for each $i = 1, \ldots, k$, there exists a uni-labeling $f_i : V(G_i) \longrightarrow R$ such that*

$$f_i\big(V(G_i)\big) \cap f_j\big(V(G_j)\big) = \varnothing \quad \text{for all } i \neq j.$$

Then the function $f : V(G) \to R$ defined by $f(v) = f_i(v), v \in V(G_i)$ is a uni-labeling of G.

Proof: Since the images of the f_i are pairwise disjoint and each f_i is injective, the union map f is injective on $V(G)$.

If $(u, v) \in E(G)$, then u and v lie in the same connected component G_i. For this i, $f(u) + f(v) = f_i(u) + f_i(v) \in U(R)$ because f_i is a uni-labeling of G_i. Thus the uni-labeling condition holds for every edge of G. Therefore f is a uni-labeling of G. □

According to Grimaldi [7], unit graph of a finite commutative ring $\mathbb{Z}_n$ is a graph whose vertices are the elements of $\mathbb{Z}_n$ and two distinct vertices x and y are adjacent if and only if $x + y$ is a unit of $\mathbb{Z}_n$.

Theorem 2.11 *Let G be a finite simple graph and let R be a finite commutative ring with unity. Then G admits a uni-labeling over R if and only if G can be embedded as a subgraph of the unit graph $G(R)$.*

Proof: ($\Rightarrow$) Suppose $f : V(G) \to R$ is a uni-labeling; by definition f is injective and for every edge $(u, v) \in E(G)$ one has $f(u) + f(v) \in U(R)$. Define $\varphi : V(G) \to V(G(R))$ by $\varphi = f$. Then φ is injective. Moreover, if $\{u, v\} \in E(G)$, the condition $f(u) + f(v) \in U(R)$ implies $\{\varphi(u), \varphi(v)\} = \{f(u), f(v)\} \in E(G(R))$. Hence φ is an injective graph homomorphism $G \hookrightarrow G(R)$, i.e., an embedding of G as a subgraph of $G(R)$.

($\Leftarrow$) Conversely, suppose $\varphi : V(G) \to V(G(R)) = R$ is an injective graph homomorphism. For every edge $\{u, v\} \in E(G)$ we have $\{\varphi(u), \varphi(v)\} \in E(G(R))$, so by definition of $G(R)$, $\varphi(u) + \varphi(v) \in U(R)$. Thus $f := \varphi$ is an injective map $V(G) \to R$ with the property that every edge-sum is a unit; in other words, f is a uni-labeling of G over R.

Therefore, G admits a uni-labeling over R if and only if G embeds as a subgraph of $G(R)$. □

Conclusion

This paper introduces the notion of uni-labeling of a graph and its empirical study demonstrates that every finite graph admits a uni-labeling with respect to some finite commutative ring. Further, we introduced the uni-labeling index $\theta^0(G)$ for various classes of graphs over finite commutative rings with unity. We established exact values of $\theta^0(G)$ for fundamental families such as cycles, paths, trees, and complete bipartite graphs, and derived general lower and upper bounds in terms of the order of the graph. In particular, for cycles C_n, we proved that $\theta^0(C_n) = n$ for all $n \neq 3$, with $\theta^0(C_3) = 4$ as the only exceptional case. For paths and trees, we showed that the optimal index coincides with the number of vertices in the graph, and we demonstrated constructive labelings over appropriate rings to achieve these bounds.

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Conflicts of interest

Both authors Pranjali and Seema Swami does not have any conflicts of interest.

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Pranjali,
Department of Mathematics,
University of Rajasthan,
JLN Marg, Jaipur-302004
India.
E-mail address: pranjali48@gmail.com

and

Seema Swami
Department of Mathematics,
University of Rajasthan,
JLN Marg, Jaipur-302004
India.
E-mail address: seemaswami960@gmail.com