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A Smooth Penalty Framework for Solving Nonlinear Inequality-Constrained Optimization Problems

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ABSTRACT: We propose a smooth approximation of the exact l_1 penalty function. This paper presents a novel smooth penalty function designed to address nonlinear programming problems with inequality-constrained. With help of this formulation, an algorithm is developed, and its convergence is rigorously established. The proposed method is tested on two numerical examples to demonstrate its effectiveness, with results compared against those obtained from existing algorithms. The findings demonstrate that the proposed approach provides reliable convergence and competitive performance for solving these types of optimization problems.

Key Words: $\epsilon-$feasibility, constrained optimization, inequality constraints, penalty methods.

Contents

1. Introduction

We study the nonlinear programming problem

$$(NLP_0) \qquad \min\ f_0(x)$$
$$\text{subject to} \quad g_i(x) \leq 0, \quad i \in I,$$
$$x \in \mathbb{R}^n,$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ ($i \in I = \{1, 2, \dots, m\}$) are continuously differentiable functions.

The feasible region is

$$G_0 = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0 \ \ \forall i \in I\},$$

which is assumed to be non-empty.

Penalty function techniques are a standard approach for treating inequality-constrained problems. The core idea is to transform (NLP_0) into an unconstrained optimization problem by augmenting the objective with penalty terms that discourage infeasibility.

A classical example is the quadratic penalty function:

$$G^{(q)}(x, \mu) = f_0(x) + \mu \sum_{i \in I} [g_i(x)]^2,$$

with penalty parameter $\mu > 0$. The corresponding unconstrained problem is

$$(NLP_1) \qquad \min_{x \in \mathbb{R}^n}\ G^{(q)}(x, \mu).$$

Although smooth, this penalty is not exact: obtaining an accurate solution often requires very large values of μ, which is computationally undesirable.

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To overcome this drawback, Zangwill introduced the exact l_1 penalty function [1]. Later on, such type of penalty function have been explored in [2,3,4,5,6,7]. Building on this, researchers have investigated k^{th}-power penalty functions as in [8,9,10,11,12,13]. where $f_0(x)$ is assumed to be positive.

Rubinov and Yang [14] proposed a further variant:

$$G_\mu^k(x,a) = (f_0(x) - a)^k + \mu \sum_{i \in I} [g_i(x)]^+, \tag{1.1}$$

under the assumptions that $(f_0(x) - a) > 0$ for all $x \in \mathbb{R}^n$, with $a \in \mathbb{R}$ and $k > 0$.

The associated problem is

$$\min_{x \in \mathbb{R}^n} \; G_\mu^k(x,a).$$

This is equivalent to

$$(NLP_\mu) \qquad \begin{aligned} &\min \; (f_0(x) - a)^k \\ &\text{subjected to } g_i(x) \le 0, i \in I, \\ &x \in \mathbb{R}^n. \end{aligned}$$

Thus, (NLP_μ) can be interpreted as the l_1 exact penalty problem of (NLP_0).

However, the functions $G^{(q)}(x,\mu)$, $G^{(k)}(x,\mu)$ for $0 < k \le 1$, and $G_\mu^k(x,a)$ are generally nondifferentiable. To enable the application of classical smooth optimization techniques such as Newton's method, smoothing approximations of exact penalty functions have been proposed [12,15,16,17,18].

Unlike most existing smooth penalty approaches, the proposed framework achieves a finer balance between smoothness and exactness. It preserves the theoretical equivalence with the original constrained problem while avoiding the numerical ill-conditioning caused by large penalty parameters. This refinement, though incremental, provides a meaningful advancement in the design of differentiable exact penalty functions.

Motivated by this, we introduce

$$p_{\epsilon,\mu}^k(t) = \begin{cases} 0, & t \le 0, \\ \dfrac{m^3\mu^3 t^{4k}}{6\epsilon^3}, & 0 \le t \le \left(\dfrac{\epsilon}{m\mu}\right)^{1/k}, \\ -\dfrac{7\epsilon}{6m\mu} + t^k + \dfrac{\epsilon}{3m\mu}\exp\Big(-\dfrac{m\mu}{\epsilon}t^k + 1\Big), & t \ge \left(\dfrac{\epsilon}{m\mu}\right)^{1/k}, \end{cases} \tag{1.2}$$

for $0 < k < \infty$, $\mu > 0$, and $\epsilon > 0$.

This construction yields a new smooth penalty function, enabling the constrained problem (NLP_0) to be approximated by a unconstrained problems in sequential form.

2. Smooth Penalty Function

We begin with the basic penalty mapping $p^k(t) : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ defined by

$$p^k(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t^k & \text{if } t > 0 \end{cases} \tag{2.1}$$

where $0 < k < \infty$.

Many important exact penalty functions have been defined using this function

Based on this penalty mapping, we have

$$G_\mu^k(x,a) = (f_0(x) - a)^k + \mu \sum_{i \in I} p^k(g_i(x)) \tag{2.2}$$

and penalty problem

$$(NP_\mu) \qquad \min G_\mu^k(x,a) \qquad \text{s.t. } x \in \mathbb{R}^n. \tag{2.3}$$

For $0 < k < \infty$ and $\mu > 0$, the function $p^k_{\epsilon,\mu}(t)$ is defined as:

$$p^k_{\epsilon,\mu}(t) = \begin{cases} 0, & \text{if } t \le 0 \\ \frac{m^3\mu^3 t^{4k}}{6\epsilon^3}, & \text{if } t \ge 0 \text{ and } -t + \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}} \ge 0 \\ -\frac{7\epsilon}{6m\mu} + t^k + \frac{\epsilon}{3m\mu}\exp\left(-\frac{m\mu}{\epsilon}t^k + 1\right), & \text{if } t \ge \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}} \end{cases} \tag{2.4}$$

where ϵ is the smoothing parameter.

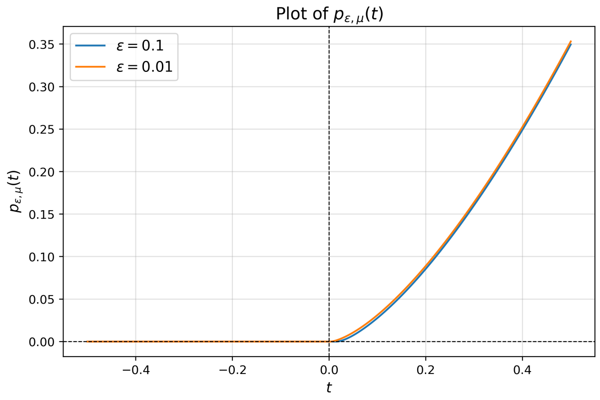

Figure 1: The behavior of $p^k_{\epsilon,\mu}(t)$ at m = 3, k = 3/2, and $\rho = 10$

We proceed to analyze the properties of $p^k_{\epsilon,\mu}(t)$.

Theorem 2.1 *For $0 < k < \infty$ and $\epsilon > 0$,*

1. *$p^k_{\epsilon,\mu}(t) \in C^1$ for $k > \frac{1}{4}$ where*

$$[p^k_{\epsilon,\mu}(t)]' = \begin{cases} 0, & \text{if } t \le 0 \\ \frac{2km^3\mu^3}{3\epsilon^3}t^{4k-1}, & \text{if } t \ge 0 \text{ and } -t + \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}} \ge 0 \\ kt^{k-1} + kt^{k-1}\exp\left(-\frac{m\mu}{\epsilon}t^k + 1\right), & \text{if } t \ge \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}} \end{cases} \tag{2.5}$$

2. *$p^k_{\epsilon,\mu}(t) =$ convereges to $p^k(t)$ as $\epsilon \to 0$.*

Proof: 1. Note that $p^k_{\epsilon,\mu}(t)$ is continuous at every point on $\mathbb{R}$ except 0 and $\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}$. So, we need to check the continuity of $p^k_{\epsilon,\mu}(t)$ at 0 and $\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}$ only.

(i) *Continuity of $p^k_{\epsilon,\mu}(t)$ at $t = 0$* is easy to verify.

(ii) *Continuity of $p^k_{\epsilon,\mu}(t)$ at $t = \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}$*:

$$\lim_{t \to \left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^-} p^k_{\epsilon,\mu}(t) =$$

$$\lim_{t \to \left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^-} \frac{m^3\mu^3 t^{4k}}{6\epsilon^3} = \frac{m^3\mu^3}{6\epsilon^3}\left(\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right)^{4k} = \frac{\epsilon}{6m\mu}.$$

$$\lim_{t \to \left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^+} p^k_{\epsilon,\mu}(t) = \lim_{t \to \left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^+} -\frac{7\epsilon}{6m\mu} + t^k + \frac{\epsilon}{3m\mu}\exp\left(-\frac{m\mu}{\epsilon}t^k + 1\right) = \frac{\epsilon}{6m\mu}$$

Also $p^k_{\epsilon,\mu}\left(\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right) = \frac{\epsilon}{6m\mu}$.

Therefore; $lim_{t\to\left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^-} p^k_{\epsilon,\mu}(t) = lim_{t\to\left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^+} p^k_{\epsilon,\mu}(t) = \frac{\epsilon}{6m\mu} = p^k_{\epsilon,\mu}\left(\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right)$

Now we prove that $p^k_{\epsilon,\mu}(t)$ is continuously differentiable. For this, we need to prove that $[p^k_{\epsilon,\mu}(t)]'1$ is continuous at 0 and $\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}$.

(i) *Continuity of* $[p^k_{\epsilon,\mu}(t)]'$ *at* $t = 0$ is again easy to verify.

(ii) *Continuity of* $[p^k_{\epsilon,\mu}(t)]'$ *at* $t = \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}$*:*

$lim_{t\to\left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^-} [p^k_{\epsilon,\mu}(t)]' =$

$lim_{t\to\left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^-} \frac{2km^3\mu^3}{3\epsilon^3}t^{4k-1} = \frac{2k}{3}\left(\frac{\epsilon}{m\mu}\right)^{1-\frac{1}{k}}$

$lim_{t\to\left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^+} [p^k_{\epsilon,\mu}(t)]' =$

$lim_{t\to\left[\left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}\right]^+} kt^{k-1} + kt^{k-1}\exp\left(-\frac{m\mu}{\epsilon}t^k + 1\right) = \frac{2k}{3}\left(\frac{\epsilon}{mu}\right)^{1-\frac{1}{k}}$

2. Now from definition of $p^k(t)$ and $p^k_{\epsilon,\mu}(t)$, we have:

$$p^k(t) - p^k_{\epsilon,\mu}(t) = \begin{cases} 0, & \text{if } t \le 0 \\ t^k - \frac{m^3\mu^3t^{4k}}{6\epsilon^3}, & \text{if } t \ge 0 \text{ and } -t + \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}} \ge 0 \\ \frac{7\epsilon}{6m\mu} - \frac{\epsilon}{3m\mu}\exp\left(-\frac{m\mu}{\epsilon}t^k + 1\right), & \text{if } t \ge \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}} \end{cases} \tag{2.6}$$

When $0 \le t \le \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}$, let $w = t^k$.

Then, $0 \le w \le \frac{\epsilon}{m\mu}$.

Consider the function

$$\psi(w) = w - \frac{m^3\mu^3w^4}{6\epsilon^3},\ 0 \le w \le \frac{\epsilon}{m\mu}$$

The derivative of $\psi(w)$ is given by

$$\psi'(w) = 1 - \frac{2m^3\mu^3t^3}{3\epsilon^3},\ 0 \le w \le \frac{\epsilon}{m\mu}$$

Obviously, $\psi'(u) > 0$ for $0 \le u \le \frac{\epsilon}{m\mu}$. Moreover, $\psi(0) = 0$ and $\psi\left(\frac{\epsilon}{m\mu}\right) = \frac{5\epsilon}{6m\mu}$. Hence, we have

$$0 \le p^k(t) - p^k_{\epsilon,\mu}(t) \le \frac{5\epsilon}{6m\mu}.$$

For $t \ge \left(\frac{\epsilon}{m\mu}\right)^{\frac{1}{k}}$, we have

$$0 < p^k(t) - p^k_{\epsilon,\mu}(t) = \frac{7\epsilon}{6m\mu} - \frac{\epsilon}{3m\mu}\exp\left(-\frac{m\mu}{\epsilon}t^k + 1\right) \le \frac{7\epsilon}{6m\mu}.$$

$$\implies \lim_{\epsilon\to 0} p^k_{\epsilon,\mu}(t) = p^k(t)$$

Moreover; from above results; it is clear that

$$p^k(t) \geq p^k_{\epsilon,\mu}(t), \ \forall\, t \in \mathbf{R}.$$

□

We assume that the parameter a is negative and the absolute value of a is sufficiently large which leads to the condition that that $f_0(x) - a > 0$ ensuring $f_0(x) - a$ to be always positive.

$$G^k_{\epsilon,\mu}(x,a) = [f_0(x) - a]^k + \mu \sum_{i=1}^{m} p^k_{\epsilon,\mu}\left(g_i(x)\right), \quad 0 < k < +\infty$$

Subsequently, $G^k_{\epsilon,\mu}(x,a)$ represents an adjusted penalty function that exhibits continuous differentiability for any $x \in \mathbb{R}^n$. It serves as a smooth approximation to $G^k_{\mu}(x,a)$. The smoothed penalty problem is as follows:

$$(NP_{\epsilon,\mu}) \quad \min G^k_{\epsilon,\mu}(x,a) \quad \text{s.t. } x \in \mathbb{R}^n.$$

Theorem 2.2 *Let $x \in \mathbb{R}^n$ be arbitrary, then*

$$0 \leq G^k_{\mu}(x,a) - G^k_{\epsilon,\mu}(x,a) \leq \frac{7\epsilon}{6}, \quad 0 < k < +\infty$$

Proof: We have

$$G^k_{\mu}(x,a) = (f_0(x) - a)^k + \mu \sum_{i \in I} p^k(g_i(x))$$

and

$$G^k_{\epsilon,\mu}(x,a) = (f_0(x) - a)^k + \mu \sum_{i=1}^{m} p^k_{\epsilon,\mu}\left(g_i(x)\right), \quad 0 < k < +\infty$$

As

$$0 \leq p^k(t) - p^k_{\epsilon,\mu}(t) \leq \frac{7\epsilon}{6m\mu}$$

so we have

$$0 \leq G^k_{\mu}(x,c) - G^k_{\epsilon,\mu}(x,c) \leq \frac{7\epsilon}{6}.$$

□

Theorem 2.3 *Let (P) and (NP) have optimal solutions as $\hat{x}$ and $\hat{x}_\mu$ respectively. If $\hat{x}_\mu$ is feasible to (P) then for same problem, it is optimal.*

Proof: Note that

$$f_0\left(\hat{x}_\mu\right) \geq f_0\left(\hat{x}\right)$$

Moreover, as per given conditions

$$0 < \left[f_0\left(\hat{x}_\mu\right) - a\right]^k = G^k_{\mu}\left(\hat{x}_\mu, a\right) \leq G^k_{\mu}\left(\hat{x}, a\right) = \left[f_0\left(\hat{x}\right) - a\right]^k, \quad 0 < k < +\infty.$$

which implies that

$$f_0\left(\hat{x}_\mu\right) \leq f_0\left(\hat{x}\right)$$

□

Theorem 2.4 *Consider $\hat{x}_\mu$ and $\hat{x}_{\epsilon,\mu}$ as the respective minimizers of (NP_μ) and $(NP_{\epsilon,\mu})$ for some $\mu > 0$ and $\epsilon > 0$. Then*

$$0 \le G_\mu^k(\hat{x}_\mu, a) - G_{\epsilon,\mu}^k(\hat{x}_{\epsilon,\mu}, a) \le \frac{7\epsilon}{6}, \quad 0 < k < +\infty$$

Proof: By Theorem 2.2,

$$0 \le G_\mu^k(\hat{x}_\mu, a) - G_{\epsilon,\mu}^k(\hat{x}_\mu, a) \le \frac{7\epsilon}{6}$$

and

$$0 \le G_\mu^k(\hat{x}_{\epsilon,\mu}, a) - G_{\epsilon,\mu}^k(\hat{x}_{\epsilon,\mu}, a) \le \frac{7\epsilon}{6}.$$

As $\hat{x}_\mu$ is optimal solution to (NP_μ), so

$$G_\mu^k(\hat{x}_\mu, a) \le G^k\mu(\hat{x}_{\epsilon,\mu}, a)$$

and similarly we have

$$G_\mu^k(\hat{x}_{\epsilon,\mu}, a) \le G_\mu^k(\hat{x}_\mu, a)$$

Hence;

$$0 \le G_\mu^k(\hat{x}_\mu, a) - G_{\epsilon,\mu}^k(\hat{x}_{\epsilon,\mu}, a) \le \frac{7\epsilon}{6}$$

From the definition of $p^k(t), p_{\epsilon,\mu}^k(t)$ and the fact that $\hat{x}_\mu, \hat{x}_{\epsilon,\mu}$ are feasible for problem (P), we have

$$\sum_{i=1}^m p^k(g_i(\hat{x}_\mu)) = \sum_{i=1}^m p_{\epsilon,\mu}^k(g_i(\hat{x}_{\epsilon,\mu})) = 0.$$

Further as $f(\hat{x}_{\epsilon,\mu}) - a > 0$, thus

$$f_0(\hat{x}_{\epsilon,\mu}) \le f_0(\hat{x}_\mu)$$

Thus, $\hat{x}_{\epsilon,\mu}$ is an optimal solution for problem (P). □

3. Algorithm

In this section, we introduce an algorithm, based on the smoothed penalty function, to compute an optimal solution of problem (NLP_0).

Definition: For $\epsilon > 0$, a point $\hat{x}_\epsilon \in X_0$ is called an ϵ-feasible solution to (NLP_0), if it satisfies $g_i(\hat{x}_\epsilon) \le \epsilon$ for all $i \in I$.

Algorithm : Algorithm for solving problem (P)

Step 1: Choose an initial point x_1^0. Select parameters $\epsilon_1 > 0, \mu_1 > 0, 0 < \lambda < 1, \beta > 1$ and a constant $a < 0$ such that $f_0(x) - a > 0, \forall x \in X_0$, let $j = 1$ and proceed to Step 2 .

Step 2: Use x_j^0 as the starting point to solve the following problem:

$$(NP_{\epsilon_j,\mu_j}) \quad \min_{x\in\mathbb{R}^n} G_{\epsilon_j,\mu_j}^k(x, a) = [f_0(x) - a]^k + \mu_j \sum_{i=1}^m p_{\epsilon_j,\mu_j}^k(g_i(x))$$

Let x_{ϵ_j,μ_j}^* be an optimal solution of (NP_{ϵ_j,μ_j}), which is obtained by the BFGS method [19].

Step 3: If $x^*_{\epsilon_j,\mu_j}$ is ϵ-feasible for problem (P), then the algorithm stops and $x^*_{\epsilon_j,\mu_j}$ is an approximate optimal solution of problem (P). Otherwise, let $\mu_{j+1} = \beta\mu_j, \epsilon_{j+1} = \lambda\epsilon_j, x^0_{j+1} = x^*_{\epsilon_j,\mu_j}$. Set $j \leftarrow j+1$ and return to Step 2.

Remark: Since $0 < \lambda < 1, \beta > 1$, it follows that as $j \to +\infty$, the sequence $\{\epsilon_j\} \to 0$ and the sequence $\{\mu_j\} \to +\infty$.

4. Numerical Examples

Now we test the feasibility of the given algorithm.
In each example, we take $\epsilon = 10^{-6}$ and $a = -100$.

Example 1. Solve problem given in " [21]

$$\begin{aligned} \min f_0(x) &= x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4 \\ \text{s.t. } g_1(x) &= 2x_1^2 + x_2^2 + x_3^2 + 2x_1 + x_2 + x_4 - 5 \le 0, \\ g_2(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8 \le 0, \\ g_3(x) &= x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10 \le 0." \end{aligned}$$

For $k = \frac{3}{2}$, let $x^0_1 = (5, 5, 5, 5)$, $\mu_1 = 10$, $\beta = 4$, $\epsilon_1 = 0.01$, $\lambda = 0.1$. The results are shown in Table 1. After five iterations, an approximate optimal solution is obtained as

$$x^* = (0.1697, 0.8357, 2.0085, -0.9650)$$

with an objective function value of

$$f(x^*) = -44.2339.$$

Table 1: Optimization results for different μ_j and ϵ_j values

Sl. No.	μ_j	ϵ_j	$x^*_{\epsilon_j,\mu_j}$	$f(x^*_{\epsilon_j,\mu_j})$	$g_1(x^*_{\epsilon_j,\mu_j})$	$g_2(x^*_{\epsilon_j,\mu_j})$	$g_3(x^*_{\epsilon_j,\mu_j})$
1	40	0.001	$(0.1873, 0.8247, 2.1502, -1.1791)$	-47.1159	0.3938	1.4209	-0.2088
2	160	0.0001	$(0.1697, 0.8321, 2.0216, -0.9880)$	-44.5074	0.0204	0.1313	-1.7289
3	640	10^{-5}	$(0.1696, 0.8353, 2.0095, -0.9664)$	-44.2517	1.22×10^{-3}	8.56×10^{-3}	-1.8731
4	2560	10^{-6}	$(0.1696, 0.8355, 2.0087, -0.9650)$	-44.2350	7.61×10^{-5}	5.37×10^{-4}	-1.8824
5	10240	10^{-7}	$(0.1697, 0.8357, 2.0085, -0.9650)$	-44.2339	4.78×10^{-6}	3.36×10^{-5}	-1.8825

Example 2. Solve the problem in " [20]

$$\begin{aligned} &\min f_0(x) = -x_1 - x_2 \\ &\text{s.t. } g_1(x) = -2x_1^4 + 8x_1^3 - 8x_1^2 + x_1 - 2 \le 0, \\ &\quad g_2(x) = -4x_1^4 + 32x_1^3 - 88x_1^2 + 96x_1 + x_2 - 36 \le 0, \\ &\quad 0 \le x_1 \le 3, \\ &\quad 0 \le x_2 \le 4." \end{aligned}$$

For $k = \frac{3}{2}$, let $x^0_1 = (1.5, 2.0), \mu_1 = 40, \beta = 10, \epsilon_1 = 0.001, \lambda = 0.1$.

From Table 2 ,we observe that in iteration 1, the optimization starts with $\mu_1 = 40$ and $\epsilon_1 = 0.001$, leading to an approximate solution $x^*_{\epsilon_1,\mu_1} = (2.1484, 4.0000)$, with an objective function value of $f(x^*_{\epsilon_1,\mu_1}) = -6.1484$. The constraint values are $g_1 = -0.0548$ and $g_2 = 0.1741$.

As the iterations progress, μ_j and ϵ_j decrease, and the solution improves. By iteration 5, with $\mu_5 = 10240$ and $\epsilon_5 = 1 \times 10^{-7}$, the approximate solution becomes $x^*_{\epsilon_5,\mu_5} = (2.0000, 4.0000)$ with $f(x^*_{\epsilon_5,\mu_5}) = -6.0000$. The constraint values approach near-zero values: $g_1 = 1.43 \times 10^{-5}$ and $g_2 = 1.65 \times 10^{-9}$, indicating that the optimization has nearly converged.

Table 2: Iteration results of the optimization process

Sl. No.	μ_j	ϵ_j	$x^*_{\epsilon_j,\mu_j}$	$f(x^*_{\epsilon_j,\mu_j})$	$g_1(x^*_{\epsilon_j,\mu_j})$	$g_2(x^*_{\epsilon_j,\mu_j})$
1	40	0.001	$(2.1484, 4.0000)$	-6.1484	-0.0548	0.1741
2	160	0.0001	$(2.0847, 4.0000)$	-6.0847	0.0223	0.0572
3	640	1×10^{-5}	$(2.0043, 4.0000)$	-6.0043	0.0041	0.0001
4	2560	1×10^{-6}	$(2.0002, 4.0000)$	-6.0002	2.31×10^{-4}	4.29×10^{-7}
5	10240	1×10^{-7}	$(2.0000, 4.0000)$	-6.0000	1.43×10^{-5}	1.65×10^{-9}

Conclusions

This work developed a smooth penalty framework for tackling nonlinear optimization problems with inequality constraints. The key idea was to replace the nonsmooth components of the classical exact penalty with a continuously differentiable approximation, thereby transforming the original constrained formulation into a sequence of smooth unconstrained subproblems. An iterative algorithm was proposed, and its convergence was established under mild assumptions. Numerical experiments on benchmark problems confirmed that the method consistently produces feasible solutions and achieves objective values comparable to, or better than, existing approaches. The results further indicate that the smoothing strategy improves numerical stability and efficiency, making the algorithm attractive for practical implementation. Future directions include extending the approach to problems with equality constraints, investigating large-scale applications, and refining the algorithm to enhance scalability and computational speed.

References

1. Zangwill, W. I. (1967). Non-linear programming via penalty functions. Management science, 13(5), 344-358.
2. Bazaraa, M. S., Sherali, H. D., & Shetty, C. M. (2006). Nonlinear programming: theory and algorithms. John wiley & sons.
3. Di Pillo, G., & Grippo, L. (1986). An exact penalty function method with global convergence properties for nonlinear programming problems. Mathematical Programming, 36(1), 1-18.
4. Han, S. P., & Mangasarian, O. L. (1979). Exact penalty functions in nonlinear programming. Mathematical programming, 17(1), 251-269.
5. Lasserre, J. B. (1981). A globally convergent algorithm for exact penalty functions. European Journal of Operational Research, 7(4), 389-395.
6. Chen, C., & Mangasarian, O. L. (1996). A class of smoothing functions for nonlinear and mixed complementarity problems. Computational Optimization and Applications, 5(2), 97-138.
7. Rosenberg, E. (1984). Exact penalty functions and stability in locally Lipschitz programming. Mathematical programming, 30(3), 340-356.
8. Huang, X. X., & Yang, X. Q. (2001). Duality and exact penalization for vector optimization via augmented Lagrangian. Journal of Optimization Theory and Applications, 111(3), 615-640.
9. Rubinov, A. M., Glover, B. M., & Yang, X. Q. (1999). Decreasing functions with applications to penalization. SIAM Journal on Optimization, 10(1), 289-313.
10. Rubinov, A. M., Glover, B. M., & Yang, X. Q. (1999). Extended Lagrange and penalty functions in continuous optimization. Optimization, 46(4), 327-351.
11. Rubinov, A. M., Glover, B. M., & Yang, X. Q. (1999). Extended Lagrange and penalty functions in continuous optimization. Optimization, 46(4), 327-351.
12. Binh, N. T. (2015). Second-order smoothing approximation to l1 exact penalty function for nonlinear constrained optimization problems. Theoretical Mathematics and Applications, 5(3), 1-17.
13. Yang, X. Q. (1995). Smoothing approximations to nonsmooth optimization problems. The ANZIAM Journal, 36(3), 274-285.
14. Rubinov, A. M., & Yang, X. Q. (2013). Lagrange-type functions in constrained non-convex optimization (Vol. 85). Springer Science & Business Media.

15. Binh, N. T., & Yan, W. (2015). Smoothing approximation to the k-th power nonlinear penalty function for constrained optimization problems. Journal of Applied Mathematics and Bioinformatics, 5(2), 1.

16. Wu*, Z. Y., Bai, F. S., Yang, X. Q., & Zhang, L. S. (2004). An exact lower order penalty function and its smoothing in nonlinear programming. Optimization, 53(1), 51-68.

17. Xu, X., Meng, Z., Sun, J., Huang, L., & Shen, R. (2013). A second-order smooth penalty function algorithm for constrained optimization problems. Computational optimization and applications, 55(1), 155-172.

18. Zenios, S. A., Pinar, M. C., & Dembo, R. S. (1995). A smooth penalty function algorithm for network-structured problems. European Journal of Operational Research, 83(1), 220-236.

19. Wright, S., & Nocedal, J. (1999). Numerical optimization. Springer Science, 35(67-68), 7.

20. Wu, Z. Y., Lee, H. W. J., Bai, F. S., & Zhang, L. S. (2005). Quadratic smoothing approximation to l_1 exact penalty function in global optimization. Journal of Industrial and Management Optimization, 1(4), 533-547.

21. Meng, Z., Dang, C., Jiang, M., & Shen, R. (2011). A smoothing objective penalty function algorithm for inequality constrained optimization problems. Numerical functional analysis and optimization, 32(7), 806-820.

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