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Line Graph associated with Equiprime Graph of a Nearring

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ABSTRACT: This paper presents a graph-theoretic exploration of three nearring-based graphs—namely, the generalized graph $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$, the equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$, and the central graph $\mathcal{C}_{\mathcal{I}}(\mathbb{N})$—each defined with respect to a fixed ideal $\mathcal{I}$ of a nearring $\mathbb{N}$. Focusing on their respective line graphs, we analyze the structural changes that emerge when the original adjacency is lifted to edge adjacencies. The motivation stems from recent advances in algebraic graph theory, where such constructions have yielded insights into ideal-related interactions within algebraic systems. Through theoretical results and illustrative examples, we demonstrate how the line graph of the equiprime graph captures nuanced connectivity patterns and contributes to a finer classification of algebraic elements. This unified approach reveals new perspectives on how algebraic structure influences graph-theoretic behavior.

Key Words: Equiprime graph, line graph, nearring, graph homomorphism, algebraic structures.

Contents

1. Introduction

Graphs arising from algebraic structures offer a visual and combinatorial perspective on algebraic concepts, revealing relationships not always apparent through traditional symbolic approaches. The concept of associating graphs to rings began with Beck [1], who defined a graph whose vertices are ring elements, with adjacency determined by multiplication resulting in zero. This idea evolved into diverse graph constructions for rings and nearrings, incorporating zero-divisors, ideals, and prime elements [6,14,15].

In the context of nearrings, which generalize rings by relaxing certain distributivity requirements, several graph-theoretic models have been introduced to analyze their structural features. The structural characteristics and behavioral patterns of zero-divisor graphs within nearrings have been the subject of significant mathematical investigation. In [5,7] and Bhavanari et al. constructed the *ideal graph* $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$ based on the interaction between nearring elements and ideals [3]. Building on this, further work introduced variations such as the equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ and the central graph $\mathcal{C}_{\mathcal{I}}(\mathbb{N})$ to better capture finer ideal-theoretic behaviors [12].

Line graphs, originally introduced in pure graph theory, represent adjacency between edges rather than vertices [9]. They have found meaningful applications in algebraic contexts when applied to the graphs associated with algebraic structures. Line graphs retain edge information from the base graph and thus encode how certain algebraic relationships are shared among elements. The notion of line graphs

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associated with nearring graphs has recently gained attention [11], highlighting unexplored aspects in their structural interplay.

In recent years, the fusion of algebraic concepts and graph-theoretic methods have revealed deep connections that not only enrich both fields but also open new avenues for applications in coding theory, cryptography, and combinatorial optimization. By translating algebraic properties into graphical language, one can often detect hidden symmetries, invariants, and structural decompositions that are less transparent in purely algebraic terms. In particular, equiprime graphs and their line graph counterparts capture delicate interactions between ideals and elements of a nearring, offering a refined perspective compared to traditional zero-divisor graphs. Moreover, these graphical approaches provide a unifying platform where tools from spectral graph theory, graph homomorphisms, and combinatorial algorithms can be applied to algebraic problems. Such perspectives motivate the systematic study of line graphs associated with nearring-based constructions, extending the existing literature on rings to the more general framework of nearrings.

This paper aims to unify and generalize previous studies by systematically examining the line graphs of $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$, $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$, and $\mathcal{C}_{\mathcal{I}}(\mathbb{N})$. We study their mutual relationships, determine conditions for their equivalence, and analyze the influence of algebraic properties like zero-symmetry, right permutability, and the IFP on the corresponding line graphs. We also investigate the impact of nearring homomorphisms on these graphical structures and introduce new results involving the subgraph structure of line graphs restricted to $\mathbb{N}\backslash\mathcal{I}$. This investigation builds on a strong foundation laid by prior works in nearring theory [13,8,2] and graph homomorphisms [10].

Throughout the paper, illustrative examples and counterexamples demonstrate the necessity of hypotheses and highlight the distinctions between various graph constructions.

2. Preliminaries

This section summarizes the fundamental definitions and results required for our study. All nearrings considered are associative and contain a multiplicative identity unless otherwise specified.

Definition 2.1 (Nearring) We refer [13,8,2] for detailed background of nearrings.
A *nearring* $(\mathbb{N}, +, \cdot)$ is a nonempty set $\mathbb{N}$ with two operations satisfying the following:

- The structure $(\mathbb{N}, +)$ forms a (not necessarily abelian) group.
- The operation $\cdot$ makes $(\mathbb{N}, \cdot)$ a semigroup.
- The structure satisfies **right distributivity**, i.e., for all $x, y, z \in \mathbb{N}$, the identity $(x + y) \cdot z = xz + yz$ holds.

Definition 2.2 (Equiprime ideal) An ideal $\mathcal{I}$ is called *equiprime* if, whenever $xny - xn0 \in \mathcal{I}$ holds for every $n \in \mathbb{N}$, then at least one among x or y is in $\mathcal{I}$. This concept was formalized in [12].

Definition 2.3 (Equiprime Graph)
Let $\mathcal{I}$ be an ideal of $\mathbb{N}$ and $p \in \mathbb{N}$. Let $EQ^{p}_{\mathcal{I}}(\mathbb{N})$ be the graph with vertex set $\mathbb{N}$ and the pair of distinct points p and $(x - y)$ are connected by an edge if and only if $prx - pry \in \mathcal{I}$ for all $r \in \mathbb{N}$ or $(x-y)rp-(x-y)r0 \in \mathcal{I}$ for all $r \in \mathbb{N}$. Then $EQ_{\mathcal{I}}(\mathbb{N}) = \cup_{p\in\mathbb{N}} EQ^{p}_{\mathcal{I}}(\mathbb{N})$ is called the *equiprime graph of* $\mathbb{N}$ *with respect to* $\mathcal{I}$. If we restrict the vertex set of $EQ_{\mathcal{I}}(\mathbb{N})$ to $\mathbb{N} \setminus \mathcal{I}$ then the graph obtained after deleting the isolated vertices if any is denoted by $EQ_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I})$. This concept was introduced in [12]

Definition 2.4 (Central Graph)
The *central graph* $\mathcal{C}_{\mathcal{I}}(\mathbb{N})$ has the vertex set $\mathbb{N}$ and two elements $x, y \in \mathbb{N}$ are connected by an edge if the product xy or yx is in $\mathcal{I}$, [3].

Definition 2.5 (Generalized Graph)
The graph $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$ is defined with vertex set $\mathbb{N}$, where two elements $x, y \in \mathbb{N}$ are adjacent if the set $x\mathbb{N}y$ or $y\mathbb{N}x$ is entirely contained in $\mathcal{I}$.

Table 1: Table of Notations

Notation	Meaning
$\mathbb{N}$	A nearring defined with binary operations $(+, \cdot)$.
$\mathcal{I}$	A nontrivial ideal contained within $\mathbb{N}$.
$\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$	Equiprime graph formed from $\mathbb{N}$ based on the ideal $\mathcal{I}$.
$\mathcal{L}(\mathcal{G})$	Line graph derived from a given graph $\mathcal{G}$.
$\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$	Line graph constructed by $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$.
$\mathcal{C}_{\mathcal{I}}(\mathbb{N})$	Central graph indicating elements whose product belongs to $\mathcal{I}$.
$\mathcal{G}_{\mathcal{I}}(\mathbb{N})$	generalized graph of $\mathbb{N}$ with respect to $\mathcal{I}$.
girth$(\mathcal{G})$	Length of the smallest cycle occurring in $\mathcal{G}$.
diam$(\mathcal{G})$	Maximum distance between any two vertices in the graph $\mathcal{G}$.

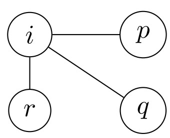

Figure 1: $\mathrm{EQ}^{i}_{\mathcal{I}}(\mathbb{N})$

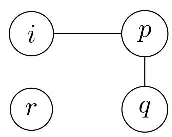

Figure 2: $\mathrm{EQ}^{p}_{\mathcal{I}}(\mathbb{N})$

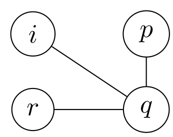

Figure 3: $\mathrm{EQ}^{q}_{\mathcal{I}}(\mathbb{N})$

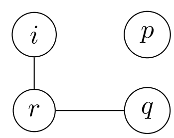

Figure 4: $\mathrm{EQ}^{r}_{\mathcal{I}}(\mathbb{N})$

Figure 5: Equiprime graphs $\mathrm{EQ}^{x}_{\mathcal{I}}(\mathbb{N})$for $x \in \mathbb{N}$ for $\mathcal{I} = \{i, q\}$

Definition 2.6 (Line Graph) The *line graph* $\mathcal{L}(\mathcal{G})$ of a graph $\mathcal{G}$ is obtained by assigning a vertex to each edge of G. Two vertices in $\mathcal{L}(\mathcal{G})$ are adjacent if and only if their corresponding edges in G share a common vertex [9].

Proposition 2.1 *[11], Let $\mathbb{N}$ be a nearring. If $\mathcal{I}$ is a c-prime ideal, then $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))$.*

3. Line Graph of the Equiprime Graph over a Nearring

Definition 3.1 Let $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ denote the equiprime graph constructed from a nearring $\mathbb{N}$ and an ideal $\mathcal{I}$. The line graph associated with it, denoted by $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, is defined such that each of its vertices corresponds to an edge in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$. Two distinct vertices in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ are connected by an edge if and only if their respective edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ share a common vertex.

Example 3.1 *Consider the nearring $\mathbb{N} = \{i, p, q, r\}$ as defined in Table 2. It can be verified that this structure forms a nearring.*

$+$	i	p	q	r
i	i	p	q	r
p	p	i	r	q
q	q	r	i	p
r	r	q	p	i

$\cdot$	i	p	q	r
i	i	i	i	i
p	i	p	i	p
q	q	q	q	q
r	q	r	q	r

Table 2: Operation tables defining a nearring structure on $\mathbb{N} = \{i, p, q, r\}$

Let $\mathcal{I} = \{i, q\}$. The equiprime graphs $\mathrm{EQ}^{x}_{\mathcal{I}}(\mathbb{N})$ for various elements $x \in \mathbb{N}$ are illustrated in Figure 5. The full equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ and corresponding line graph $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ are shown in Figure 6 and 7 respectively.

Proposition 3.1 *Let $\mathcal{I}$ be a proper ideal of $\mathbb{N}$. Then $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ is connected and its diameter satisfies* $\mathrm{diam}(\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))) \leq 3$.

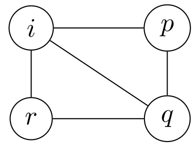

Figure 6: $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ with $\mathcal{I} = \{i, q\}$

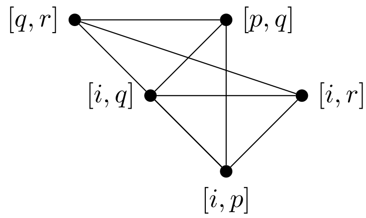

Figure 7: $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ for $\mathcal{I} = \{i, q\}$

Proof:

Since $0 \in \mathbb{N}$ is adjacent to all other vertices in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$, every edge in the graph involves 0 or is connected through it. Let $A = [p, q]$ and $B = [r, s]$ be two distinct vertices in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, corresponding to edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$.

If A and B share a vertex in the original graph (that is, $p = r$, $p = s$, $q = r$, or $q = s$), then they are adjacent in the line graph, giving a path of length 1. If not, then there exists a path through vertex 0, such as: $[p, q] \sim [p, 0] \sim [0, s] \sim [r, s]$, which is a path of length 3. Thus, in all cases, any two vertices in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ have the distance at most 3 between them, proving the claim. □

Proposition 3.2 *Let $\mathcal{I} = \{0\}$ be a 3-prime ideal of a simple, zero-symmetric integral nearring $\mathbb{N}$. If $|\mathbb{N}| = n$, then $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ forms a complete graph.*

Proof: Since $\mathbb{N}$ is both zero-symmetric and integral, the element 0 is linked to all nonzero elements in the graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$. Suppose an edge $\langle a, b\rangle$ exists with $a, b \neq 0$. Then either $anb - an0 \in \mathcal{I}$ or $bna - bn0 \in \mathcal{I}$ for every $n \in \mathbb{N}$. Given the zero-symmetry property, $an \cdot 0 = 0$ and $bn \cdot 0 = 0$, implying anb and bna belong to $\mathcal{I}$.

Since $\mathcal{I} = \{0\}$ is 3-prime, either $a \in \mathcal{I}$ or $b \in \mathcal{I}$ must hold, which contradicts our assumption that both a and b are nonzero. Hence, the only edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ are those that include 0 — that is, of the form $(0, x_i)$ for $x_i \in \mathbb{N} \setminus \{0\}$, making the graph a star centered at 0.

Therefore, in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, each vertex corresponds to an edge sharing the vertex 0 in the original graph, and thus every pair of vertices is adjacent. This proves that the line graph is complete. □

Remark 3.1 The above result no longer holds if the nearring fails to be integral or zero-symmetric. In such cases, the equiprime graph need not be a star graph, and hence $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ may not be complete. We provide following example.

Example 3.2 *Consider the nearring $\mathbb{N} = \{i, p, q, r\}$ defined by the binary operations $+$ and $\cdot$ as given in Table 2 (see Example 3.1) and the ideal 3-prime ideal $\mathcal{I} = \{i\}$.*

The corresponding equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ is illustrated in Figure 8, while its associated $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ appears in Figure 9.

Observe that although $\mathcal{I} = \{i\}$ is a 3-prime, the nearring does not satisfy the conditions required for the completeness of the line graph:

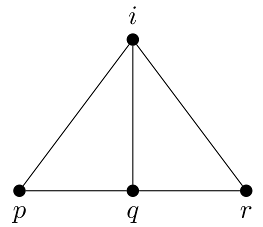

Figure 8: Equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$

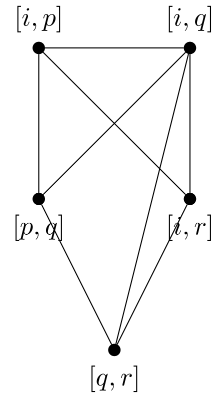

Figure 9: $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$

- *$\mathbb{N}$ is not zero-symmetric since $p \cdot q = i$ but $q \cdot p \neq i$,*
- *$\mathbb{N}$ is not integral because $p \cdot q = i$ even though $p \neq i$ and $q \neq i$.*

As a result, the equiprime graph is not a star graph, and consequently, $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ is not complete.

Proposition 3.3 *Let $\mathcal{I} = \{0\}$ be a 3-prime ideal of a simple, zero-symmetric integral nearring $\mathbb{N}$ of order n. Then $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ is a complete graph.*

Proof: Since $\mathbb{N}$ is simple, its only ideals are $\{0\}$ and $\mathbb{N}$ itself. If $\mathcal{I} = \mathbb{N}$, then every pair of elements in $\mathbb{N}$ is adjacent in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$, making it a complete graph, and so is its line graph.

Now consider the nontrivial case $\mathcal{I} = \{0\}$. Under the given assumptions (integrality, zero-symmetry, and 3-primality), Proposition 3.2 applies, and hence $\mathcal{L}(\mathrm{EQ}_{\{0\}}(\mathbb{N}))$ is complete. □

Proposition 3.4 *Let $\mathbb{N}$ be an integral nearring of order n, and let $\mathcal{I} = \{0\}$ be an equiprime ideal. Then $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ is a complete graph.*

Proof: Since $\mathbb{N}$ is integral, no two nonzero elements multiply to zero. Let $a, b \in \mathbb{N} \setminus \{0\}$ and suppose that (a, b) is an edge in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$. Then either anb or bna lies in $\mathcal{I}$ for all $n \in \mathbb{N}$. As $\mathcal{I}$ is equiprime, this implies either $a \in \mathcal{I}$ or $b \in \mathcal{I}$, which contradicts the assumption that $a, b \neq 0$.

Therefore, every edge in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ must involve the zero element. The graph is thus a star centered at 0, and its line graph becomes a complete graph since all edges are incident at 0. □

Proposition 3.5 *Let $\mathbb{N}$ be an integral, zero-symmetric nearring of order n, and let $\mathcal{I} = \{0\}$ be a 3-prime ideal of $\mathbb{N}$. Then $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ has infinite girth, i.e., $\mathrm{girth}(\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))) = \infty$, if and only if $n \in \{2, 3\}$.*

Proof: ($\Rightarrow$) Suppose $\mathrm{girth}(\mathcal{L}(\mathrm{EQ}_{\{0\}}(\mathbb{N}))) = \infty$. This implies that the line graph contains no cycles. Now assume $n > 3$. Then the equiprime graph contains at least four vertices, and since 0 is adjacent to all other elements, the graph includes at least three distinct edges such as $[0, x_1], [0, x_2], [0, x_3]$, which share the common vertex 0. This structure introduces cycles in the line graph, contradicting the assumption. Hence, $n \leq 3$.

($\Leftarrow$) Conversely, if $n = 2$, then $\mathbb{N} = \{0, a\}$ and the only edge in $\mathrm{EQ}_{\{0\}}(\mathbb{N})$ is $(0, a)$. The line graph has a single vertex and no edges, thus no cycles.

If $n = 3$, let $\mathbb{N} = \{0, a, b\}$. Then only edges involving 0 are present in the equiprime graph (since it is integral and $\mathcal{I}$ is 3-prime). Hence, $[0, a]$ and $[0, b]$ represent the sole vertices within the line graph, and they are not adjacent as they share only the vertex 0 in the original graph. This produces no cycle, so the girth is again infinite.

Therefore, $\mathrm{girth}(\mathcal{L}(\mathrm{EQ}_{\{0\}}(\mathbb{N}))) = \infty$ if and only if $n = 2$ or $n = 3$. □

Remark 3.2 The previous result highlights that the girth becomes finite once the nearring has four or more elements. For instance, in Example 3.1, where $|\mathbb{N}| > 3$, we observe that $\mathrm{girth}(\mathcal{L}(\mathrm{EQ}_{\{0\}}(\mathbb{N}))) \neq \infty$.

Proposition 3.6 *Let N be an integral, zero-symmetric nearring and $|N| = n$. Let I be a 3-prime ideal of simple nearring N. Then $gr(\mathcal{L}(EQ_I(N))) = \infty$ if and only if $|N| = 2$ or $|N| = 3$.*

Proposition 3.7 *Let $\mathbb{N}$ be an integral nearring of order n, and let $\mathcal{I} = \{0\}$ be an equiprime ideal of $\mathbb{N}$. Then $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ has infinite girth if and only if $n = 2$ or $n = 3$.*

Proposition 3.8 *Let $\mathcal{I}$ be an equiprime ideal of a nearring $\mathbb{N}$. Then the set $P = \{\langle a, b\rangle \mid a \in \mathcal{I}$ or $b \in \mathcal{I}\}$ is a dominating set of vertices in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.*

Proof: Let (a, b) be any edge in the equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$. According to the definition of an equiprime ideal, for all $n \in \mathbb{N}$, either $anb - an0 \in \mathcal{I}$ or $bna - bn0 \in \mathcal{I}$. This implies that at least one of a or b lies in $\mathcal{I}$. Thus, every edge in the graph has at least one endpoint from $\mathcal{I}$.

Therefore, all such edges (a, b) where either $a \in \mathcal{I}$ or $b \in \mathcal{I}$, together form an edge-dominating set in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$. Consequently, the corresponding vertices $\langle a, b\rangle$ in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ dominate the remaining vertices through adjacency. Hence, P is a vertex dominating set. □

Remark 3.3 The preceding result is valid only when $\mathcal{I}$ is an equiprime ideal; otherwise the set $P = \{[a, b] \mid a \in \mathcal{I}$ or $b \in \mathbb{N}\}$ might not act as a dominating set in the corresponding line graph. For example, as shown in Example 2.6, the ideal $\mathcal{I} = \{0\}$ does not qualify as equiprime, since for instance $bnc - bn0 = 0 \in \mathcal{I}$ holds for all $n \in \mathbb{N}$, yet neither b nor c is an element of $\mathcal{I}$. Consequently, under such conditions, the set $\{[0, n] \mid n \in \mathbb{N}\}$ fails to dominate all vertices in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.

Proposition 3.9 *Let $a, b \in \mathbb{N}$, and let $\mathcal{I}$ be a proper ideal of $\mathbb{N}$. If $|\mathbb{N}| > 3$, then the vertex $[a, b]$ in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ cannot be a pendant vertex.*

Proof: Since the element 0 shares adjacency with every vertex in the equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$, all edges of the form $[0, x]$ for $x \in \mathbb{N} \setminus \mathcal{I}$ appear as vertices in the line graph.

Given $[a, b] \in \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, it follows from the structure of $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ that this edge shares a vertex (either a or b) with $[0, a]$ or $[0, b]$. Hence, $[a, b]$ must be adjacent to at least one other vertex in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, and thus it cannot be pendant. □

Remark 3.4 The condition $|\mathbb{N}| > 3$ stated earlier is not always essential. For smaller nearrings, a pendant vertex may occur. For instance, consider the nearring $\mathbb{N} = \mathbb{Z}_3$, representing integers modulo 3, and let the ideal be $\mathcal{I} = \{0\}$. The equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ and its line graph are depicted in Figures 10 and 11, respectively.

In this case, the line graph contains exactly two vertices corresponding to the edges $[0, 1]$ and $[0, 2]$, which are joined by a single edge. Each of these is adjacent to only one other vertex, making both of them pendant. This demonstrates that the size condition in the previous proposition is indeed necessary.

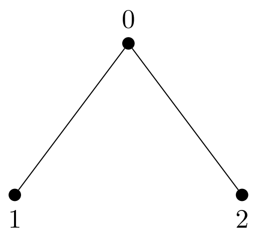

Figure 10: Equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{Z}_3)$

Figure 11: $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{Z}_3))$

Proposition 3.10 *Let $\mathcal{I}$ be an equiprime ideal of a nearring $\mathbb{N}$, and let $a, b \in \mathbb{N}$ such that $[a, b]$ is a vertex in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. Then either $a \in \mathcal{I}$ or $b \in \mathcal{I}$.*

Proof: If either $a = 0$ or $b = 0$, then the result holds trivially since the zero element is contained in every ideal of $\mathbb{N}$. Now assume $a \neq 0$ and $b \neq 0$.

Since $[a, b]$ is a vertex in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, it corresponds to the edge (a, b) in the equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$. By definition of the equiprime graph, we know that either $anb - an0$ or $bna - bn0$ is in $\mathcal{I}$ for all $n \in \mathbb{N}$.

Without loss of generality (WLOG), assume $anb - an0 \in \mathcal{I}$ for all $n \in \mathbb{N} \implies a \in \mathcal{I}$ or $b \in \mathcal{I}$. Thus, the claim follows in all cases. □

Remark 3.5 The conclusion of the preceding proposition may not hold when $\mathcal{I}$ fails to be an equiprime ideal. To illustrate, take $\mathcal{I} = \{0\}$ in Example 3.2, which is not equiprime. In this case, the pair (b, c) forms an edge in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$, resulting in $[b, c]$ being included as a vertex in the corresponding line graph. However, neither b nor c is contained in $\mathcal{I}$, demonstrating that the equiprime assumption is essential for the conclusion to be valid.

4. Interrelations between $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$, $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, and $\mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))$

Proposition 4.1 *Let $\mathbb{N}$ be a nearring and $\mathcal{I}$ a 3-prime ideal of $\mathbb{N}$. Then $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) \subseteq \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.*

Proof: Since $V(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = V(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})) = \mathbb{N}$, their line graphs are defined on the same vertex set.

Let $\langle x, y\rangle$ and $\langle z, w\rangle$ be adjacent vertices in $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$. Then the edges $(x \sim y)$ and $(z \sim w)$ in $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$ share a common vertex. We may assume WLOG, $x = z$.

From the definition of $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$, we have: $x\mathbb{N}y \subseteq \mathcal{I}$ and $x\mathbb{N}w \subseteq \mathcal{I}$. Since $\mathcal{I}$ is a 3-prime ideal, it follows that either $x \in \mathcal{I}$ or $y \in \mathcal{I}$, and either $x \in \mathcal{I}$ or $w \in \mathcal{I}$.

Case 1: If $x \in \mathcal{I}$, then for all $n \in \mathbb{N}$, $xny - xn0 \in \mathcal{I}$, $\quad xnw - xn0 \in \mathcal{I}$, implying $(x \sim y)$ and $(x \sim w)$ are edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$.

Case 2: If $y, w \in \mathcal{I}$, then $ynx - yn0 \in \mathcal{I}$, $\quad wnx - wn0 \in \mathcal{I}$, so $(y \sim x)$ and $(w \sim x)$ are edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$.

Hence, in either case, $\langle x, y\rangle$ and $\langle z, w\rangle$ are adjacent in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. Therefore, $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) \subseteq \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. □

Example 4.1 *This example illustrates that the inclusion $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) \subseteq \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ may be strict, even when $\mathcal{I}$ is a 3-prime ideal of the nearring $\mathbb{N}$.*

Consider the 3-prime ideal $\mathcal{I} = \{i\}$ of the nearring $\mathbb{N} = \{i, p, q, r\}$ as defined in Table 2.

The graphs $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$ and $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ are shown in Figure 12 and Figure 13 , and the corresponding $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$ and $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ are shown in Figure 14 and Figure 15.

In this illustration, the vertex $\langle y, z\rangle$ and its associated edges are present in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, but absent in $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$. Therefore, $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) \subset \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.

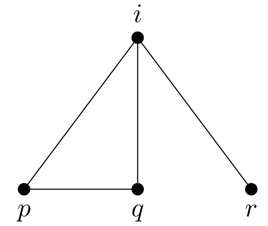

Figure 12: Graph $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$

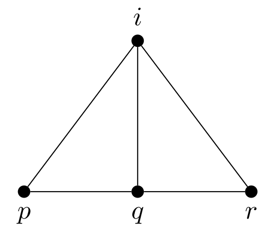

Figure 13: Graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$

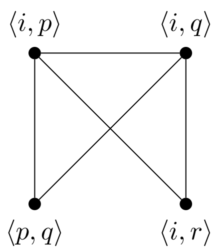

Figure 14: Line graph $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$

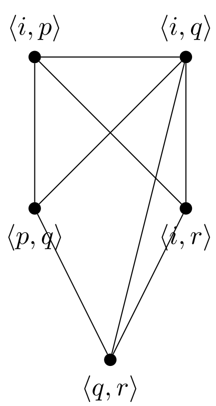

Figure 15: Line graph $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$

Proposition 4.2 *Let $\mathcal{I}$ be an equiprime ideal of the nearring $\mathbb{N}$. Then $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.*

Proof: Since all equiprime ideals satisfy the condition of being 3-prime, by Proposition 4.1, we have $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) \subseteq \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.

Now, let $\langle x, y\rangle$ and $\langle z, w\rangle$ be adjacent vertices in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. Then the corresponding edges (x, y) and (z, w) in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ share a common vertex. WLOG, assume $x = z$.

From the definition of $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$, we have: $xny - xn0 \in \mathcal{I}$ and $xnw - xn0 \in \mathcal{I}$ for all $n \in \mathbb{N}$. Since $\mathcal{I}$ is equiprime, this implies: - Either $x \in \mathcal{I}$ or $y \in \mathcal{I}$, and - Either $x \in \mathcal{I}$ or $w \in \mathcal{I}$.

If $x \in \mathcal{I}$, then $x\mathbb{N}y \subseteq \mathcal{I}$ and $x\mathbb{N}w \subseteq \mathcal{I}$, so both edges are in $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$.

If $y \in \mathcal{I}$ and $w \in \mathcal{I}$, then $y\mathbb{N}x \subseteq \mathcal{I}$ and $w\mathbb{N}x \subseteq \mathcal{I}$, again both edges are in $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$.

Hence, $\langle x, y\rangle$ and $\langle z, w\rangle$ are adjacent in $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$, establishing the reverse inclusion.

Therefore, $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. □

Proposition 4.3 *Let $\mathcal{I}$ be an ideal in the nearring $\mathbb{N}$. Then the line graph of the generalized graph $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$ coincides with the line graph of the equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$, that is, $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, provided at least one of the following conditions is satisfied:*

1. *$\mathbb{N}$ is zero-symmetric,*
2. *$\mathbb{N}$ satisfies right distributivity, or*
3. *$\mathcal{I}$ is a totally reflexive ideal of $\mathbb{N}$.*

Proof: This result is proved under the assumption that $\mathbb{N}$ is zero-symmetric. The arguments under the other two conditions follow analogously.

Assume $\mathbb{N}$ is zero-symmetric and $\mathcal{I}$ is an ideal of $\mathbb{N}$. Let $\langle x, y\rangle$ and $\langle z, w\rangle$ be adjacent vertices in $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$. Then the edges (x, y) and (z, w) in $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$ share a common vertex. WLOG, let $x = z$.

From the definition of $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$: $x\mathbb{N}y \subseteq \mathcal{I}$ or $y\mathbb{N}x \subseteq \mathcal{I}$, $x\mathbb{N}w \subseteq \mathcal{I}$ or $w\mathbb{N}x \subseteq \mathcal{I}$.

Case 1: If $x\mathbb{N}y \subseteq \mathcal{I}$ and $x\mathbb{N}w \subseteq \mathcal{I}$, then for all $n \in \mathbb{N}$, $xny, xnw \in \mathcal{I}$.
Since $\mathbb{N}$ is zero-symmetric: $xny - xn0 \in \mathcal{I}$, $xnw - xn0 \in \mathcal{I}$, so (x, y) and (x, w) are edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$, hence $\langle x, y\rangle$ and $\langle x, w\rangle$ are adjacent in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.

Case 2: If $y\mathbb{N}x \subseteq \mathcal{I}$ and $w\mathbb{N}x \subseteq \mathcal{I}$, then for all $n \in \mathbb{N}$, $ynx, wnx \in \mathcal{I}$. By zero-symmetry: $ynx - yn0 \in \mathcal{I}$, $wnx - wn0 \in \mathcal{I}$, so (y, x) and (w, x) are edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ and $\langle y, x\rangle$, $\langle w, x\rangle$ are vertices in the line graph.

Therefore: $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) \subseteq \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.

Now, assume $\langle x, y\rangle$ and $\langle z, w\rangle$ are adjacent in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. Then (x, y) and (z, w) are edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ sharing a vertex. Let $x = z$.

From the definition of $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$: $xny - xn0 \in \mathcal{I}$ or $ynx - yn0 \in \mathcal{I}$, $xnw - xn0 \in \mathcal{I}$ or $wnx - wn0 \in \mathcal{I}$.

Case 3: For all $n \in \mathbb{N}$, if $xny - xn0 \in \mathcal{I}$ and $xnw - xn0 \in \mathcal{I}$, then zero-symmetry implies $xny, xnw \in \mathcal{I}$, so $x\mathbb{N}y, x\mathbb{N}w \subseteq \mathcal{I}$ and hence $(x, y), (x, w) \in \mathcal{G}_{\mathcal{I}}(\mathbb{N})$.

Case 4: If $ynx - yn0 \in \mathcal{I}$ and $wnx - wn0 \in \mathcal{I}$ for all $n \in \mathbb{N}$, then $ynx, wnx \in \mathcal{I}$, i.e., $y\mathbb{N}x, w\mathbb{N}x \subseteq \mathcal{I}$.

Thus: $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})) \subseteq \mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$.

Hence, the line graphs are equal: $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. □

Remark 4.1 The equality $\mathcal{L}\left(\mathcal{G}_{\mathcal{I}}(\mathbb{N})\right) = \mathcal{L}\left(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})\right)$, as stated in Proposition 4.3, does not necessarily hold under general conditions. This is illustrated in Example 4.2, where the nearring $\mathbb{N}$ fails to satisfy one or more of the following properties: zero-symmetry, distributivity, or total reflexivity. The absence of any of these conditions can lead to a divergence between the line graphs of $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$ and $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$.

Proposition 4.4 *Let $\mathcal{I}$ be an equiprime ideal of a right permutable nearring $\mathbb{N}$. Then the following equality holds: $\mathcal{L}\left(\mathcal{G}_{\mathcal{I}}(\mathbb{N})\right) = \mathcal{L}\left(\mathcal{C}_{\mathcal{I}}(\mathbb{N})\right) = \mathcal{L}\left(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})\right).$*

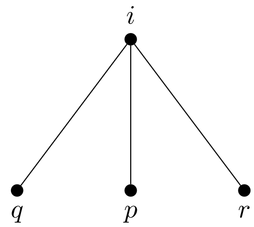

Figure 16: Graph $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$

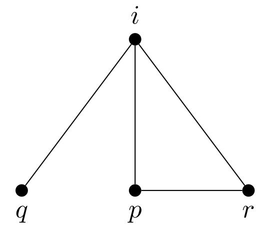

Figure 17: Graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$

Proof: Suppose $x, y \in \mathbb{N}$ are such that $xy \in \mathcal{I}$. Then, for every $n \in \mathbb{N}$, we have $xyn \in \mathcal{I}$, i.e., $xy\mathbb{N} \subseteq \mathcal{I}$. Due to the right permutability of $\mathbb{N}$, it follows that $x\mathbb{N}y \subseteq \mathcal{I}$, meaning that $x\mathbb{N}y$ is contained in $\mathcal{I}$. Since $\mathcal{I}$ is equiprime, we conclude that either $x \in \mathcal{I}$ or $y \in \mathcal{I}$.

This observation confirms that $\mathcal{I}$ is a 3-prime ideal. Further, the right permutability of $\mathbb{N}$ ensures that 3-primality implies c-primality. Hence, $\mathcal{I}$ also satisfies the condition of being c-prime.

Now, by Proposition 2.9, we know that every equiprime ideal is 3-prime, and under right permutability, 3-primality implies c-primality. Proposition 4.3 asserts that $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, while Proposition 2.9 implies $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))$.

Combining both results, we conclude: $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. □

Proposition 4.5 *Let $\mathcal{I}$ be an equiprime ideal of a nearring $\mathbb{N}$. If $\mathcal{I}$ satisfies the IFP, then: $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.*

Proof: Let $x, y \in \mathbb{N}$ such that $xy \in \mathcal{I}$. Since $\mathcal{I}$ satisfies the IFP, it follows that $xny \in \mathcal{I}$ for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$: $xny - xn0 \in \mathcal{I}$, since $xn0 \in \mathcal{I}$ as constant terms lie in every ideal.

As $\mathcal{I}$ is equiprime, $x \in \mathcal{I}$ or $y \in \mathcal{I}$. Therefore, $\mathcal{I}$ is also a c-prime ideal.

By Proposition 2.1, this implies:

$$\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N})). \tag{4.1}$$

Furthermore, since $\mathcal{I}$ is equiprime, Proposition 4.2 gives:

$$\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})). \tag{4.2}$$

Combining equations (4.1) and (4.2), we conclude: $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. □

Proposition 4.6 *Let $\mathcal{I}$ be a 3 - prime ideal of $\mathbb{N}$. Suppose $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$ and $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ share the same vertex set, i.e., $V(\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))) = V(\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})))$, then $\mathcal{I}$ is an equiprime ideal of $\mathbb{N}$.*

Proof: Let $x, y \in \mathbb{N}$ be such that $xny - xn0 \in \mathcal{I}$ for all $n \in \mathbb{N}$. Then $(x, y) \in E(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, so $\langle x, y\rangle \in V(\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})))$.

Given the vertex sets are equal, we have $\langle x, y\rangle \in V(\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})))$, which implies $(x, y) \in E(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$.

Hence, $x\mathbb{N}y \subseteq \mathcal{I}$ or $y\mathbb{N}x \subseteq \mathcal{I}$.

Since $\mathcal{I}$ is 3-prime, it follows that $x \in \mathcal{I}$ or $y \in \mathcal{I}$. Therefore, $\mathcal{I}$ is an equiprime ideal of $\mathbb{N}$. □

Remark 4.2 We now demonstrate through counterexamples that the conditions in the preceding propositions are necessary.

Example 4.2 *Let $\mathbb{N} = \{i, p, q, r\}$ be a nearring as defined in Table 2. Let $\mathcal{I} = \{i\}$. The graphs $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$ and $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ are shown in Figures 18 and 19, respectively.*

Observe that $V(\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))) \neq V(\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})))$. Moreover, $\mathcal{I} = \{i\}$ is not an equiprime ideal of $\mathbb{N}$, since $qna - qni \in \mathcal{I}$ for all $n \in \mathbb{N}$, but $q \notin \mathcal{I}$ and $p \notin \mathcal{I}$.

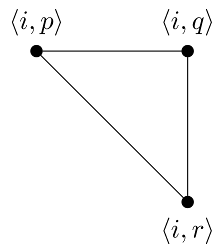

Figure 18: $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$

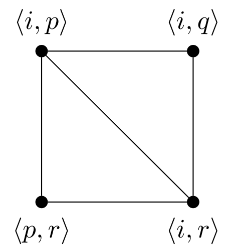

Figure 19: $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$

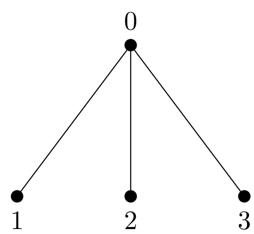

Figure 20: Graph $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$

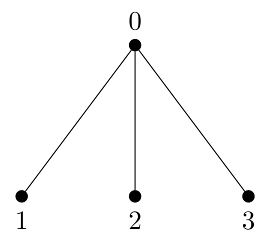

Figure 21: Graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$

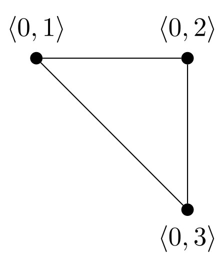

Figure 22: $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$

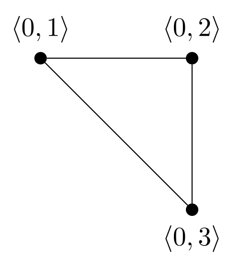

Figure 23: $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$

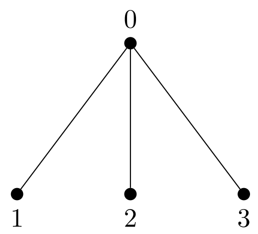

Figure 24: $\mathcal{C}_{\mathcal{I}}(\mathbb{N}) = \mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$

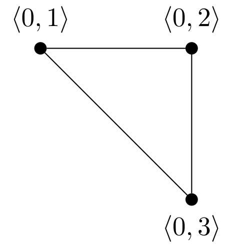

Figure 25: $\mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$

Example 4.3 *Let $\mathbb{N} = \mathbb{Z}_4$, and let $\mathcal{I} = \{0\}$. The graphs $\mathcal{G}_{\mathcal{I}}(\mathbb{N})$ and $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ are shown below: Here, $V(\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))) = V(\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})))$, but $\mathcal{I} = \{0\}$ is not a 3-prime ideal of $\mathbb{N}$ since $2n2 = 0 \in \mathcal{I}$ for all $n \in \mathbb{N}$, while $2 \notin \mathcal{I}$. Hence, $\mathcal{I}$ is not equiprime.*

Proposition 4.7 *Let $\mathcal{I}$ be an equiprime ideal of a nearring $\mathbb{N}$. If $V(\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))) = V(\mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N})))$, then $\mathcal{I}$ is a c-prime ideal of $\mathbb{N}$.*

Proof: Assume $x, y \in \mathbb{N}$ such that $xy \in \mathcal{I}$. By definition of $\mathcal{C}_{\mathcal{I}}(\mathbb{N})$, we get $(x, y) \in E(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))$. Since vertex sets are equal, i.e., $V(\mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))) = V(\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})))$, it follows that the edge (x, y) also belongs to $E(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$. Hence, for all $n \in \mathbb{N}$, we have $xny - xn0 \in \mathcal{I}$. As $\mathcal{I}$ is equiprime, this implies $x \in \mathcal{I}$ or $y \in \mathcal{I}$. Therefore, $\mathcal{I}$ satisfies the definition of a c-prime ideal. □

Remark 4.3 The conclusion drawn in Proposition 4.7 may not remain valid if the ideal $\mathcal{I}$ is not equiprime. In particular, even when the vertex sets of the corresponding line graphs are identical, it is still possible for $\mathcal{I}$ to fail being c-prime.

Example 4.4 *Let $\mathbb{N} = \mathbb{Z}_4$, and $\mathcal{I} = \{0\}$. Here, the equiprime graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ and the central graph $\mathcal{C}_{\mathcal{I}}(\mathbb{N})$ coincide, as depicted in Figure 24, and their respective line graphs also match, shown in Figure 25.*

We observe that $V(\mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))) = V(\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})))$. However, $\mathcal{I} = \{0\}$ is not equiprime, because for all $n \in \mathbb{N}$: $2n2 - 2n0 = 0 \in \mathcal{I}$, but $2 \notin \mathcal{I}$. Additionally, $\mathcal{I}$ fails to be c-prime since $2 \cdot 2 = 0 \in \mathcal{I}$, but $2 \notin \mathcal{I}$.

5. Graph Homomorphisms Induced by Nearring Homomorphisms

Proposition 5.1 *Let $\eta : \mathbb{N}_1 \to \mathbb{N}_2$ be a surjective nearring homomorphism, and let $\mathcal{I}$ be an ideal of $\mathbb{N}_1$. Then η induces a graph homomorphism from $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}_1))$ to $\mathcal{L}(\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2))$.*

Proof: Suppose $\langle x, y\rangle$ and $\langle y, z\rangle$ are adjacent vertices in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}_1))$. Then (x, y) and (y, z) are edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}_1)$, meaning: $xny - xn0 \in \mathcal{I}$ and $ynz - yn0 \in \mathcal{I}$, $\forall n \in \mathbb{N}_1$. Applying η and using its homomorphic property: $\eta(xny - xn0) = \eta(x)\eta(n)\eta(y) - \eta(x)\eta(n)\eta(0) \in \eta(\mathcal{I})$, which shows that $(\eta(x), \eta(y))$ is an edge in $\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2)$. Similarly, $(\eta(y), \eta(z))$ is also an edge in $\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2)$.

Thus, the images $\langle\eta(x), \eta(y)\rangle$ and $\langle\eta(y), \eta(z)\rangle$ are adjacent vertices in $\mathcal{L}(\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2))$, completing the proof. □

Proposition 5.2 *Let $\eta : \mathbb{N}_1 \to \mathbb{N}_2$ be a surjective nearring homomorphism, and let $\mathcal{I}$ be an ideal of $\mathbb{N}_1$. Then η induces a graph homomorphism from $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}_1))$ to $\mathcal{L}(\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2))$.*

Proof: Suppose $\langle x, y\rangle$ and $\langle y, z\rangle$ are adjacent vertices in $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}_1))$. Then (x, y) and (y, z) are edges in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}_1)$, meaning: $xny - xn0 \in \mathcal{I}$ and $ynz - yn0 \in \mathcal{I}$, $\forall n \in \mathbb{N}_1$. Applying η and using its homomorphic property: $\eta(xny - xn0) = \eta(x)\eta(n)\eta(y) - \eta(x)\eta(n)\eta(0) \in \eta(\mathcal{I})$, which shows that $(\eta(x), \eta(y))$ is an edge in $\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2)$. Similarly, $(\eta(y), \eta(z))$ is also an edge in $\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2)$.

Thus, the images $\langle\eta(x), \eta(y)\rangle$ and $\langle\eta(y), \eta(z)\rangle$ are adjacent vertices in $\mathcal{L}(\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2))$, completing the pf. □

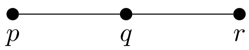

Figure 26: $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I})$

Figure 27: $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I}))$

Proposition 5.3 *Let $\eta : \mathbb{N}_1 \to \mathbb{N}_2$ be a surjective homomorphism of nearrings, and let $\mathcal{I}$ be an ideal of $\mathbb{N}_1$. Then the set $P = \{\langle \eta(x), \eta(y) \rangle \mid x, y \in \mathcal{I}\}$ forms a vertex dominating set in $\mathcal{L}(\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2))$.*

Proof: From earlier work (Theorem 3.6(iii) of Kedukodi et al.) it follows that the edge set $\{(x, y) \mid x \in \mathcal{I}, y \in \mathbb{N}_1\}$ forms an edge-dominating set in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}_1)$.

By Proposition 5.1, the homomorphism η maps this edge-dominating set to $\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2)$, and hence its image under η is: $\{\langle \eta(x), \eta(y) \rangle \mid x \in \mathcal{I}, y \in \mathbb{N}_1\}$. In particular, for $x, y \in \mathcal{I}$, the set P contains vertices whose incident edges dominate the structure of $\mathcal{L}(\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N}_2))$ through adjacency. Thus, P is a vertex dominating set. □

Definition 5.1 Let $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I})$ denotes the equiprime graph constructed from the elements of a nearring $\mathbb{N}$ excluding its ideal $\mathcal{I}$. $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I}))$ is a graph in which each vertex corresponds to an edge in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I})$, and an edge is drawn between two such vertices whenever their corresponding edges in the equiprime graph meet at a common vertex.

Example 5.1 *Let $\mathbb{N} = \{i, p, q, r\}$ be a nearring as defined in Table 2. Let $\mathcal{I} = \{i\}$. The graph of $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I})$ and $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I}))$ can be seen in Figure 26 and Figure 27 respectively.*

Proposition 5.4 *$\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I}))$ is a subgraph of $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$.*

Proposition 5.5 *If $\mathcal{I}$ is an equiprime ideal of the nearring $\mathbb{N}$, then $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I}))$ is an empty graph.*

Proof: Suppose for contradiction, that (x, y) is an edge in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I})$ for some $x, y \in \mathbb{N} \setminus \mathcal{I}$. Then, by definition, for some $n \in \mathbb{N}$, $xny - xn0 \in \mathcal{I}$ or $ynx - yn0 \in \mathcal{I}$. WLOG, suppose $xny - xn0 \in \mathcal{I}$ for every $n \in \mathbb{N}$. Since $\mathcal{I}$ is equiprime, $x \in \mathcal{I}$ or $y \in \mathcal{I}$ — a contradiction, as both were chosen outside $\mathcal{I}$. Thus, no such edge exists in $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I})$, and its line graph must be empty. □

Remark 5.1 The converse of Proposition 5.5 does not necessarily hold. That is, if $\mathcal{I}$ is not equiprime, $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I}))$ may still be non-empty.

For instance, let $\mathcal{I} = \{0\}$,(as in example 5.5) which is not equiprime in $\mathbb{N}$ since, for every $n \in \mathbb{N}$, it is possible that $ynz - yn0 = 0 \in \mathcal{I}$ while $y, z \notin \mathcal{I}$. Hence, $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N} \setminus \mathcal{I}))$ is clearly not empty.

6. Applications

The study of graph-theoretic structures over nearrings offers both structural clarity and functional insights in various mathematical and applied domains. In particular, analyzing the interconnections among $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, $\mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))$, and $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$ reveals patterns that are relevant in algebraic computation, error analysis, and secure system design.

1. **Fault Tracing in Algebraic Computation:** In computational models built on nearrings, certain products yield zero divisors that may indicate computation errors. The graph $\mathrm{EQ}_{\mathcal{I}}(\mathbb{N})$ identifies such interactions. Its line graph $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ helps trace how these faults could influence neighboring operations. For instance, when $\mathbb{N} = \mathbb{Z}_4$ and $\mathcal{I} = \{0\}$, the connectivity highlights how operations involving 0 can affect broader computation chains.

2. **Structure-Based Cryptographic Analysis:** The central graph $\mathcal{C}_{\mathcal{I}}(\mathbb{N})$ collects all element pairs whose product lies in the ideal $\mathcal{I}$. A dense set of such connections may signal algebraic weakness in systems relying on nearring-based operations. Examining $\mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))$ aids in identifying and avoiding structurally predictable patterns in cryptographic routines.

3. **Ideal Classification via Graph Comparison:** In some nearrings, it is observed that $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N})) = \mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ if and only if the ideal $\mathcal{I}$ satisfies both 3-prime and equiprime properties. This equality can be utilized in software implementations to algorithmically verify algebraic conditions.

4. **Control Structures in Network Topology:** As shown in Proposition 4.3, the set $P = \{\langle \eta(x), \eta(y) \rangle : x, y \in \mathcal{I}\}$ serves as a dominating set in $\mathcal{L}\left(\mathrm{EQ}_{\eta(\mathcal{I})}(\mathbb{N})\right)$. This property is useful in designing networks where minimal control or observation points can efficiently monitor larger systems.

5. **Error-Correcting Code Design:** The boundary region between $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$ and $\mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))$ can be exploited in constructing error-correcting codes. Vertices corresponding to minimal dominating sets help identify redundancy patterns, ensuring robustness in transmission channels. This provides an algebraically grounded method of detecting and correcting transmission errors.

6. **Complexity Reduction in Algorithmic Verification:** When verifying algebraic properties of large-scale computational systems, direct symbolic computation may be infeasible. By encoding nearring behaviors into $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$, algorithmic verification reduces to graph traversal problems. This translation lowers computational complexity in automated theorem provers and symbolic computation systems.

7. Conclusion

This study presents a detailed graph-theoretic exploration of line graphs arising from different ideal-based structures over nearrings. By examining the interrelations among $\mathcal{L}(\mathcal{G}_{\mathcal{I}}(\mathbb{N}))$, $\mathcal{L}(\mathrm{EQ}_{\mathcal{I}}(\mathbb{N}))$, and $\mathcal{L}(\mathcal{C}_{\mathcal{I}}(\mathbb{N}))$, we established precise conditions under which these graphs coincide or differ. In particular, the behavior of these line graphs was characterized based on the nature of the ideal $\mathcal{I}$—whether it is 3-prime, equiprime, or c-prime.

We introduced new results regarding the structural consequences of properties like zero-symmetry, total reflexivity, right permutability, and the IFP . Furthermore, the concept of a line graph over the equiprime graph of $N \setminus \mathcal{I}$ was formalized and analyzed, offering new insights into the effect of excluding ideal elements from the vertex set.

Graph homomorphisms induced by nearring homomorphisms were also investigated, demonstrating how algebraic mappings between nearrings can translate into structural correspondences between their associated line graphs.

These findings not only deepen the understanding of algebraic-graph theoretic connections in nearring theory but also open potential applications in cryptographic schemes, secure networks, and algebraic coding, where ideal-based decomposition and graph equivalence can be leveraged for design and analysis.

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