



## Fuzzy Secure Double Resolving Set and Co-Secure Double Resolving Set and its Application

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**ABSTRACT:** In this article, we present a novel concept in fuzzy graph theory named fuzzy secure double resolving set. The main contribution of this paper is the introduction of various fuzzy secure resolving sets such as fuzzy secure resolving set, fuzzy co-secure resolving set, fuzzy secure resolving number, fuzzy secure double resolving set, fuzzy co-secure double resolving set, fuzzy edge resolving set, fuzzy edge secure resolving set, fuzzy bi-resolving set, and fuzzy global resolving set. The primary contribution of this document is the implementation of the fuzzy secure double resolving set in uncertain graphs. It guarantees the identification and monitoring of nodes, even in the event of some node failures. This document defines fuzzy secure resolving set, examines its characteristics, and offers instances to illustrate its functional application. We additionally offer examples to assist in grasping how it functions. This idea is beneficial since it guarantees that even if certain nodes in the graph malfunction or cease to operate, the system is still capable of accurately identifying and monitoring other nodes. This is particularly beneficial for practical issues such as sensor networks, communication systems, and fields where high reliability is essential even in the presence of potential faults or unpredictability. We also present some properties, corollaries, results, theorems and application related to fuzzy secure double resolving sets.

**Keywords:** Fuzzy graph, strength of connectedness, fuzzy double-resolving set, fuzzy secure double resolving set, fuzzy co-secure resolving set.

### Contents

<b>1 Introduction</b>	<b>1</b>
<b>2 Preliminaries</b>	<b>3</b>
<b>3 Fuzzy Secure Resolving Set and Fuzzy Co-Secure Resolving Set</b>	<b>4</b>
<b>4 Fuzzy Secure Double Resolving Set and Fuzzy Co-Secure Double Resolving Set</b>	<b>4</b>
<b>5 Fuzzy Edge Resolving Set</b>	<b>7</b>
<b>6 Application of Fuzzy Secure Double Resolving Set in Rail way Network Monitoring and Management</b>	<b>8</b>
<b>7 Conclusion</b>	<b>10</b>

### 1. Introduction

Exploration of graph theory within fuzzy sets, as proposed by Zadeh, offers a robust mathematical structure for representing systems defined by imprecision, vagueness, and uncertainty, like intricate communication networks or hierarchical frameworks. A key idea in graph theory is the resolving set, crucial for the unique identification and localization of each node in a network. This concept was extended to the fuzzy resolving set, which employs the strength of connectedness to obtain distinct representation of vertices in the ambiguous fuzzy context. In many real-world applications, such as network monitoring, sensor placement, and crisis management, a higher degree of robustness and identification accuracy is required. This need motivates the introduction of more sophisticated concepts: this study presents and thoroughly examines the innovative idea of the fuzzy secure double resolving set. Primary contributions of this research are: An extensive analysis of associated security parameters, such as fuzzy secure resolving sets, fuzzy co-secures resolving sets, and their relevant fuzzy secure resolving numbers. The official

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introduction and thorough characteristics of the fuzzy secure double resolving set and the fuzzy co-secure double resolving set.

### 1.1 Motivation of the work

- **Uncertainty and Fuzziness in the Real World**

The majority of real-world systems, such as sensor networks, social networks, and communication systems, don't function in binary terms. Relationships and relationships frequently have varying degrees of dependability, strength, and trust. Fuzzy graphs are useful in this situation: They make it possible to simulate unclear relationships and partial truths

- **Requirement for Individual Identification**

By measuring a vertex's strength of connectedness from a chosen set of reference nodes, resolving sets aid in the unique identification of nodes in network analysis. This is essential for: Communication network routing, Sensor network localization, Cybersecurity intrusion detection, Distances, however, are not clear in fuzzy surroundings. Therefore, a fuzzy resolving set that can manage ambiguity and guarantee unique identification is required.

- **Redundancy and Security**

A layer of fault tolerance is added by the "secure" feature: in the event that a node is compromised or fails, the network can still be resolved by a backup node. In critical infrastructure, where a single node failure shouldn't bring down the entire system, this is crucial. Each node is resolved by two different paths thanks to the "double" portion, which improves: Redundancy: A node can be identified in multiple ways. Resilience: Defense against attacks or single-point failures.

- **Integrating All Three: Significance of It**

The fuzzy secure double resolving set is driven by the necessity to: Accurately model intricate, unpredictable systems. Guarantee strong identification even in the presence of noisy or incomplete data. Ensure safety and error resilience in critical applications.

#### Literature review

A fuzzy graph extends standard graph theory by allowing vertices and edges to have degrees of membership (values between 0 and 1), rather than being entirely present or absent. This enables the representation of relationships that are unclear or imprecise. In graph theory, a resolving set is a subset of vertices that uniquely identifies every other vertex in the graph based on their distances from the vertices in the subset. Lotfi Asker Zadeh introduced fuzzy mathematics in 1965 [39, 40]. Later, in 1975, fuzzy graph theory was introduced by Rosenfeld. Subsequently, Bhattacharya [5] presented fuzzy automorphisms, defined morphisms between fuzzy graphs, and made several observations on fuzzy graphs. The characterization of fuzzy trees and the complement of a fuzzy graph were defined by Sunitha and Vijayakumar [36]. Several fuzzy graphs were defined by Mordeson and Nair [27]. The types of arcs in a fuzzy graph were established by Mathew and Sunitha [24, 25]. Operations on fuzzy graphs were defined by Mordeson and Chang-Shyh. Rao, Y., Lei, S., Talebi, A.A. and Mojahedfar, M., discussed a novel concept of level graph in interval-valued fuzzy graphs with application [26]. Slater introduced the idea of a graph's resolving number in 1975. Sooryanarayana and Badekar discussed certain varieties of resolving sets in graphs [34], while Subramanian and Arasappan introduced secure resolving sets in graphs [35]. The concepts of the fuzzy resolving set and fuzzy resolving number were introduced by Shanmugapriya R. and D. Mary Jiny in 2019 [17, 18]. They also discussed the fuzzy super resolving number for some special graphs [19]. Shanmugapriya R. and Hemalatha P.K. explored independency in fuzzy resolving sets of labeling graphs [20], while Vasuki M. and Shanmugapriya R. introduced fuzzy resolving domination [21], motivated by the need for robots to uniquely determine their position based on safety levels in a fuzzy graph-structured environment. Several types of sets in graph theory are related to the idea of security. Shanmugapriya R and Sangeetha P introduced fuzzy double resolving set in fuzzy graphs [22], Shanmugapriya R and P. Sangeetha discussed neutrosophic resolving set and its application in neutrosophic graphs [23], Subramanian H. and Arasappan S in 2018, discussed secure resolving sets in

graphs [35]. Sunil, M. P., & Kumar, J. S. introduced on null vertex in bipolar fuzzy graphs [37]. Talavera, et.al. (2024) explore Aggregation of fuzzy graphs [38]. We used the concept of fuzzy sets to describe the fuzzy randomized signal and determine its lifetime, as stated by Vijayabalan D et al. [44]. Many stochastic orders of the CTS and DTS are also found, along with the likelihood ratio, mean residual life, mean residual life order, and other important reliability operators. Vijayabalan D et al. [42] investigate a novel method for comparing the expectations of stochastic models in ambiguous settings. Stochastic models are essential to both actuarial science and economic theory. The main advantage of this study is that it facilitates comprehension of the novel ideas of stochastic comparison of stochastic models based on the exponential order. We created a new definition, resolved the preservation characteristics and theorem, and put it into practice using the fuzzy mean inactive time order notion. Stochastic models are used for many purposes. In order to design a practical application of fuzzy models, Vijayabalan D et al. [43] studied the new notion of general distribution. In order to clarify the effect of corticosterone, we discussed multiple states fuzzy mathematical models in the present fuzzy Survival and Hazard functions, and associated alpha-cut sets are provided by Vijayabalan D et al. [45]. To demonstrate this, we used fuzzy survival and hazard models based on two-parameter Weibull distributions. As the experiment termination alpha value increases, we found that the higher fuzzy hazard model for the impact of corticosterone release decreases and the smaller fuzzy survival model for the impact of corticosterone release increases. In this paper, fuzzy secure resolving sets and fuzzy cosecure resolving sets are introduced and studied. The resolving set and the secure resolving set represent the minimum number of transmitters required for a robot to always be aware of its position within the network modelled by the graph. The robot uses "landmarks," or transmitters placed at certain sites (vertices of the graph), to determine its location within the network. As noted in previous research, determining the resolving set of a graph is an NP-hard problem. Moreover, secure resolving set theory finds applications in wireless communication networks, electrical networks, chemical structures, and commercial networks. Arumugam introduced the concepts of co-secure and secure domination in graphs [4].

### Organization of the paper .

The paper is organized as follows. Section 2 presents the necessary definitions. Section 3 introduces and illustrates the concepts of the Fuzzy secure resolving set and the Fuzzy cosecure resolving set. Section 4 defines the Fuzzy secure double resolving set and the Fuzzy double cosecure resolving set, accompanied by illustrations and theorems. Section 5 covers the definitions, illustrations, and theorems related to the Fuzzy edge resolving set, Fuzzy edge secure resolving set, Fuzzy bire-solving set, and the Global resolving set. Finally, Section 6 discusses the real-life applications of the Fuzzy secure double resolving set.

Table 1: names and notations

Name	Notation
Fuzzy resolving set	$F_{RS}$
Fuzzy resolving number	$F_{RN}$
Fuzzy double resolving set	$F_{DRS}$
Fuzzy double resolving number	$F_{DRN}$
Fuzzy secure double resolving set	$F_{SDRS}$
Fuzzy co-secure double resolving set	$F_{CSDRS}$
Fuzzy super secure double resolving set	$F_{SSDRS}$
Fuzzy super co-secure double resolving set	$F_{SCSDRS}$

## 2. Preliminaries

**Definition 2.1** The membership function  $\mu_A(x)$  of a fuzzy set  $A$  is defined as  $\mu_A : X \rightarrow [0, 1]$ . Thus, every element  $x \in X$  has a membership degree  $\mu_A(x) \in [0, 1]$ . The fuzzy set  $A$  is completely determined by the set of ordered pairs  $A = \{(x, \mu_A(x)) \mid x \in X\}$ .

**Definition 2.2** A Fuzzy Graph is a pair  $G = (\varphi, \psi)$  where  $V$  is a nonempty set of vertices together with the functions  $\varphi : V \rightarrow [0, 1]$  and  $\psi : V \times V \rightarrow [0, 1]$ , such that for all  $a, b \in V$ ,  $\psi(a, b) \leq \min\{\varphi(a), \varphi(b)\}$ .

**Definition 2.3** The path's strength is determined by the membership value of its weakest edge in the path  $P .\psi^{sc}(p_1, p_n)$  represents the strength of the connection between the two vertices  $p_1$  to  $p_n$ , which is the strongest of all the tracks between  $p_1$  to  $p_n$ .

**Definition 2.4** Let  $G = (\varphi, \psi)$  be connected  $F_G$ ,  $S$  be a subset of  $V$  is said to be  $F_{RS}$  if every element in  $V \setminus S$  with respect to  $S$  having distinct representation then that subset is said to be a  $F_{RS}$  of  $G$ . Least size of  $F_{RS}$  is called  $F_{RN}$ . Which is representing as  $f_r(G)$ .

**Definition 2.5** Let  $G$  be a fuzzy graph,  $S$  be a fuzzy subset of  $G$ , if every two distinct elements in  $G \setminus S$  there exist two distinct elements in  $S$  such that their difference between strength of connectedness are distinct then the Fuzzy subset  $S$  is referred to as an  $F_{DRS}$  of  $G$ . Least size of  $F_{DRS}$  is called  $F_{DRN}$ , it is denoted by  $f_{2r}(G)$ .

### 3. Fuzzy Secure Resolving Set and Fuzzy Co-Secure Resolving Set

**Definition 3.1** A subset  $S$  of fuzzy graph  $G = (\varphi, \psi)$  is said to be a  $F_{SRS}$  of  $G$  if  $S$  be a fuzzy subset of  $G$ , for any  $p$  in  $V(G) \setminus S$ , there exists  $r$  in  $S$ , then the resulting set  $(S \setminus \{r\}) \cup \{p\}$  still yields a  $F_{RS}$ . Least size of  $F_{SRS}$  is called  $F_{SRN}$  which is denoted by  $f_{s_r}(G)$ . Limitation of  $F_{SRN}$  is  $(2 \leq f_{s_r}(G) \leq (n - 1))$ .

If any  $p$  in  $S$  there exists  $r$  in  $V(G) \setminus S$ , replace  $p$  by  $r$  in  $S$  then the resulting set  $(S \setminus \{p\}) \cup \{r\}$  is a  $F_{RS}$  then that fuzzy subset  $S$  is called a  $F_{CSRS}$ . Least size of  $F_{CSRS}$  is called  $F_{CSRN}$  which is represented by  $f_{cs_r}(G)$ . Limitation of  $F_{CSRN} : 2 \leq f_{cs_r}(G) \leq (n - 1)$ .

#### Illustration 3.2

In the below fuzzy graph vertex set  $V = \{v_1, v_2, v_3, v_4, v_5\}$  and edge set  $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_1v_4\}$  then the Minimum Fuzzy resolving sets are  $\{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}$ .  
 Minimum Fuzzy secure resolving sets are  $\{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}$ .  
 Minimum Fuzzy co-secure resolving sets are  $\{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}$ .  
 In this graph all the Minimum resolving sets are Minimum secure resolving sets as well as Minimum co-secure resolving sets also  $f_{s_r}(G) = 2, f_{cs_r}(G) = 2$ .

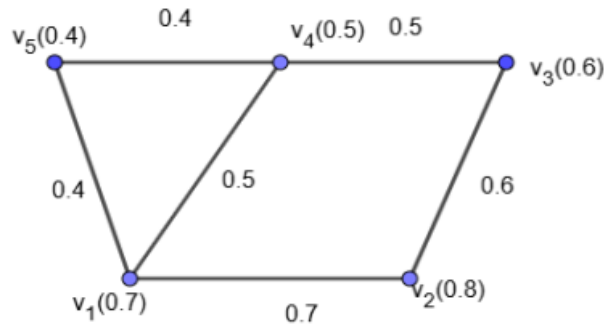


Figure 1: Fuzzy Graph

### 4. Fuzzy Secure Double Resolving Set and Fuzzy Co-Secure Double Resolving Set

**Definition 4.1** A subset  $S$  of fuzzy graph  $G = (\varphi, \psi)$  is said to be a  $F_{SDRS}$  of  $G$  if  $S$  be a fuzzy subset of  $G$ , for any  $p$  in  $V(G) \setminus S$  there exists  $r$  in  $S$ , then the resulting set  $(S \setminus \{r\}) \cup \{p\}$  yields a  $F_{DRS}$ . Least size of  $F_{SDRS}$  is called  $F_{SDRN}$  which is denoted by  $f_{s_{2r}}(G)$ . Limitation of  $F_{SDRN}$  is  $(2 \leq f_{s_{2r}}(G) \leq (n - 1))$ . If any  $p$  in  $S$  there exists  $r$  in  $V(G) \setminus S$ , replace  $p$  by  $r$  in  $S$  then the resulting

set  $(S \setminus \{p\}) \cup \{r\}$  is a  $F_{DRS}$  then that fuzzy subset  $S$  is called a  $F_{CSDRS}$ . Least size of  $F_{CSDRS}$  is called  $F_{CSDRN}$  which is represented by  $f_{cs2r}(G)$ . Limitation of  $F_{CSDRN} : 2 \leq f_{cs2r}(G) \leq (n - 1)$ .

Let  $S$  be a fuzzy super double resolving set of a fuzzy graph  $G = (\varphi, \psi)$  We say that  $S$  is a fuzzy super secure double resolving set if, for every  $p \in V(G)$  there exists a vertex  $r \in S$  such that the modified set  $(S \setminus \{r\}) \cup \{p\}$  is still an  $F_{SDRS}$  of  $G$ . The least cardinality of an  $F_{SDRS}$  is called the fuzzy super secure double resolving number, denoted by  $f_{s2r}(G)$ . If any  $p$  in  $S$  there exists  $r$  in  $V(G)$ , then the resulting set  $(S \setminus \{p\}) \cup \{r\}$  yields is a fuzzysuper double resolving set that fuzzy subset  $S$  is called fuzzy super co-secure double resolving set ( $F_{CSDRS}$ ). Least size of  $F_{CSDRS}$  is called Fuzzy super co-secure double resolving number ( $f_{s2r}(G)$ ).

**Illustration 4.2**

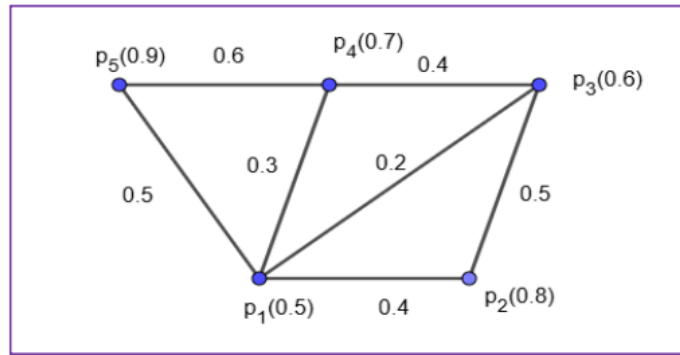


Figure 2: Fuzzy Graph

In this fuzzy graph minimum fuzzy double resolving sets are

$$\{p_1, p_2\}, \{p_2, p_4\}, \{p_2, p_5\}, \{p_3, p_4\}, \{p_3, p_5\}.$$

Minimum fuzzy secure double resolving sets are

$$\{p_2, p_4\}, \{p_2, p_5\}, \{p_1, p_3\}, \{p_2, p_3\}.$$

Minimum fuzzy co-secure double resolving sets are

$$\{p_2, p_4\}, \{p_2, p_5\}, \{p_3, p_5\}, \{p_1, p_3\}, \{p_1, p_4\}, \{p_1, p_5\}, \{p_2, p_3\}.$$

**Corollary 4.1**  $F_{SDRS}$  set always  $F_{SRS}$  but converse need not be true.

**Corollary 4.2**  $f_{s2r}(G_1 \cup G_2) = \max\{f_{s2r}(G_1), f_{s2r}(G_2)\}$

**Illustration 4.3**

Fuzzy secure double resolving sets on union of two  $F_G$

In  $G_1$  fuzzy secure double resolving sets are  $\{v_1, v_2\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_2, v_4\}$ , so  $(f_{s2r}(G_1) = 2)$ .

In  $G_2$  do not have fuzzy secure resolving set of cardinality 2,

so fuzzy secure resolving sets of  $G_2$  are

$$\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_2, v_4, v_5\}, \{v_1, v_2, v_4\}, \{v_5, v_2, v_3\}, \{v_3, v_4, v_1\}, \text{ therefore } (f_{s2r}(G_2) = 3).$$

Fuzzy secure resolving sets of  $(G_1 \cup G_2)$  are

$$\{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_1, v_3, v_5\}, \{v_3, v_4, v_5\}, \{v_4, v_5, v_1\}, \{v_4, v_5, v_2\},$$

so  $(f_{s2r}(G_1 \cup G_2) = 3)$ .

**Corollary 4.3** Let  $G$  be a  $F_G$  If  $H$  is a fuzzy secure double resolving set of  $G$ , then  $H$  does not necessarily have to be a Fuzzy secure double resolving set of  $G^c$ .

**Theorem 4.1**  $F_{SDRS}$  need not to be a fuzzy super secure double resolving set but fuzzy super secure double resolving set is always a  $F_{SDRS}$ .

**Proof:** Let  $G$  be a  $F_G$  with  $n$  nodes and let  $S = \{p_1, p_2, p_3, \dots, p_m\}$  be  $F_{DRS}$  of  $G$  that is every pair of vertices in  $(V(G) \setminus S)$  there exist two elements in  $S$  such that difference between strength of connectedness should be distinct but every pair of vertices in  $V$  need not necessary there exist two vertices in  $S$  such that satisfies  $F_{DRS}$  condition so that  $S$  does not need to be fuzzy super double resolving set. Now let  $S = \{p_1, p_2, p_3, \dots, p_m\}$  be fuzzy super double resolving set of  $G$  then every pair of vertices in  $V$  there exist two vertices in  $s$  and satisfies  $F_{DRS}$  conditions. So that every pair of vertices in  $(V(G) \setminus S)$  there exist two vertices in  $S$  and meets the conditions of a  $F_{DRS}$  therefore all Fuzzy super double resolving set of  $G$  is  $F_{DRS}$  of  $G$ .

**Corollary 4.4** Let  $G$  be  $F_G$  with four vertices,  $H$  be a  $F_{SDRS}$  of  $G$  then  $(V \setminus H)$  is also  $F_{SDRS}$  of  $G$ .

**Theorem 4.2** If  $F_G$  with four vertices, edge membership value is not constant then  $f_r(G) = f_{sr}(G) = f_{csr}(G) = f_{2r}(G) = f_{s2r}(G) = f_{cs2r}(G) = 2$ .

**Proof:** Let  $G$  be  $F_G$  with  $V(G) = \{a, b, c, d\}$  minimum cardinality of fuzzy secure double resolving set is 2,  $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$  are subset of  $V$  with cardinality 2 from these 6 sets definitely four sets are fuzzy double resolving sets, suppose  $S_1 = \{a, b\}, S_2 = \{a, c\}$  is fuzzy double resolving set then  $(V \setminus S_1) = \{c, d\}, (V \setminus S_2) = \{b, d\}$  is also fuzzy double resolving set of  $G$ . These four sets are fuzzy secure double resolving sets because we take  $c$  from  $(V \setminus S)$  there exist  $b$  in  $S_1$ , if  $b$  is replaced by  $c$  then resulting set  $\{a, c\}$  is fuzzy double resolving set similarly take  $d$  from  $(V \setminus S)$ , there exist  $a$  in  $S_1$ , if  $a$  is replaced by  $d$  then resulting set  $\{d, c\}$  is fuzzy double resolving set so that  $S_1 = \{a, b\}$  is fuzzy secure double resolving set we do the same process for all the remaining 3 sets the conditions should satisfied moreover every fuzzy secure double resolving set is fuzzy secure resolving set also hence the proof.

**Theorem 4.3** Union of two fuzzy secure double resolving set need not be fuzzy secure double resolving set.

**Proof:** Suppose  $G$  be fuzzy graph  $|v(G)| = 4$  and  $G^*$  be cycle graph in this case we get two fuzzy secure double resolving set of cardinality two if take union of these two sets we get a new set of cardinality three for that case complement of the set contains only one element this is contradiction to definition of fuzzy double resolving set.

**Corollary 4.5** If  $G$  be fuzzy graph with 4 vertices and  $G^*$  is a cycle, in a fuzzy graph having only two different edge membership value, opposite edge membership values are distinct that case we cannot get fuzzy secure and co-secure double resolving set.

**Proof:** In this case we cannot able to find fuzzy double resolving set.

**Corollary 4.6** Let  $G$  be  $F_G$ ,  $H$  be a secure fuzzy double resolving set of  $G$  then  $(V \setminus H)$  need not be secure fuzzy double resolving set of  $G$ .

**Corollary 4.7** If  $G$  be fuzzy graph with 4 vertices and  $G^*$  is a cycle, in a fuzzy graph having two different edge membership value and opposite edge membership values are distinct for that case Fuzzy secure and co-secure double resolving set does not exist.

**Proof:** In this case fuzzy double resolving set do not exist so we cannot get fuzzy secure and cosecure double resolving set.

**Corollary 4.8** All secure and co-secure fuzzy double resolving sets are fuzzy double resolving sets, but the converse is not true

**Corollary 4.9** In a star graph  $S_n$  of all the vertices and edges membership values are distinct then  $f_{s2r}(G) = 2$

**Proof:** In a star graph  $S_n$  of all the vertices and edges membership values are distinct then we have at least two fuzzy double resolving set of cardinality two that two fuzzy double resolving sets are fuzzy secure double resolving sets.

**Corollary 4.10** *In a star graph  $S_n$  alternative edges having same membership values then  $f_{s2r}(G) = (n - 2)$ .*

**Corollary 4.11** *If  $F_G$  having four vertices, edge membership values is not constant then all fuzzy double resolving sets are fuzzy secure double resolving sets as well as fuzzy co secure double resolving sets. If a fuzzy double-resolving set is present in  $FG$ , there must be at least two of them.*

**Result 1** If  $F_G$  having four vertices, edge membership values is not constant then every fuzzy double resolving sets are fuzzy secure double resolving sets as well as fuzzy co secure double resolving sets

**Result 2**

If a fuzzy double-resolving set exists in  $FG$ , then the graph must contain at least two distinct fuzzy double-resolving sets.

**Result 3**

If  $n \geq 2$ ,  $f_{s2r}(P_{2n}) = f_{2r}(P_{2n}) = n$

**Result 4**  $n \geq 2$ ,  $f_{s2r}(P_{2n+1}) = f_{2r}(P_{2n+1}) = n + 1$

## 5. Fuzzy Edge Resolving Set

**Definition 5.1** Let  $G = (\varphi, \psi)$  be  $F_G$  with  $n$  vertices  $|n| \leq 3$ ,  $|E| \geq 2$  and  $H$  be a subset of  $V$  is said to be edge resolving set of  $G$  the representation of  $H$  with respect to edges are distinct.

**Illustration 5.1**

Here the vertex set  $V = \{s_1, s_2, s_3, s_4\}$ , edge set  $E = \{e_1, e_2, e_3, e_4\}$ .  
the subset  $H = \{s_1, s_2\}$  is a edge resolving set of  $G$ .

$H$  with respect to  $e_1$

$$\begin{aligned} &= \{s_1, s_2\} \text{ with respect to } \{s_1, s_2\} \\ &= \{\min(\psi^{sc}(s_1, s_1), (\psi^{sc}(s_1, s_2), \min((\psi^{sc}(s_2, s_1), (\psi^{sc}(s_2, s_2)) \\ &= \{\min(0, 0.6), \min(0.6, 0)\} \end{aligned}$$

so that  $H$  with respect to  $e_1 = \{0, 0\}$

Similarly  $H$  with respect to  $e_2 = \{0.6, 0\}$ ,

$H$  with respect to  $e_3 = \{0.6, 0.7\}$ ,

$H$  with respect to  $e_4 = \{0, 0.6\}$

The subset  $H$  has unique depiction in relation to edges so that  $H$  is a fuzzy edge resolving set of  $G$ .

Strength of connectedness matrix of the above  $F_G$  is

$$\begin{pmatrix} 0 & 0.6 & 0.6 & 0.6 \\ 0.6 & 0 & 0.7 & 0.7 \\ 0.6 & 0.7 & 0 & 0.8 \\ 0.6 & 0.7 & 0.8 & 0 \end{pmatrix}$$

**Definition 5.2**  $G = (\varphi, \psi)$  be  $F_G$ . If the subset  $H$  of  $V$  both vertex resolving set as well as edge resolving set then that set  $H$  is said to be bi-resolving set of  $G$ .

**Illustration 5.2**

In Figure 4,  $H = \{s_1, s_2\}$ , so  $(V \setminus H) = \{s_3, s_4\}$

$H$  with respect to  $s_3 = (\psi^{sc}(s_1, s_3), \psi^{sc}(s_2, s_3)) = (0.7, 0.7)$

$H$  with respect to  $s_4 = (\psi^{sc}(s_1, s_4), \psi^{sc}(s_2, s_4)) = (0.5, 0.6)$

From this  $H$  is represented differently in relation to  $(V \setminus H)$ .

Such that  $H$  is vertex resolving set of  $G$ ,  $H$  is vertex resolving set as well as edge resolving set of  $G$ , so  $H$  is bi-resolving set of  $G$ .

**Corollary 5.1** *In a star graph  $S_n$  alternative edges having same membership values then  $f_{s2r}(G) = (n - 2)$ .*

**Definition 5.3**  $G = (\varphi, \psi)$  be  $F_G$ . If the subset  $H$  of  $V$  is vertex resolving set as well as edge resolving set then that set  $H$  is said to be global resolving set.

**Illustration 5.3**

Strength of connected matrix of  $G$

$$\begin{pmatrix} 0 & 0.3 & 0.2 & 0.3 \\ 0.3 & 0 & 0.2 & 0.5 \\ 0.2 & 0.2 & 0 & 0.2 \\ 0.3 & 0.3 & 0.2 & 0 \end{pmatrix}$$

Strength of connected matrix of  $G^c$

$$\begin{pmatrix} 0 & 0.3 & 0.2 & 0.2 \\ 0.3 & 0 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0 \end{pmatrix}$$

**Theorem 5.1** *In a complete fuzzy graph, there does not exist a fuzzy global resolving set.*

**Proof:**  $G = (\varphi, \psi)$  be complete  $F_G$ . By definition, all the edges are strong, that is, the edge membership value of the complete  $F_G$  is the minimum of the two endpoints' membership values. However, in the complement of the complete  $F_G$ , all the edges are isolated because, by the definition of the complement fuzzy graph, the edge membership value is the difference between the minimum of the endpoints' membership values and the fuzzy graph edge membership value. From this there is no path between any two vertices; therefore, we cannot obtain a fuzzy resolving set in the complement fuzzy graph. Hence, the proof.

## 6. Application of Fuzzy Secure Double Resolving Set in Rail way Network Monitoring and Management

Railway networks are an intricate system. Railway network management has many components like Maintenance of tracks, bridges, stations, and signal-in systems, Scheduling and dispatching trains, Real-time monitoring of train movements using technologies like GPS and signalling systems, Ensuring ad hence to timetables and managing delays, Using centralized traffic control (CTC) or automatic block signalling (ABS) to manage train movements, Safety Management, Managing passenger ticketing, reservations, and customer service and so on. The efficiency, safety, and dependability of railway network management can be affected by a number of issues. These problems frequently have connections and need for calculated fixes. Reducing traffic, fostering sustainable transportation, and bolstering economic growth all depend on effective railway network management. Significant changes in the design and operation of rail networks are being fuelled by technological advancements. To identify issues early, railway components (tracks, signals, and stations) require routine monitoring. However, the system may effectively categorize and resolve fault sites even when sensor data is ambiguous or incomplete by employing a fuzzy secure double resolving set. This is because sensor or inspection data may contain uncertainty or fuzziness. This minimizes service interruption and guarantees timely maintenance. In railway network management, the use of fuzzy secure double resolving sets (FSDRS) enhances the system's efficiency, robustness, and adaptability, especially when it comes to managing uncertainties and guaranteeing safe operations.

**Problem:** Consider a railway network in which train movements are tracked by sensors placed at strategic nodes, or stations. The fuzzy secure double resolving set (FSDRS) makes sure that the network can still identify the impacted segment safely and uniquely in the event that one sensor fails or produces unclear data as a result of a technical issue or interference, enabling prompt remedial action.

Strength of connectedness matrix of the above fuzzy graph

$$\begin{bmatrix} - & 0.7 & 0.3 & 0.5 & 0.5 & 0.4 \\ 0.7 & - & 0.3 & 0.5 & 0.5 & 0.4 \\ 0.3 & 0.3 & - & 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 & - & 0.5 & 0.4 \\ 0.5 & 0.5 & 0.3 & 0.5 & - & 0.4 \\ 0.4 & 0.4 & 0.3 & 0.4 & 0.4 & - \end{bmatrix}$$

Table 2: Vertex Membership Value of a Graph

Vertex	Membership Value
A	0.7
B	0.8
C	0.3
D	0.5
E	0.6
F	0.4

Table 3: Edge membership value of a Graph

Edge	Membership Value
AB	0.7
BC	0.3
CD	0.3
DE	0.5
EF	0.4
FA	0.3
BE	0.5
CF	0.2

### Solution of the problem having two steps

- Selection of a double resolving set
- Dynamic replacement

#### Selection of a Double Resolving Set

Identify a minimal set of key stations (e.g., major hubs or interchanges) that can uniquely resolve all other stations. For example, in a network of 6 stations are (A, B, C, D, E, F), we cannot able to get fuzzy double resolving set of cardinality 2. A, C, D, B, E, F, A, C, E, B, D, F, A,D, F, B, C, E, A, E, F, B, C, D these are fuzzy double resolving set of cardinality 3 in this FG these are the minimum fuzzy double resolving set in this FG If one set is fuzzy double resolving set then its complement also fuzzy double resolving set of the FG.

#### Dynamic Replacement:

If a station in  $S = A, C, D$  becomes non-operational (e.g., D is under maintenance), replace it with another station (E). The new set  $S = A, C, E$  ensures that all stations remain uniquely identifiable, maintaining network operations. so that S is secure Fuzzy double resolving set of G. Similar manner if  $S = A, D, F$  is fuzzy double resolving sets replaced D by E the new set  $S = A, E, F$  is also Fuzzy double resolving sets so that S is a fuzzy secure double resolving set of G. Even during disruptions, the fuzzy secure resolving set enables railway man-agers to track trains and resolve stations using the modified set S. For managing railway networks, the fuzzy secure resolving set offers a stable and flexible architecture. It improves by guaranteeing distinct identification and permitting dynamic replacements. The ability to bounce back from set-backs, efficiency of operations, Cost efficiency. This concept is vital for modern railway systems, where adaptability and robustness are crucial in the face of increasing complexity and uncertainty. In this application, we consider only six railway stations and determine the fuzzy secure double resolving set of G. However, if the number of stations increases as in large transportation networks with dense graphs this concept can still be applied to obtain optimized solutions. suppose 'n' number of vertices (railway stations) In this case, the fuzzy double resolving set and the fuzzy secure double resolving set can be efficiently determined using a computing network algorithm and a MATLAB program.

### Comparative analysis of FRS, FDRS, and FSDRS in railway networks

- **FRS:**  
Good for simple networks with clean data. Fails under noise, outages, or tampering.
- **FDRS:**  
Better discriminability with witness pairs. Still fragile if both witnesses are affected by the same error.
- **FSDRS:**  
Adds resilience and security. Maintains distinctness under uncertainty,

sensor dropout, correlated errors, and adversarial attacks. Ensures timely fault detection and safe operations in complex railway system FRS ensures uniqueness, FDRS strengthens discriminability, but FSDRS uniquely guarantees robustness and security under uncertainty. This makes FSDRS the most practical and valuable tool for railway network management, where data imperfections are inevitable and safety is paramount.

#### Limitations and scope

Network systems may be made more resilient and flexible with the use of fuzzy secure double resolving sets. The system can reliably identify and distinguish between network components even in the event of node failures or unclear information thanks to the fuzzy secure double resolving set, which combines fuzzy logic ideas with secure double resolving. This is particularly helpful in contexts where data may be noisy, unreliable, or incomplete, such as sensor networks, distributed computing, and cybersecurity. While fuzzy secure double resolving ensures that the capacity to resolve problems remains dependable even in the face of potential failures, the fuzzy component allows for flexible understanding of distances and relationships among nodes. Thus, fuzzy secure double resolving improves overall reliability in complex systems, strengthens network monitoring, and improves error management. Although fuzzy secure double resolving sets provide significant advantages in terms of adaptation and robustness, they also have significant drawbacks. The extra computing complexity required to create and maintain these sets is a significant disadvantage, particularly in large or dynamic networks. Fuzzy logic and fuzzy secure double resolving sets must be integrated using complex algorithms, which may be resource-intensive and challenging to correctly execute. Furthermore, uncertainty may be introduced by interpreting fuzzy values, making it difficult to get consistent outcomes across many applications or systems. Accessibility for general practitioners may be limited by the need for specific expertise in fuzzy mathematics and fuzzy secure double resolving set design. In essential systems, these issues may delay real-time decision-making and impede scaling.

## 7. Conclusion

In this article, we have presented the fuzzy secure resolving set, fuzzy co-secure resolving set, fuzzy secure double resolving set, fuzzy co-secure double resolving set, fuzzy super secure double resolving set in FG, fuzzy edge resolving set in FG, some theorems and corollaries, and results, and we have also discussed the application of the fuzzy secure double resolving set in railway network monitoring and management. In fuzzy graphs, together with their related characteristics. Implementing a fuzzy secure double resolving set can enhance system efficiency, minimise downtime, and improve overall dependability. Even when specific vertices are impacted. Through mistakes or ambiguity, the fuzzy secure double resolving set maintains its effectiveness, rendering it a useful instrument for fuzzy secure double resolving set construction. This flexibility is especially beneficial for network planning, communication systems, and various intricate systems where elements might malfunction or act in unforeseen ways. In the future, we aim to focus on identifying fuzzy double resolving sets in fuzzy digraphs.

**Conflicts of interest:** The authors declare that there is no conflict of interest.

**Data availability:** Not applicable

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**Some Families of the Eulerian Integrals Involving Generalized Hypergeometric Functions**

Shoukat Ali, Deepak Kumar, Subrat Parida and Sunil Dutt Purohit\*

ABSTRACT: In this paper, we established a family of unified Eulerian integrals involving generalized hypergeometric function by employing generalized summation theorems for the series  ${}_3F_2$  such as generalized Watson’s theorem, generalized Dixon’s theorem and generalized Whipple’s theorem. Various special cases are deduced as consequences of our main theorems.

Key Words: Eulerian integral, generalized hypergeometric function, generalized Watson’s theorem, generalized Dixon’s theorem and generalized Whipple’s theorem.

**Contents**

<b>1 Introduction</b>	<b>1</b>
<b>2 Integrals Involving Hypergeometric Functions</b>	<b>2</b>
2.1 Main Results	4
<b>3 Applications of Integrals</b>	<b>6</b>
3.1 Special Cases of (2.5)	7
3.2 Special Cases of (2.6)	7
3.3 Limiting Case of (2.8)	9
<b>4 Conclusion</b>	<b>9</b>

**1. Introduction**

The generalized hypergeometric function  ${}_rF_s[z]$  ( $r, s \in \mathbb{N}_0$ ) for which the infinite series form reads [19]

$${}_rF_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right] = \sum_{k \geq 0} \frac{\prod_{j=1}^r (a_j)_k}{\prod_{j=1}^s (b_j)_k} \frac{z^k}{k!},$$

where  $(a)_k = a(a + 1) \cdots (a + k - 1)$  signifies the Pochhammer symbol and  $a_j \in \mathbb{C}, b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ . The series converges for all  $z \in \mathbb{C}$  if  $r \leq s$ . It is divergent for all  $z \neq 0$  when  $r > s + 1$ . Finally, if  $r = s + 1$ , the series converges on the unit circle  $|z| = 1$  when  $\Re(\sum b_j - \sum a_j) > 0$ .

In the theory of hypergeometric and generalized hypergeometric functions, there exist a remarkably large number of hypergeometric summation formulas which can be expressed in terms of the Gamma functions. In particular, for specified values of the argument, usually, 1, -1 and 1/2, the hypergeometric function  ${}_2F_1$  and generalized hypergeometric function  ${}_3F_2, {}_4F_3$  and  ${}_5F_4$  reduce to the well-known classical summation theorems such as the Gauss, Gauss second, Bailey and Kummer ones for the  ${}_2F_1$  series, as well as the Watson, Dixon, Whipple for the  ${}_3F_2$ , second Whipple for the  ${}_4F_3$  and Dougall’s summation for the  ${}_5F_4$  series play an important role in the theory of generalized hypergeometric functions (cf. [19]).

In 1961, for  $\Re(\varepsilon) > 0, \Re(\varrho) > 0$ , MacRobert [14] defined the extension of the classical Beta integral as

$$\int_0^1 z^{\varepsilon-1} (1-z)^{\varrho-1} \{1+mz+n(1-z)\}^{-\varepsilon-\varrho} dz = (1+m)^{-\varepsilon} (1+n)^{-\varrho} \frac{\Gamma(\varepsilon)\Gamma(\varrho)}{\Gamma(\varepsilon+\varrho)}, \tag{1.1}$$

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Figure 3: Fuzzy Graph



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## Optimal Control Strategies for Mitigating Cardiovascular and Type 2 Diabetes Risks in Psoriasis and Psoriatic Arthritis: A Mathematical Model and Numerical Simulations

Oumaima Bouaoultaine\*, Maryem El karchani and Nadia Idrissi Fatmi

**ABSTRACT:** In this study, we investigate a novel mathematical model that describes the dynamics of psoriasis and psoriatic arthritis. The goal of this paper is to mitigate the risks associated with psoriasis and psoriatic arthritis on cardiovascular health and type 2 diabetes through control strategies involving three variables: awareness programs for patients with psoriasis or psoriatic arthritis, medical follow-up, and promoting a healthy and regular lifestyle. We demonstrate the existence of optimal controls and describe them using states and adjoint functions, primarily based on Pontryagin's maximum principle. Numerical simulations for various scenarios validate the effectiveness of the optimization approach.

**Key Words:** Psoriasis, psoriatic arthritis, type 2 diabetes, cardiovascular health, mathematical model, optimal controls, pontryagin maximum principle, numerical simulations.

### Contents

<b>1 Introduction</b>	<b>1</b>
<b>2 The Mathematical Model Formulation</b>	<b>3</b>
<b>3 Model Analysis</b>	<b>6</b>
3.1 The solution's positivity	6
3.2 Boundedness of the solution	8
3.3 The existence and uniqueness of the solution	9
<b>4 The Problem of Optimal Control</b>	<b>11</b>
4.1 The optimal control problem	12
4.2 Existence of the optimal control	12
4.3 Characterization of the optimal control	13
<b>5 Numerical Simulation</b>	<b>15</b>
<b>6 Conclusion</b>	<b>20</b>

### 1. Introduction

Psoriasis is a chronic autoimmune skin disease that manifests as thick, red, scaly patches [1,2]. These lesions, often itchy, can appear on various parts of the body, including the elbows, knees, scalp, and lower back [3]. The exact causes of psoriasis are not fully understood, but they involve a combination of genetic and environmental factors, as well as an abnormal immune response. Triggers can include stress, infections, certain medications, and skin trauma. The consequences of psoriasis go beyond skin symptoms, potentially affecting patients' quality of life, causing pain, and being associated with other conditions such as psoriatic arthritis, cardiovascular diseases, and mental health disorders [4]. The World Health Organization (WHO) [1] estimates that approximately 2 – 3% of the global population is affected by psoriasis, amounting to over 125 million people.

Psoriatic arthritis is a type of inflammatory arthritis [1]. This condition is characterized by joint inflammation, causing pain, stiffness, and swelling, mainly in the fingers, toes, knees, and ankles [5]. Similar to psoriasis, an abnormal immune response is central to the development of psoriatic arthritis. The consequences of this disease can be severe, including permanent joint damage, reduced mobility, and

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Figure 4: Fuzzy Graph



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**Spray-Invariant Sets in Infinite-Dimensional Manifolds \***

Kaveh Eftekharinasab

**ABSTRACT:** We introduce the concept of spray-invariant sets on infinite-dimensional manifolds, where any geodesic of a spray starting in the set stays within it for its entire domain. These sets, possibly including singular spaces such as stratified spaces, exhibit different geometric properties depending on their regularity: sets that are not differentiable submanifolds may show sensitive dependence, for example, on parametrization, whereas for differentiable submanifolds invariance is preserved under reparametrization. This framework offers a broader perspective on geodesic preservation than the rigid notion of totally geodesic submanifolds, with examples arising naturally even in simple settings, such as linear spaces equipped with flat sprays.

**Key Words:** Sprays, geodesics, spray-invariant sets, infinite-dimensional manifolds, adjacent and second-order adjacent cone, singular spaces.

**Contents**

<b>1</b>	<b>Sprays</b>	<b>2</b>
<b>2</b>	<b>Spray-Invariant Sets</b>	<b>5</b>
2.1	Automorphisms Preserving Spray Invariance . . . . .	17
2.2	Orbit Types and Spray Invariance . . . . .	18
<b>3</b>	<b>Spray-Invariant Sets for <math>MC^k</math>-Fréchet Manifolds</b>	<b>20</b>
<b>4</b>	<b>Aspects of Banach and Hilbert Manifolds</b>	<b>23</b>

**Introduction**

This work studies subsets of infinite-dimensional manifolds, including singular spaces such as stratified spaces, where any geodesic of a spray starting in the subset remains within it for the entire duration of its definition. The behavior of such sets, which we call *spray-invariant*, depends strongly on their regularity. For instance, for sets that are not differentiable submanifolds, reparametrization of geodesics may affect whether they remain within the set. In contrast, for differentiable submanifolds, this invariance is preserved. The motivation for studying spray-invariant sets with less regularity stems from the observation that such sets can arise naturally even in simple settings like linear spaces equipped with flat sprays.

We focus on the intrinsic properties of sprays and work within the broader context of spray geometry. This approach does not require the existence of a spray induced by a Finsler (or Riemannian) metric or compatibility with such a structure. Consequently, we can analyze the dynamics of geodesics in the setting of infinite-dimensional manifolds, where traditional Finsler (or Riemannian) geometric tools are either unavailable or inapplicable. We primarily focus on the more general context of Fréchet manifolds; however, our results are applicable to Hilbert and Banach manifolds as well.

Given a subset  $S$  of a manifold  $M$  and a spray  $\mathcal{S}$  on  $M$ , we define the *admissible set*  $A_{\mathcal{S},S}$  (Definition 2.5) as the collection of all tangent vectors  $v \in TM$  such that the projection  $\tau(v) \in S$ , and  $\mathcal{S}(v)$  belongs to the second-order adjacent cone of  $S$  at  $\tau(v)$ . In Theorem 2.1, we prove that if  $S$  is closed, then a geodesic  $g(t)$  lies entirely in  $S$  if and only if its tangent vector  $g'(t)$  belongs to  $A_{\mathcal{S},S}$  for all  $t$  in its domain. This equivalence establishes  $A_{\mathcal{S},S}$  as a fundamental invariant for analyzing the behavior of geodesics. Building on this, we define a spray-invariant set as follows: a subset  $S$  is spray-invariant for the spray  $\mathcal{S}$  if, for every geodesic  $g: I \rightarrow M$  of  $\mathcal{S}$  with initial tangent  $g'(0) \in A_{\mathcal{S},S}$ , the entire trajectory remains within  $S$ , i.e.,  $g(t) \in S$  for all  $t \in I$ , where  $I$  is the maximal interval of existence. Example 2.2 provides an instance

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Figure 5: Fuzzy Graph



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**Finite Integral Involving Incomplete Alpha-Functions, Generalized Extended Hurwitz's Zeta Function, Incomplete Gamma Function and Elliptic Integrals of the First Kind**

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**ABSTRACT:** In this study, we derive a general finite integral that incorporates the generalized Hurwitz-Lerch Zeta function of two variables, the incomplete Gamma function, elliptic integrals of the first kind, and incomplete Alpha-functions. The paper concludes with a discussion of several corollaries and observations.  
**Key Words:** Incomplete Gamma function, incomplete Alpha-function, Mellin-Barnes integrals contour, generalized Hurwitz-Lerch Zeta function, incomplete Gamma function, elliptic integrals of first kind.

**Contents**

<b>1 Introduction and Preliminaries</b>	<b>1</b>
1.1 Required integral	5
<b>2 Main Integral</b>	<b>5</b>
<b>3 Special Cases</b>	<b>7</b>
<b>4 Concluding Remarks</b>	<b>10</b>
<b>5 Compliance with Ethical Standards</b>	<b>10</b>

**1. Introduction and Preliminaries**

The development of incomplete special functions is closely tied to the evolution of special functions, which play a crucial role in various fields such as mathematics, physics, and engineering. These functions often arise as solutions to differential or integral equations, as well as in the analysis of probability and statistics. Incomplete special functions extend classical functions by truncating or limiting the domain of integration or summation.

The incomplete Gamma function and incomplete hypergeometric function were analyzed by Srivastava et al. [2], focusing on their specific characteristics. Recently, Srivastava et al. [3] introduced and analyzed the incomplete  $H$ -function and incomplete  $F$ -function. Further contributions to the study of the incomplete  $H$ -function, incomplete  $F$ -function, and integrals involving the incomplete  $H$ -function have been made by several researchers, including Banaś et al. [7], Banaś and Kumar [8], and Banaś et al. [9]. In more recent studies, Kumar et al. [1] examined improper integrals related to the incomplete Alpha-functions, while Kumar et al. [18] studied the Borus integral involving a class of polynomials and incomplete  $H$ -functions. Additionally, Kumar et al. [11] investigated the Borus integral with the generalized multi-index Mittag-Leffler function and incomplete  $F$ -functions.

In the present study, we investigate a generalized finite integral that involves the generalized Hurwitz-Lerch Zeta function of two variables, the incomplete Gamma function, elliptic integrals of the first kind, and incomplete Alpha-functions.

The Gamma function, first introduced by Leonhard Euler in 1729 as an extension of the factorial function [10], is represented by Euler's integral formula, given as

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \quad (\Re(\alpha) > 0), \tag{1.1}$$

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Figure 6: Fuzzy Graph