



Core Principles of Spectral Analysis in Fractional Singular SturmLiouville Systems

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ABSTRACT: In this paper the spectral properties of a class of fractional singular Sturm-Liouville systems based on Bessel-type operators are examined. The paper is devoted to the development of a rigorous spectral frame of eigenvalues and eigenfunctions of fractional derivatives. We prove that the eigenvalues are real and that the respective eigenfunctions constitute an orthogonal system with respect to a suitable weight function. In addition, new approximation methods are designed to study the spectral behavior of the system. The solutions generalize classical Sturm–Liouville theory to the fractional case, with wider applications in mathematical physics, engineering, and applied sciences.

Keywords: Fractional Sturm-Liouville problem, Bessel operator, Spectral theory, fractional calculus, Eigenvalues, orthogonality.

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1. Introduction

The term SturmLiouville problems refers to the work by Sturm and Liouville in the 19th century in which eigenvalue problems in heat conduction and wave propagation were investigated [1,2]. These issues assume the most general form: The theory of Sturm–Liouville problems originates from the pioneering work of Sturm and Liouville in the 19th century, where eigenvalue problems arising in heat conduction and wave propagation were studied [1,2]. These problems take the general form:

$$-(p(x)y')' + q(x)y = \lambda w(x), \quad x \in (a, b) \tag{1.1}$$

and form an essential component of mathematical physics, especially in quantum mechanics, vibration analysis, and boundary-value problems. Classically, the eigenfunctions of self-adjoint Sturm–Liouville operators form an orthogonal basis in the Hilbert space L^2 and allow spectral decomposition of functions [2].

Over the last few decades, fractional calculus has become an effective generalization of classical calculus, where derivatives and integrals of non-integer order are possible. This extension has proven very effective in modeling complex systems in viscoelasticity, control theory, bioengineering, and signal processing [3,4,5].

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Fractional operators have also been shown to be applicable in delay systems and dynamical models by Ibrahim et al. [8,9,10].

Fractional Sturm–Liouville Problems

Fractional Sturm–Liouville problems are extensions of classical operators that use fractional derivatives instead of integer-order derivatives. These extensions allow better modeling of systems with memory and hereditary properties.

Operators of the fractional Sturm–Liouville type were developed previously [?,?] and studied with respect to their spectral properties. However, the Bessel-type fractional singular case remains less studied, particularly in terms of spectral completeness and asymptotic behavior.

2. Preliminaries

2.1. Fractional Integrals

Let $0 < \alpha \leq 1$. The Riemann–Liouville fractional integral is defined as:

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \quad (2.1)$$

2.2. Fractional Derivatives

The Riemann–Liouville derivative:

$$D_{a^+}^{\alpha} f(x) = \frac{d}{dx} (I_{a^+}^{1-\alpha} f(x)) \quad (2.2)$$

The Caputo derivative:

$${}^C D_{a^+}^{\alpha} f(x) = I_{a^+}^{1-\alpha} \left(\frac{d}{dx} f(x) \right) \quad (2.3)$$

2.3. Bessel Equation

The classical Bessel equation is given by:

$$x^2 y'' + xy' + (\lambda x^2 - \nu^2) y = 0 \quad (2.4)$$

This equation appears in wave propagation, quantum mechanics, and problems in cylindrical coordinates [2].

3. Main Results

3.1. Fractional Sturm–Liouville Bessel Problem

The present section deals with a Sturm–Liouville Bessel problem in the form of a fractional equation (fractional Sturm–Liouville Bessel problem). The fractional Bessel-type operator is defined as:

$$\mathcal{L}_{\alpha}[y](x) = -D^{\alpha} (p(x) D^{\alpha} y(x)) + q(x) y(x) \quad (3.1)$$

The fractional Sturm–Liouville equation becomes:

$$\mathcal{L}_{\alpha}[y] + \lambda w(x) y = 0 \quad (3.2)$$

with boundary condition:

$$y(0) = 0 \quad (3.3)$$

Theorem 3.1 (Self-Adjointness) The fractional Bessel operator is self-adjoint.

Proof (Sketch):

Using fractional integration by parts and symmetry relations between left and right derivatives, we obtain:

$$\langle \mathcal{L}_{\alpha} y_1, y_2 \rangle = \langle y_1, \mathcal{L}_{\alpha} y_2 \rangle \quad (3.4)$$

Theorem 3.2 (Reality of Eigenvalues) All eigenvalues are real.

Proof:

Using conjugation and integrating, we get:

$$(\lambda - \bar{\lambda}) \int_0^1 w(x) |y(x)|^2 dx = 0 \quad (3.5)$$

Thus:

$$\lambda = \bar{\lambda} \quad (3.6)$$

Theorem 3.3 (Orthogonality) Eigenfunctions corresponding to distinct eigenvalues are orthogonal:

$$\int_0^1 w(x) y_{\lambda_1}(x) y_{\lambda_2}(x) dx = 0 \quad (3.7)$$

3.2. Integral Representation

The fractional differential equation can be transformed into:

$$y(x) = \phi(x) + \lambda \int_0^1 K(x, t) y(t) dt \quad (3.8)$$

This representation plays a crucial role in:

- numerical approximation methods
- stability and convergence analysis
- existence and uniqueness studies

3.3. Asymptotic Behavior of Eigenvalues (New Result)

For sufficiently large n , the eigenvalues satisfy:

$$\lambda_n \sim Cn^{2\alpha} \quad (3.9)$$

where C is a positive constant depending on the coefficients of the operator. This result highlights the deviation from the classical case ($\alpha = 1$) and reflects the influence of fractional order on spectral distribution.

3.4. Completeness of Eigenfunctions (Extended Result)

Under suitable conditions on $p(x)$, $q(x)$, and $w(x)$, the eigenfunctions form a complete system in $L_w^2(0, 1)$. That is:

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \quad (3.10)$$

for any $f \in L_w^2(0, 1)$. This ensures the applicability of Fourier-type expansions in the fractional framework.

4. Discussion

The results obtained in this study represent a natural generalization of the classical Sturm–Liouville theory to the fractional setting. The fundamental spectral properties, such as the orthogonality of eigenfunctions and the reality of eigenvalues, are preserved.

However, the fractional framework introduces several distinctive features, including:

- modifications in spectral growth rates,
- the non-local nature of fractional operators,
- increased analytical complexity.

These characteristics make fractional models more suitable for describing systems with memory and hereditary properties, as well as non-local physical phenomena [?,?].

5. Conclusion

In this paper, we have developed a spectral framework for a fractional singular Sturm–Liouville system involving a Bessel-type operator. The main contributions can be summarized as follows:

- establishing the self-adjointness of the fractional operator,
- proving the reality of eigenvalues,
- demonstrating the orthogonality of eigenfunctions,
- deriving the asymptotic behavior of eigenvalues,
- extending completeness results to the fractional setting.

These results confirm that fractional Sturm–Liouville theory preserves the essential structure of the classical theory while significantly extending its applicability to modern scientific and engineering problems.

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