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Metric Dimension of Signed Unit Graphs

Pranjali*, Balkrishan Agrawal

ABSTRACT: This paper determines the metric dimension of the signed unit graph $G_\Sigma(R)$ associated with a finite commutative ring R. Explicit formulas and bounds have been established for the metric dimension of signed unit graph for several classes of rings. Further, we have characterized the rings for which the metric dimension of the signed unit graph is equal to metric dimension of its underlying graph.

Key Words: Local ring, signed graph, unit graph, resolving set, metric dimension.

Contents

1. Introduction

Unless mentioned or defined otherwise, for all terminology and notation in graph theory and abstract algebra not specifically mentioned or defined in this paper, the reader is referred to the standard textbooks [5,7]. In this paper, we consider only finite and commutative rings with $1 \neq 0$.

"Let $G = (V(G), E(G))$ be a finite, simple, undirected and connected graph of order $n = |V(G)|$. The distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of the shortest u-v path in G. A vertex $u \in V(G)$ resolves a pair $\{x, y\} \subset V(G)$ if $d(u, x) \neq d(u, y)$. A set of vertices $S \subseteq V(G)$ is a *resolving set* of G if every pair of vertices of G is resolved by some vertex in S. The *metric dimension* of G, denoted by $\dim(G)$ or $\beta(G)$, is the minimum cardinality of a resolving set of G. The concept of resolving set and metric dimension were first introduced by Slater in 1975 [12], and independently by Harary and Melter in 1976 [6]."

"The graph G equipped with a *signature* σ is called a *signed graph*, denoted $\Sigma := (G, \sigma) = (V, E, \sigma)$, where $G = (V, E)$ is the underlying graph, and $\sigma : E \to \{-, +\}$ is a signature function that assigns a positive or negative sign to each edge. A signed graph is referred to as positive homogeneous (negative homogeneous) if all its edges are positive (negative), and as non-homogeneous otherwise. The sign of a path P, denoted $\sigma(P)$, in a signed graph Σ is the product of the signs of the edges along the path, i.e., $\sigma(P) = \prod_{e \in E(P)} \sigma(e)$. A *cycle* in Σ is considered positive (negative) if the product of the signs of all its edges is positive (negative). A signed graph is balanced if every cycle within it is positive. Given a signed graph Σ, the negation $-\Sigma$ is the signed graph obtained by negating the sign of each edge. A signed graph Σ is said to be anti-balanced if its negation $-\Sigma$ is balanced. A graph is called geodetic if there is a unique shortest path between every pair of vertices. For more information on signed graphs, refer to the bibliography by Zaslavsky [13]."

Inspired by the applications of signed graphs and unit graphs, recent work by Pranjali et al. [10] introduced the concept of *signed unit graphs* associated with rings. They characterized the commutative rings for which $G_\Sigma(R)$ and its *negation* $\eta(G_\Sigma(R))$ are balanced. The formal definition is as follows:

Definition 1.1 [10] "A signed unit graph is an ordered pair $G_\Sigma(R) := (G(R), \ \sigma)$, where $G(R)$ is the unit graph of a commutative ring R and for an edge (a, b) of $G_\Sigma(R)$, σ is defined as

$$\sigma(a,b) = \begin{cases} + \,, & \text{if } \ a \in U(R) \text{ or } b \in U(R); \\ - \,, & \text{otherwise.''} \end{cases}$$

* Corresponding author.
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The unit graph $G(\mathbb{Z}_6)$, $G(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and $G(\mathbb{Z}_2 \times \mathbb{Z}_5)$ and their corresponding signed graphs are shown in Figure 1, Figure 2 and Figure 3, respectively in which positive edges are drawn as solid line segment and negative edges as dotted line segment.

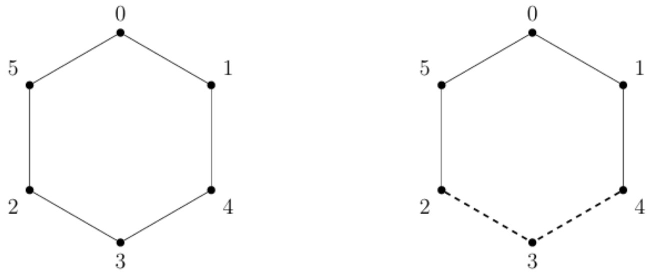

Figure 1: The unit graph $G(\mathbb{Z}_6)$ and associated signed unit graph $G_\Sigma(\mathbb{Z}_6)$

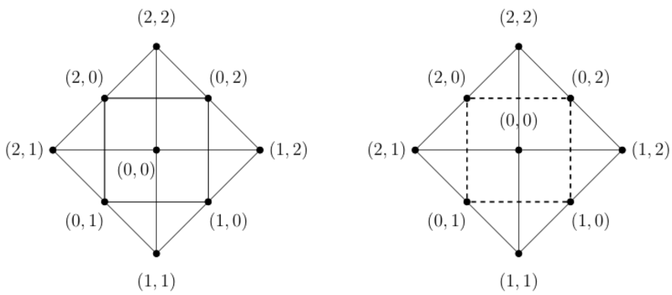

Figure 2: The unit graph $G(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and associated signed unit graph $G_\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_3)$

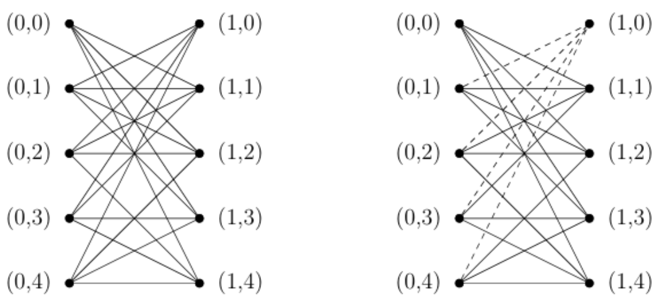

Figure 3: The unit graph $G(\mathbb{Z}_2 \times \mathbb{Z}_5)$ and associated signed unit graph $G_\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_5)$

In this work, we focus on determining the metric dimension of signed unit graphs associated with finite commutative rings. Recently, Shahul K. Hameed et al. [4] introduced the concept of signed distances for a signed graph Σ as follows:

Definition 1.2 [4] "Let $p(u, v)$ be the shortest path between two given vertices u and v, and let $P(u, v)$ be the collection of all such shortest paths. Then

(1) $\sigma_{\max}(u, v) = -1$ if all shortest u-v paths are negative, and $+1$ otherwise.

(2) $\sigma_{\min}(u, v) = +1$ if all shortest u-v paths are positive, and -1 otherwise.

(3) $d_{\max}(u,v) = \sigma_{\max}(u,v)d(u,v)$.

(4) $d_{\min}(u,v) = \sigma_{\min}(u,v)d(u,v)$.

Two vertices u and v in a connected signed graph Σ are said to be distance compatible, or simply compatible, if $d_{\max}(u,v) = d_{\min}(u,v)$."

Turning to metric dimension of graph; Let $G = (V, E)$ be a finite, simple, and connected graph. The *metric dimension* of G is defined as the cardinality of the smallest ordered subset $W \subseteq V$ such that for every pair of distinct vertices $u, v \in V$, there exists a vertex $w \in W$ for which the length of the shortest path from w to u differs from that to v. In other words, W uniquely resolves all pairs of vertices in G. The concept of metric dimension was first introduced by Slater [12] in 1975 . The formal definition is given below.

Definition 1.3 [12] "Let G be a connected graph with n vertices, and let $W = \{w_1, w_2, \ldots, w_n\}$ be a subset of $V(G)$. For any vertex $v \in V(G)$, the *representation* of v with respect to W is defined as

$$r(v \mid W) = \big(d(v,w_1),\, d(v,w_2),\, \ldots,\, d(v,w_n)\big),$$

where $d(v, w_i)$ denotes the shortest distance between v and w_i for each $i = 1, \ldots, n$). If every pair of distinct vertices of G has a distinct representation with respect to W, then W is called a *resolving set* of the graph G."

The concept of metric dimension was also introduced independently by Harary and Melter [6] in 1976, where *metric generators* were referred to as *resolving sets*. The formal definition is as given:

Definition 1.4 [6] A resolving set of minimum cardinality is called a *metric basis* of G, and its cardinality is known as the *metric dimension* of G, denoted by $\dim(G)$, $\beta(G)$, or $m_d(G)$.

In view of Definition 1.4, one can encounter the following problem:

Problem 1 For a given positive integer k, determine whether there exists a *signed unit graph* of order $n(n > k)$ with metric dimension k. If such graphs exist, characterize the commutative rings whose signed unit graphs possess metric dimension k.

In general, the metric dimension of a graph satisfies the following inequality:

$$1 \leq \dim(G) \leq n - 1. \tag{1.1}$$

Shahul Hameed et al. [4] have established the following inequality for the metric dimension of a signed graph:

$$1 \leq \dim(\Sigma) \leq \dim(G). \tag{1.2}$$

That is the metric dimension of a signed graph is less than or equal to that of its underlying graph.

In light of Inequalities (1.1) and (1.2), the metric dimension of a *signed unit graph* satisfies the following bounds:

$$1 \leq \dim(G_\Sigma(R)) \leq \dim(G(R)). \tag{1.3}$$

In light of Inequality (1.3), the following problem arises:

Problem 2 Determine the signed unit graphs and associated commutative rings R for which

$$\dim(G_\Sigma(R)) = \dim(G(R)).$$

The following example shows the existence of such ring for which both parameter are equal:

Example 1 Let $R \cong \mathbb{Z}_8$. Then the unit graph $G(R)$ is isomorphic to the complete bipartite graph $K_{4,4}$, and since all edges of $K_{4,4}$ are positive, the signed unit graph satisfies $G_\Sigma(R) \cong G(R)$. Consequently, the metric dimension is $\dim(G(R)) = 4 + 4 - 2 = 6 = \dim(G_\Sigma(R))$.

There are several applications of metric dimension which have been explored in various fields. For instance, its use in the navigation of robots within networks is discussed in [8], while applications to chemistry are presented in [2]. Furthermore, problems in pattern recognition and image processing, including those involving hierarchical data structures, are examined in [9]. Given the wide range of contexts in which the problem of distinguishing the vertices of a graph arises, numerous variants of the original concept of metric dimension have emerged in the specialized literature.

In the upcoming theorems, we address Problem 1.

Theorem 1.1 *Let R be a commutative ring with $1 \neq 0$ and let $G_\Sigma(R)$ be its signed unit graph. Then $\dim(G_\Sigma(R)) = 1$ if and only if $R \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof: It is well known that a connected graph has metric dimension 1 if and only if it is a path P_n with $n \geq 2$. Hence, if $\dim(G(R)) = 1$, then $G(R) \cong P_n$ for some $n \geq 2$, in particular every vertex has degree ≤ 2. Note that in $G(R)$, the degree of 0 is $|U(R)|$, so $|U(R)| \leq 2$. Now we shall tackle both cases for $|U(R)|$.

If $|U(R)| = 1$, then necessarily $\mathrm{char}(R) = 2$. If R is not a field, then R is local with nonzero maximal ideal M and $1 + M \subseteq U(R)$, provide more than one unit, a contradiction. Hence R is a field of characteristic 2, so $R \cong \mathbb{Z}_2$. For $\mathbb{Z}_2$, the unit graph is $K_2 \cong P_2$, which has metric dimension 1.

If $|U(R)| = 2$, then elements are not self inverse, so $\mathrm{char}(R) \neq 2$. If R is not a field, then again in the local case $1 + M \subseteq U(R)$ yields $|U(R)| > 2$, and in $R \cong R_1 \times R_2$ one has $|U(R)| = |U(R_1)||U(R_2)| \geq 4$, both contradictions. Hence R is a (finite) field, say $|R| = q$, with $|U(R)| = q - 1 = 2$, so $q = 3$ and $R \cong \mathbb{Z}_3$. For $\mathbb{Z}_3$, the unit graph is the path P_3, which has metric dimension 1.

Conversely, for $R = \mathbb{Z}_2$ and $R = \mathbb{Z}_3$, the unit graphs are P_2 and P_3, respectively and their corresponding signed graphs are all-positive. Therefore, $\dim(G_\Sigma(R)) = 1$ in each case. □

Theorem 1.2 *Let R be a commutative ring with $1 \neq 0$ and $G_\Sigma(R)$ be its signed unit graph. If R is isomorphic to any of the following listed rings: $\mathbb{F}_5$ or $\mathbb{Z}_4$ or $\mathbb{F}_2[x]/(x^2)$ or $\mathbb{Z}_6$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then $\dim\big(G_\Sigma(R)\big) = 2$.*

Proof: Let R be isomorphic to any of the following listed rings: $\mathbb{F}_5$ or $\mathbb{Z}_4$ or $\mathbb{F}_2[x]/(x^2)$ or $\mathbb{Z}_6$. Then in view of [11, Theorem 2.2], $\dim\big(G_\Sigma(R)\big) = 2$, so due to inequality (1.3), $\dim\big(G_\Sigma(R)\big) \leq 2$. According to Theorem 1.1, $\dim\big(G_\Sigma(R)\big) \neq 1$. Thus all the listed rings have $\dim\big(G_\Sigma(R)\big) = 2$. Next if we take $\mathbb{Z}_3 \times \mathbb{Z}_3$, then its unit graph and corresponding signed unit graph are shown in Figure 2. Consider the set $W = \{(1,1), (1,2), (2,2)\}$. The metric representations of the vertices with respect to W are:

$$
\begin{aligned}
&r((0,0)|W) = (1,1,1), \quad r((1,0)|W) = (1,1,2), \quad r((0,1)|W) = (1,2,2),\\
&r((2,0)|W) = (2,2,1), \quad r((0,2)|W) = (2,1,1), \quad r((2,1)|W) = (2,2,2),\\
&r((1,1)|W) = (0,2,2), \quad r((1,2)|W) = (2,0,2), \quad r((2,2)|W) = (2,2,0).
\end{aligned}
$$

Since each vertex has a unique metric representation, W is a resolving set. Therefore, W is a minimum resolving set, and $\dim(G(R)) = 3$. In order to find $\dim(G_\Sigma(R))$, we consider the set $W = \{(2,0), (0,2)\}$. The metric representations with respect to W are:

$$
\begin{aligned}
&r((0,0)|W) = (2,2), \quad r((1,0)|W) = (2,-1), \quad r((1,1)|W) = (-2,-2),\\
&r((1,2)|W) = (-2,1), \quad r((2,1)|W) = (1,-2), \quad r((2,2)|W) = (1,1),\\
&r((0,1)|W) = (-1,2), \quad r((2,0)|W) = (0,-1), \quad r((0,2)|W) = (-1,0).
\end{aligned}
$$

Since each vertex of $G_\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ has a unique metric representation, W is a minimum resolving set for $G_\Sigma(R)$, and hence $\dim(G_\Sigma(R)) = 2$. □

The above result has also drawn our attention that there exists a ring, namely, $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ for which $\dim\big(G_\Sigma(R)\big) < \dim\big(G(R)\big)$.

Theorem 1.3 *Let R be a commutative ring with $1 \neq 0$ and $G_\Sigma(R)$ be its signed unit graph. If R is isomorphic to $\mathbb{F}_7$ or $\mathbb{F}_4$ or $\mathbb{Z}_2 \times \mathbb{F}_4$, then* $\dim\big(G_\Sigma(R)\big) = 3$.

Proof: It is known from [11] that finite commutative rings with unit graphs having metric dimension 3 are isomorphic to rings $\mathbb{F}_7$, $\mathbb{F}_4$, or $\mathbb{Z}_2 \times \mathbb{F}_4$.

For the field $\mathbb{F}_7$ and $\mathbb{F}_4$, the metric dimension of $G_\Sigma(R)$ is same as of $G(R)$ due to homogeneous positive nature of $G_\Sigma(R)$.

For $\mathbb{Z}_2 \times \mathbb{F}_4$ there does not exist a set W with $|W| = 2$, which perform as a resolving set for $G_\Sigma(R)$. Thus due to inequality (1.3), $\dim\big(G_\Sigma(\mathbb{Z}_2 \times \mathbb{F}_4)\big) = 3$. Hence the result. □

2. Metric Dimension of Signed Unit Graphs Over Local Rings

In this section, we determine the metric dimension of the signed unit graphs associated with local rings.

Theorem 2.1 *Let R be a finite local ring with unity. Then,*

$$\dim(G_\Sigma(R)) = \dim(G(R)).$$

Proof: Let R be a local ring. In view of [10], the signed unit graph $G_\Sigma(R)$ is an all-positive signed graph. Hence,

$$G_\Sigma(R) \cong G(R),$$

and thus the metric dimension of the signed unit graph is same as of its unit graph. □

Theorem 2.2 *Let R be a finite local ring with nonzero maximal ideal M, and suppose that $|R/M| = q$. Then*

$$\dim\big(G_\Sigma(R)\big) = |R| - q.$$

Furthermore, if R has $\mathbb{Z}_2$ as a quotient, then

$$\dim\big(G_\Sigma(R)\big) = |R| - 2.$$

Proof: Let R be a finite local ring with maximal ideal $M \neq 0$ and $|R/M| = q$. The vertices of the unit graph $G(R)$ split into q cosets of M. All elements within the same coset have same neighbour, so each coset forms a twin class. In order to find resolving set atleast $|M| - 1$ vertices must be chosen from $|M|$. Thus every resolving set has at least $q(|M| - 1) = |R| - q$ vertices. Now it remains to show that such chosen W is minimum resolving set.

Now take all except one element from each coset form the set W with $|W| = |R| - q$. If two vertices belong to the same coset, one of them lies in W and separates the pair. If they lie in different cosets, a vertex chosen from the coset $(-a)+M$ will distinguish any pair $x \in a+M$, $y \in b+M$ with $a+M \neq b+M$. Hence W resolves $G(R)$, and $\dim(G(R)) = |R| - q$.

In the signed unit graph $G_\Sigma(R)$, each edge is positive since no two nonunits are adjacent. Thus,

$$\dim(G_\Sigma(R)) = \dim(G(R)) = |R| - q.$$

Suppose R is local ring having $\mathbb{Z}_2$ as a quotient, then $G(R)$ is complete bipartite $K_{\frac{|R|}{2},\frac{|R|}{2}}$. Invoking Theorem 2.1, $\dim(G_\Sigma(R)) = \dim(G(R)) = |R| - 2$. □

Theorem 2.3 *Let $\mathbb{F}$ be a finite field with characteristic p. Then,*

$$\dim(G_\Sigma(\mathbb{F})) = \begin{cases} |\mathbb{F}| - 1, & \text{if } p = 2, \\ \dfrac{|\mathbb{F}| - 1}{2}, & \text{if } p \neq 2. \end{cases}$$

Proof: Let $\mathbb{F}$ be a finite field with characteristic p and the associated signed unit graph is $G_\Sigma(\mathbb{F})$. Based on the different values of p, we have the following cases:

Case 1: When $p = 2$, then $G_\Sigma(\mathbb{F}_{2^n})$ is an all positive signed graph whose underlying graph is K_{2^n}. Thus, $G(\mathbb{F}_{2^n}) \cong G_\Sigma(\mathbb{F}_{2^n})$. By the [11, Proposition 2.5]

$$\dim(G(\mathbb{F}_{2^n})) = \dim(G_\Sigma(\mathbb{F}_{2^n})) = |\mathbb{F}| - 1.$$

Case 2: When $p \neq 2$, clearly

$$G(\mathbb{F}_{p^n}) \cong K_{p^n} \setminus \{e_1, e_2, \ldots, e_k\}, \text{ where } k = \frac{p^n - 1}{2}.$$

Note that in view of the Definition 1.1, $G_\Sigma(\mathbb{F}_{p^n})$ is isomorphic to $G(\mathbb{F}_{p^n})$. Hence,

$$\dim(G(\mathbb{F}_{p^n})) = \dim(G_\Sigma(\mathbb{F}_{p^n})) = \frac{|\mathbb{F}| - 1}{2}.$$

□

Theorem 2.4 *Let R be a finite commutative ring with $1 \neq 0$, and $R/J \cong \mathbb{Z}_2$ for some ideal J of R. Suppose $G(R)$ is connected, then* $\dim(G_\Sigma(R)) = \dim(G(R))$.

Proof: Let R be a finite commutative ring with $1 \neq 0$. Suppose there exists an ideal J with $R/J \cong \mathbb{Z}_2$. Our aim is to show that $\dim(G_\Sigma(R)) = \dim(G(R))$. Now there are two cases when R is local or R is non local

Case 1: Suppose R is local then J is the unique maximal ideal. Then $J = J(R)$ and $U = R \setminus J$. Because $R/J \cong \mathbb{Z}_2$, the vertex set of $G(R)$ is partitioned into the two cosets J and $u + J$. For x, y in the same coset we have $x + y \in J$, so no edges lie inside a coset. Every edge of $G(R)$ joins J to $u + J$, and at least one end vertex is a unit. Therefore $G(R)$ is complete bipartite and due to [10], $G_\Sigma(R)$ is an all-positive bipartite signed graph. This indicates that a set resolves $G_\Sigma(R)$ if and only if it resolves $G(R)$, and therefore $\dim(G_\Sigma(R)) = \dim(G(R))$.

Case 2: Next let R be a non local ring then J is not the unique maximal ideal. However J is still a maximal ideal as $R/J \cong \mathbb{Z}_2$. Now as done in Case 1, R again decomposes into the two cosets J and $u + J$, and no edge lies inside a coset; hence $G(R)$ is bipartite with partite sets J and $u + J$. In this case negative edges of $G_\Sigma(R)$ are exactly those between a vertex of J and a *zero-divisor* of $u + J$. Now we shall make use of switching. Consider the *switching* $\tau : V(G_\Sigma(R)) \to \{1, -1\}$ defined by

$$\tau(v) = \begin{cases} -1, & \text{if } v \in (u + J) \setminus U, \\ +1, & \text{otherwise.} \end{cases}$$

Under the above switching function, the sign of every edge (v_1, v_2) has changed by the factor $\tau(v_1)\tau(v_2)$. By construction, after switching, the edges incident to a unit remains positive, and the negative edges incident to a zero-divisors in $u + J$ are flipped to positive. Thus every edge of switched signed unit graph becomes positive. Thus the switched signed graph has all-positive edges, so its signed distances equal the ordinary distances.

For any vertices v, w, the sign of every v–w walk changes under switching by the endpoint factor $\tau(v)\tau(w)$; consequently the switched signed distance $\widetilde{d}_\Sigma(v, w)$ satisfies

$$\widetilde{d}_\Sigma(v, w) = \tau(v)\tau(w)\, d_\Sigma(v, w),$$

and the distance is unchanged. Choose W as a resolving set of $G(R)$, then W resolves the switched signed unit graph $\widetilde{G}_\Sigma(R)$ as it is an all-positive. From [4] it is known that metric dimension of signed graph is invariant under switching, by which we conclude that $\dim(\widetilde{G}_\Sigma(R)) = dim(G_\Sigma(R))$. Therefore $\dim(G_\Sigma(R)) = \dim(G(R))$.

Thus in both cases we get the desired result. □

Theorem 2.5 *Let R be a finite commutative ring with $1 \neq 0$ and $|R| < 12$ and R does not have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient. Then* $\dim(G_\Sigma(R)) \in \{1, 2, 3, 4, 5, 6, 7\}$.

Proof: Let R be a finite commutative ring with $1 \neq 0$ and $|R| < 12$. If the order of R belongs to $\{2, 3, 5, 7, 11\}$, then R is isomorphic to a field. By Theorem 2.3, the corresponding metric dimensions are $1, 1, 2, 3,$ and 5, respectively.

Now, consider the remaining cases where $|R| \in \{4, 6, 8, 9, 10\}$.

For $|R| = 4$, the possible rings (up to isomorphism) are $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{F}_4$, and $\mathbb{Z}_2[x]/\langle x^2 \rangle$. Among these, the signed unit graphs of all rings except $\mathbb{Z}_2 \times \mathbb{Z}_2$ are connected. The latter is disconnected and hence excluded from consideration. The metric dimensions of $G_\Sigma(\mathbb{Z}_4)$ and $G_\Sigma(\mathbb{Z}_2[x]/\langle x^2 \rangle)$ are both 2, while $\dim(G_\Sigma(\mathbb{F}_4)) = 3$.

For $|R| = 6$, by Theorem 1.2, the metric dimension of $G_\Sigma(\mathbb{Z}_6)$ is 2.

When $|R| = 8$, there exist ten commutative rings with unity, of which six are local and four are non-local. Among the local rings, five have $\dim(G_\Sigma(R)) = 6$, while for the field $\mathbb{F}_8$, we obtain $\dim(G_\Sigma(\mathbb{F}_8)) = 7$. Among the non-local rings, three namely $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/\langle x^2 \rangle$ are excluded from consideration. For the remaining non-local ring $\mathbb{Z}_2 \times \mathbb{F}_4$, Theorem 1.3 gives $\dim(G_\Sigma(\mathbb{Z}_2 \times \mathbb{F}_4)) = 3$.

For $|R| = 9$, there are four non-isomorphic rings: $\mathbb{F}_9$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/\langle x^2 \rangle$, and $\mathbb{Z}_3 \times \mathbb{Z}_3$. Using Theorem 2.2, we have $\dim(G_\Sigma(\mathbb{Z}_9)) = \dim(G_\Sigma(\mathbb{Z}_3[x]/\langle x^2 \rangle)) = 6$. For $\mathbb{F}_9$, $\dim(G_\Sigma(\mathbb{F}_9)) = 4$, and from Theorem 1.2, $\dim(G_\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 2$.

Finally, when $|R| = 10$, we have $R \cong \mathbb{Z}_{10}$. In this case, the set $W = \{0, 1, 2, 6\}$ forms a resolving set. For each vertex $v \in V(G(\mathbb{Z}_{10}))$, the corresponding distance vector $r(v \mid W) = (d(v, 0), d(v, 1), d(v, 2), d(v, 6))$ is unique, which implies that $\dim(G_\Sigma(\mathbb{Z}_{10})) = 4$.

Hence, for every finite commutative ring R with $1 \neq 0$ and $|R| < 12$ that does not have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, we have $\dim(G_\Sigma(R)) \in \{1, 2, 3, 4, 5, 6, 7\}$. □

Theorem 2.6 *Let R be a finite commutative ring with identity and $G(R)$ be its unit graph and let $J = J(R)$ be its Jacobson radical. Then*

$$\max\{1,\ |R| - |R/J|\} \ \leq\ \dim\big(G_\Sigma(R)\big) \ \leq\ \dim\big(G(R)\big) \ \leq\ |R| - 1.$$

Moreover, the lower bound is tight for every finite local (non-field) ring ($J \neq 0$), i.e., when R is local with residue field $\kappa = R/J$,

$$\dim\big(G_\Sigma(R)\big) = |R| - |\kappa|.$$

Proof: The natural homomorphism $f : R \to R/J$ partitions R into cosets of J. Elements in the same cosets are twins in $G(R)$, so any resolving set must contain all but at most one vertex from each coset, which gives the lower bound $|R| - |R/J|$. Since the graph under consideration are connected this gives $\dim(G) \geq 1$, hence $\max\{1, |R| - |R/J|\}$. The upper bound is easily obtained as complete graph has the largest metric dimension. If R is local with residue field κ, then by taking all vertices except one from each coset $r + J$ resolves the graph, which gives $|R| - |\kappa|$. □

Remark 2.1 The metric dimension of the signed unit graph formed by the direct product of different rings can vary; for instance, when $R_1 \cong \mathbb{Z}_3$ and $R_2 \cong \mathbb{Z}_3$, the signed unit graph of $\mathbb{Z}_3$ has a metric dimension of 1, while the metric dimension of the signed unit graph $G_\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is 2. Next, if $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \mathbb{Z}_4$, the signed unit graphs of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ have metric dimensions of 1 and 2 respectively, but the metric dimension of $G_\Sigma(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is not defined as it is disconnected.

Conclusion

In this paper, we have investigated the metric dimension of signed unit graphs $G_\Sigma(R)$ associated with finite commutative rings. The concept generalize the notion of metric dimension of unit graph. One of the major outcome of study is that for local rings, the metric dimension of the unit graph and the associated signed unit graph coincide.

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Conflicts of interest

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Pranjali,
Department of Mathematics,
University of Rajasthan,
JLN Marg, Jaipur-302004
India.
E-mail address: pranjali48@gmail.com

and

Balkrishan Agrawal
Department of Mathematics,
University of Rajasthan,
JLN Marg, Jaipur-302004
India.
E-mail address: balkrishanagrawal16@gmail.com