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r-Lucky Labeling of Graphs

Rini Angeline Sahayamary A., Teresa Arockiamary S.

ABSTRACT: A new variant of lucky labeling namely r-lucky labeling has been studied in this paper. This variant is of two types: Type I and Type II r-lucky labeling. A few basic graphs are investigated and obtained the results.

Keywords: Labeling, lucky labeling, d-lucky labeling, cycle graph, ladder graph.

Contents

1. Introduction

The labeling technique studied in this research paper is a variation of lucky labeling. As an alternative to conventional graph labelings, the concept was first presented by Czerwiński, Grytczuk, and Żelazny in 2004. The $d-$lucky labeling, $e-$lucky labeling are some of the variants of lucky labeling that exists in literature. The $d-$lucky labeling was introduced by Mirka Miller *et* al. and it was studied by many others, a few are listed here [3] [4] [5]. In this way we introduce another variant of lucky labeling namely r-lucky labeling which refers to the residual. This technique has two types called the Type 1 $r-$lucky labeling and Type 2 $r-$lucky labeling and their definitions are as follows:

Definition 1.1 ***Type 1 r−lucky labeling***
"Let $l : V(G) \to \{1, 2, ..., k\}$ be a labeling of the vertices of a graph G by positive integers. A labeling l is said to be of type 1 $r-lucky$ labeling, if $r(u) = l(u) + \Sigma_{v\in N(u)} |l(u) - l(v)|$ where $l(u)$ and $l(v)$ denotes the labels of u and v, and $N(u)$ denotes the open neighborhood of u. If $r(u) \neq r(v)$, for every pair of adjacent vertices u and v in G, G satisfies $r-$lucky labeling. The $r-lucky$ number of a graph G, denoted by $\eta_r(G)$, is the least positive integer k such that G has $r-$lucky labeling with $\{1, 2, ..., k\}$ as the set of labels".

Definition 1.2 ***Type 2 r−lucky labeling***
"Let $l : V(G) \to \{1, 2, ..., k\}$ be a labeling of the vertices of a graph G by positive integers. A labeling l is said to be of type 2 $r-lucky$ labeling, if $r(u) = d(u) + \Sigma_{v\in N(u)} |l(u) - l(v)|$ where $d(u)$ denotes the degrees of u, $N(u)$ denotes the open neighborhood of u. If $r(u) \neq r(v)$, for every pair of adjacent vertices u and v in G, G satisfies r-lucky labeling. The $r-lucky$ number of a graph G, denoted by $\eta_r(G)$, is the least positive integer k such that G has a $r-$lucky labeling with $\{1, 2, ..., k\}$ as the set of labels".

New labeling techniques are usually studied for basic graphs [6] [7] [8] [9] initially. In the same way, a study on basic graphs was made and the results are presented in section 4. If the graph satisfies both type 1 and type 2 we say that the graph admits r-lucky labeling.

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2. Preliminaries

In this paper, we would recall some of the basic definitions say lucky labeling and $d-$lucky labeling that are required to understand the $r-$lucky labeling better manner and some of the definition of graphs.

Definition 2.1 *[1] "Let $f : V(G) \rightarrow \mathbb{N}$ be a labeling of the vertices of a graph G by positive integers. Let $S(v)$ denote the sum of the labels over all the neighbors of a vertex v in G. If v is an isolated vertex of G we put $S(v) = 0$. A labeling f is lucky if $S(u) \neq S(v)$ for every pair of adjacent vertices u and v. The lucky number of a graph G, denoted by $\eta(G)$ is the least positive integer k such that G has a lucky labeling with $\{1, 2, ...k\}$ as a set of labels".*

Definition 2.2 *[2] "Let l: V(G) →{1,2,...,k} be a labeling of the vertices of a graph G by positive integers. Define $c(u) = d(u) + \Sigma_{v \in N(u)} l(v)$ where $d(u)$ denotes the degree of u and $N(v)$ denotes the open neighborhood of v. We define a labeling l as d-lucky labeling if $c(u) \neq c(v)$, for every pair of adjacent vertices u and v in G. The d-lucky number of a graph G, denoted by $\eta_{dl}(G)$, is the least positive integer k such that G has a d-lucky labeling with {1,2,...,k} as the set of labels".*

Definition 2.3 *"A path in a graph is a sequence of vertices where each adjacent pair is connected by an edge. The path graph is denoted by P_n. The length of a path is the number of edges it contains".*

Definition 2.4 *[7] "A cycle is a closed path where the first and last vertices are the same. The cycle graph is denoted by C_n. A graph with no cycles is called acyclic".*

Definition 2.5 *"A tree is a connected acyclic graph. A wheel is a graph obtained from a cycle by adding a new vertex and edges joining it to all the vertices of a cycle".*

Definition 2.6 *[8] "The ladder graph L_n is an undirected connected graph with $2n$ vertices and $3n - 1$ edges. It is the cartesian product of path P_n with n vertices and complete graph K_2".*

Definition 2.7 *[6] "The star graph has n edges and $(1+n)$ vertices. It is represented by $K_{1,n}$. $V(K_{1,n}) = \{u, u_1, u_2, ..., u_n\} \bigcup \{v\}$ and $E(K_{1,n}) = \{vu_i\}$ where $n \epsilon N$ and $1 \leq i \leq n$".*

The basic graphs used in this paper are path graph, cycle graph, wheel graph, ladder graph and star graph.

3. Figures

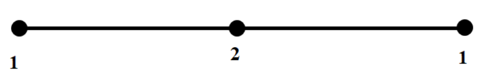

Figure 1:
$r-$lucky labeling of P_3

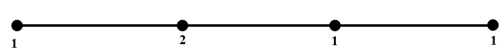

Figure 2:
$r-$lucky labeling of P_4

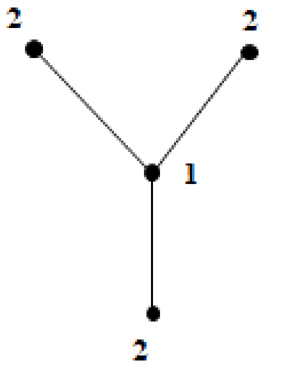

Figure 3:
$r-$lucky labeling of S_4

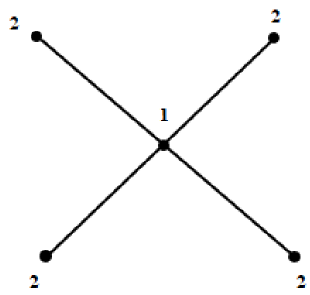

Figure 4:
$r-$lucky labeling of S_5

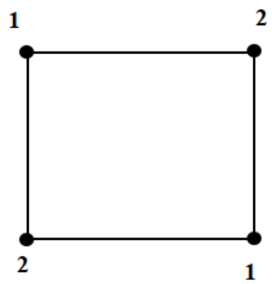

Figure 5:
Type 1 $r-$lucky labeling of C_4

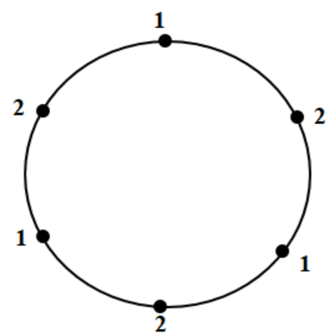

Figure 6:
Type 1 $r-$lucky labeling of C_6

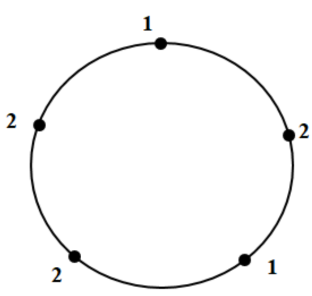

Figure 7:
Type 1 $r-$lucky labeling of C_5

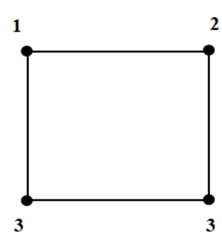

Figure 8:
Type 2 $r-$lucky labeling of C_4

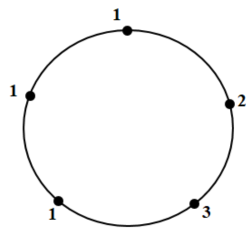

Figure 9:
Type 2 $r-$lucky labeling of C_5

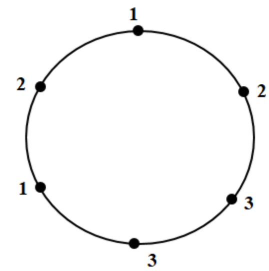

Figure 10:
Type 2 $r-$lucky labeling of C_6

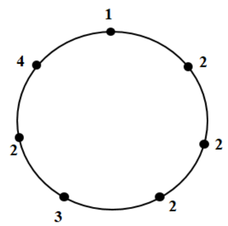

Figure 11:
Type 2 $r-$lucky labeling of C_7

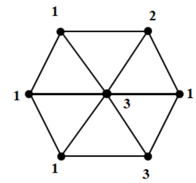

Figure 12:
Type 2 $r-$lucky labeling of W_7

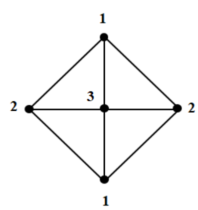

Figure 13:
Type 2 $r-$lucky labeling of W_5

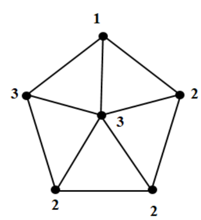

Figure 14:
Type 2 $r-$lucky labeling of W_6

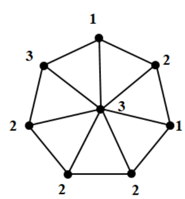

Figure 15:
Type 2 $r-$lucky labeling of W_8

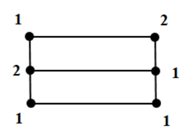

Figure 16:
Type 2 $r-$lucky labeling of L_3

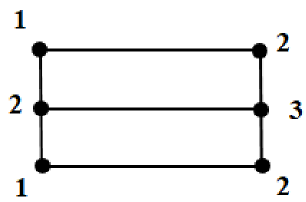

Figure 17:
Type 2 $r-$lucky labeling of L_3

4. r − Lucky Labeling of Graphs

Theorem 4.1 *The path graph P_n admits $r-$lucky labeling for $n \geq 3$.*

Proof: To prove the graph P_n is $r-$lucky, it has to satisfy the condition $r(u) \neq r(v)$. We claim $r(u) \neq r(v)$. For illustration, we take P_3, P_4 and P_5. We call the vertices of P_n as $u_1, u_2, ..., u_n$. We begin to label the vertices of P_3 with 1 and 2 alternatively. By the condition of Type 1 labeling we have $r(u_1) = 2$, $r(u_2) = 4$, $r(u_3) = 2$. This implies $r(u_1) \neq r(u_2)$ and $r(u_2) \neq r(u_3)$. This proves that our claim $r(u)$ and $r(v)$ are distinct for all pair of vertices which are adjacent vertices in P_3. In a similar way, type 2 labeling holds for P_3. See Figure 1.

We now give the illustration for P_4. Label the vertices with 1 and 2 as shown in Figure 2.

By applying the condition of type 1 and type 2 labeling, we get $r(u_1) = 2$, $r(u_2) = 4$, $r(u_3) = 3$ and $r(u_4) = 2$. This implies $r(u_1) \neq r(u_2)$, $r(u_2) \neq r(u_3)$ and $r(u_3) \neq r(u_4)$. This holds our claim $r(u) \neq r(v)$ for any n and for all pair of adjacent vertices in P_n. Therefore we proved that the graph P_n is $r-$lucky and the $r-$lucky number is 2. We also say that the graph is a Euler lucky graph. □

Remark: The path graph P_2 satisfying $r-$lucky labeling is a trivial case.

Theorem 4.2 *The star graph $K_{1,n}$ admits $r-$lucky labeling for $n > 3$.*

Proof: The star graph $K_{1,n}$ is a graph on $n-$vertices with one center vertex of degree $(n-1)$ and $(n-1)$ pendant vertices of degree 1.

By construction, we label the center vertex u_1 as 1 and $(n-1)$ pendant vertices u_2, u_3,...,u_n as 2 such that it satisfies the condition of type 1 and type 2 $r-$lucky labeling for any $n > 3$ and for all pairs of adjacent vertices in $K_{1,n}$.

For illustration, we consider $K_{1,4}$. By the condition of type 1: $r(u_1) = 4$, $r(u_2) = r(u_3) = r(u_4) = 3$ and by the condition of type 2: $r(u_1) = 6$, $r(u_2) = r(u_3) = r(u_4) = 2$. This implies $r(u_1) \neq r(u_2)$, $r(u_1) \neq r(u_3)$ and $r(u_1) \neq r(u_4)$ for both type 1 and 2. See Figure 3.

Similarly for $K_{1,5}$. By the condition of type 1: $r(u_1) = 5$, $r(u_2) = r(u_3) = r(u_4) = 3$ and by the condition of type 2: $r(u_1) = 8$, $r(u_2) = r(u_3) = r(u_4) = 2$. This implies $r(u_1) \neq r(u_2)$, $r(u_1) \neq r(u_3)$, $r(u_1) \neq r(u_4)$ and $r(u_1) \neq r(u_5)$ for both type 1 and 2 for both type 1 and 2. See Figure 4.

From the above illustration, we can say that

$$l(u_i) = \begin{cases} 1 & for\ i = 1, \\ 2 & for\ all\ i, \quad 2 \leq i \leq n \end{cases}$$

and $r(u) \neq r(v)$ for every pair of adjacent vertices. This implies that star graph satisfies $r-$lucky labeling for any $n > 3$. □

Theorem 4.3 *The Cycle graph C_n admits $r-$lucky labeling for $n \geq 4$.*

Proof: We prove the result for type 1 and type 2 separately.
Type 1: Claim: $r(u) \neq r(v)$
Case 1: n even. For illustration, consider the graph C_4 in Figure 5. Label the vertices with 1 and 2 alternatively. By the condition, we have $r(u_1) = 3$, $r(u_2) = 4$, $r(u_3) = 3$, $r(u_4) = 4$ such that $r(u_1) \neq r(u_2) \neq r(u_3) \neq r(u_4) \neq r(u_1)$ for all pair of vertices adjacent in G.

Similarly, we can label the graph C_6. By the condition, we have $r(u_1) = 3$, $r(u_2) = 4$, $r(u_3) = 3$, $r(u_4) = 4$, $r(u_5) = 3$, $r(u_6) = 4$ such that $r(u_1) \neq r(u_2) \neq r(u_3) \neq r(u_4) \neq r(u_5) \neq r(u_6) \neq r(u_1)$ for all pair of vertices adjacent in G. See Figure 6. A similar argument holds for all n even. Therefore for all pair of adjacent vertices our claim $r(u) \neq r(v)$ holds.

Case 2: n odd, $n \geq 5$.

To prove our claim, we take C_5. We begin to label the vertices with 1 and 2 alternatively.

That is u_1 and u_3 as 1 and u_2 and u_4 as 2 and label u_5 as 3. By condition as in case 1 here we have $r(u_1) \neq r(u_2) \neq r(u_3) \neq r(u_4) \neq r(u_5) \neq r(u_1)$ for all pair of adjacent vertices in G which satisfies the condition of type 1 $r-$lucky labeling. See Figure 7.

In the same way, the vertices of C_7, u_1, u_3, u_5 are labeled as 1, u_2, u_4, u_6 as 2 and u_7 as 3. Therefore $r(u_1) \neq r(u_2) \neq r(u_3) \neq r(u_4) \neq r(u_5) \neq r(u_6) \neq r(u_7) \neq r(u_1)$ for all pairs of adjacent vertices in C_7. Our claim holds for any n odd.

From the above cases, we infer that the graph C_n satisfies type 1 $r-$lucky labeling with the following set of labels.

For Case (1):

$$l(u_i) = \begin{cases} 1 \;\; for \;\; 1 \leq i \leq n-1, & i \;\; odd \\ 2 \;\; for \;\; 2 \leq i \leq n, & i \;\; even \end{cases}$$

For Case (2):

$$l(u_i) = \begin{cases} 1 \;\; for \;\; 1 \leq i \leq n-2, & i \;\; odd \\ 2 \;\; for \;\; 2 \leq i \leq n-1, & i \;\; even \\ 3 \;\; for \;\; i = n \end{cases}$$

The $r-$lucky number for C_n for n even is 2 and for odd n is 3.

Type 2:

Claim: $r(u) \neq r(v)$

Case 1: $4 \leq n < 7$

For illustration, we consider C_4, C_5 and C_6. See Figure 8, 9 and 10. To prove the condition of type 2, we label the vertices from the set $1, 2, 3$ such that $r(u_1) \neq r(u_2) \neq r(u_3) \neq r(u_4) \neq r(u_1)$. This implies that $r(u) \neq r(v)$ for all pair of adjacent vertices. This argument holds for $n = 4, 5, 6$. Therefore the graph C_n satisfies type 2 $r-$lucky labeling and $\eta_r(G) = 3$.

Case 2: $n \geq 7$

For illustration, Consider C_7. See Figure 11 . To prove our claim, we label the vertices from the set $\{1, 2, 3, 4\}$ such that $r(u_1) \neq r(u_2) \neq r(u_3) \neq r(u_4) ... r(u_7) \neq r(u_1)$.

This proves that $r(u) \neq r(v)$ for all pair of adjacent vertices. This argument holds for C_7. Therefore the graph C_n satisfies type 2 $r-$lucky labeling for $n \geq 7$ and $\eta_r(G) = 4$.

□

Theorem 4.4 *The Wheel graph W_n admits $r-$lucky labeling for $n \geq 5$.*

Proof: The wheel graph W_n is a graph on n vertices in which center vertex of degree $n-1$ is connected to all the vertices of degree 3. To prove W_n satisfies $r-$lucky labeling, we claim $r(u) \neq r(v)$.

Now consider the graph W_7 for illustration. See Figure 12. There are six vertices of degree 3 and one center vertex with degree 6. We label the vertices with 1 and 2 initially to check our claim. This implies that there is atleast one pair of adjacent vertex for which $r(u) = r(v)$. Therefore we label the vertices with $\{1, 2, 3\}$. Take a pair of adjacent vertex with label 3 and 1. The vertex with label 3 has degree 6 and label 1 has degree 3. Therefore $r(u) = 11$ and $r(v) = 4$. This shows that $r(u) \neq r(v)$ for this pair of vertex. By a similar argument we can prove for all pair of vertices adjacent in W_7. This implies that the graph W_7 satisfies type 1 $r-$lucky labeling.

Similarly we prove type 2 labeling in two cases for n even and odd separately.

Case 1: n odd, $n \geq 5$:

The graph W_n is labeled from the set of labels $\{1, 2, 3\}$ in the following manner.

$$l(u_i) = \begin{cases} 1 & for \quad i\ odd \\ 2 & for \quad i\ even \\ 3 & for \quad center\ vertex \end{cases}$$

For illustration, we have W_5 shown in Figure 13.

This satisfies the condition $r(u) \neq r(v)$ for every pair of adjacent vertices.

Case 2: n even, $n \geq 6$

The graph W_n is labeled from the set of labels $\{1, 2, 3\}$ in a particular pattern which differs from case (i) and type 1 labeling.

The labels of W_6 are as follows:

$$l(u_i) = \begin{cases} 1 & for \quad i = 1 \\ 2 & for \quad 2 \leq i \leq 4 \\ 3 & for \quad i = 5, 6 \end{cases}$$

The labels of W_8 are as follows:

$$l(u_i) = \begin{cases} 1 & for \quad i = 1, 3 \\ 2 & for \quad i = 2, 4 \leq i \leq 6 \\ 3 & for \quad i = 7, 8 \end{cases}$$

Figure 14 and 15 shows W_6 and W_8 respectively. In the same way, we can label W_n for any n even and thus it satisfies the condition $r(u) \neq r(v)$ for every pair of adjacent vertices. Therefore we can say that W_n satisfies type 1 and type 2 $r-$lucky labeling for all pair of adjacent vertices in W_n and the least positive integer satisfied by the graph is 3 and it is denoted by $\eta_r(G) = 3$.

□

Theorem 4.5 *The Ladder graph L_n admits $r-$lucky labeling for $n \geq 3$.*

Proof: The graph L_n is a graph on $2n$ number of vertices and $3n - 2$ edges. To prove that L_n satisfies $r-$lucky labeling, we claim $r(u) \neq r(v)$.

For illustration, we consider L_3 with six vertices and their labels are shown in the Figure 16 and 17.

Take a pair of adjacent vertices u_1 of degree 2 and u_3 of degree 3. By the condition of type 1: $r(u_1) = 3$ and $r(u_2) = 5$ similarly by the condition of type 2: $r(u_1) = 3$ and $r(u_2) = 7$. This implies that $r(u) \neq r(v)$ for both type 1 and type 2. The least positive integer satisfied by L_3 for type 1 is 2 and for type 2 is 3. This proves our claim. Thus the graph L_n satisfies $r-$lucky labeling for all pairs of adjacent vertices in L_n. □

5. Conclusion

The new labeling technique was applied to basic graphs and found that certain graphs satisfies both type 1 and type 2 $r-$lucky labeling on the otherhand certain graphs satisfy either the one. Application of this technique will be extended to some family of graphs to study their behaviour.

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Rini Angeline Sahayamary A.,
Department of Mathematics, Stella Maris College,
University of Madras,
India.
E-mail address: riniangeline@gmail.com

and

Teresa Arockiamary S.,
Department of Mathematics, Stella Maris College,
University of Madras,
India.
E-mail address: teresa_aroc@yahoo.co.in