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Conformal Ricci Soliton in (ε)-Kenmotsu Manifolds

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ABSTRACT: The present study examines a semi-symmetric (ε)-Kenmotsu manifold with M-projective curvature tensor, admitting conformal Ricci soliton, pseudo projective curvature tensor method. We have discovered that the (ε)-Kenmotsu manifold admitting conformal Ricci soliton is a quadratic equation that is M-projective Ricci symmetric. We have proved that a M-projective and pseudo projective semi symmetric (ε)-Kenmotsu manifold admitting conformal Ricci soliton is ϱ-Einstein manifold.

Keywords: (ε)-Kenmotsu manifold, conformal Ricci soliton, pseudo projective curvature tensor, M-projective curvature tensor.

Contents

1. Introduction

Hamilton first suggested the concept of Ricci Soliton in the middle of the 1980s. In addition to being self-similar solutions of Hamilton's Ricci flow, Ricci solitons are inherent extensions of Einstein metrices. Hamilton, R. S. [7]. It frequently appears as the boundaries of singularity dilations in the Ricci flow. Numerous geometers have studied the Ricci soliton in contact manifolds ([10], [5]). The equation for the Ricci soliton is provided by

$$\pounds_B + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor, g is the Riemannian metric, B is a vector field, λ is a scalar, and $\pounds_B$ is the Lie derivative.

Fischer [8] presented a novel idea in 2005 termed conformal Ricci flow, This variation of the conventional Ricci flow equation takes the place of the unit volume limitations with a scalar curvature constraint. Because conformal geometry constrains the scalar curvature and because the resulting formulas are the vector field sum of a Ricci flow equation and a conformal flow equation, they are known as the "conformal Ricci flow equations". These new formulas are provided by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg, \tag{1.2}$$
$$R(g) = -1.$$

where p is a non-dynamical scalar field and $R(g)$ is the manifold's scalar curvature.

This conformal Ricci flow equation is guaranteed by the idea of conformal Ricci soliton, which was first presented by A. Bhattacharyya and N. Basu [4].

$$\pounds_B g + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{1.3}$$

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In this work, we have demonstrated $W^*(\zeta, B).S = 0$ is a quadratic equation. We study (ε)-Kenmotsu manifold admitting conformal Ricci soliton and $R(\zeta, B).W^* = 0$ is ϱ-Einstein manifold, where W^* is M-projective curvature tensor. We have discovered that $R(\zeta, B).\tilde{P} = 0$ is an ϱ-Einstein manifold, where $\tilde{P}$ is pseudo projective curvature tensor.

2. Preliminaries

A smooth manifold (M, g) of n-dimension is a (ε)-almost contact metric manifold if

$$\begin{aligned} \varpi^2 B &= -B + \varrho(B)\zeta, & (2.1)\\ \varrho(\zeta) &= 1, & (2.2)\\ g(\zeta, \zeta) &= \varepsilon, & (2.3)\\ \varrho(B) &= \varepsilon g(B, \zeta), & (2.4)\\ g(\varpi B, \varpi D) &= g(B, D) - \varepsilon\varrho(B)\varrho(D), & (2.5) \end{aligned}$$

for each and every vector field B, D on M, where ε is either 1 or -1 according to whether ζ is a space or time like vector field, and rank that ϖ is $(n-1)$. If

$$d\varrho(B, D) = g(B, \varpi D), \tag{2.6}$$

for every $B, D \in TM$, Next, we state that $M(\varpi, \zeta, \varrho, g, \varepsilon)$ is an almost contact metric manifold. In addition, we have

$$\varrho \circ \varpi = 0, \quad \varpi\zeta = 0. \tag{2.7}$$

If a manifold with a ε-contact metric satisfies

$$(\nabla_B \varpi)D = -g(B, \varpi D)\zeta - \varepsilon\varrho(D)\varpi B, \tag{2.8}$$

here ∇ indicates the Riemannian connection of g, The manifold M is then known as a (ε)-Kenmotsu manifold. [6]. If and only if

$$\nabla_B \zeta = \varepsilon(B - \varrho(B)\zeta). \tag{2.9}$$

a (ε)-almost contact metric manifold is a (ε)-Kenmotsu. Furthermore, w.r.t. the LC-connection on a (ε)-Kenmotsu manifold M, the following connections satisfy [6]

$$\begin{aligned} (\nabla_B \varrho)D &= g(B, D) - \varepsilon\varrho(B)\varrho(D), & (2.10)\\ R(B, D)\zeta &= \varrho(B)D - \varrho(D)B, & (2.11)\\ R(\zeta, B)D &= \varrho(D)B - \varepsilon g(B, D)\zeta, & (2.12)\\ \varrho(R(B, D)F) &= \varepsilon[g(B, F)\varrho(D) - g(D, F)\varrho(B)], & (2.13)\\ S(B, \zeta) &= -(n-1)\varrho(B), & (2.14)\\ Q\zeta &= -\varepsilon(n-1)\zeta, & (2.15) \end{aligned}$$

where $g(QB, D) = S(B, D)$.
It yields to

$$S(\varpi B, \varpi D) = S(B, D) + \varepsilon(n-1)\varrho(B)\varrho(D). \tag{2.16}$$

3. (ε)-Kenmotsu manifold admitting cr soliton $W^*(\zeta, B).S = 0$

R. S. Mishra and G. P. Pokhariyal [9] described on the Riemannian manifold M-projective curvature tensor W^* as

$$\begin{aligned} W^*(B,D)F = R(B,D)F - \frac{1}{2(n-1)}&[S(D,F)B - S(B,F)D \\ &+g(D,F)QB - g(B,F)QD]. \end{aligned} \tag{3.1}$$

From (3.1), we've obtained

$$\begin{aligned} W^*(\zeta,B)D = R(\zeta,B)D - \frac{1}{2(n-1)}&[S(B,D)\zeta - S(\zeta,D)B \\ &+g(B,D)Q\zeta - g(\zeta,D)QB]. \end{aligned} \tag{3.2}$$

Using (2.12), (2.14) and (2.15) in (3.2), we have

$$\begin{aligned} W^*(\zeta,B)D &= \varrho(D)B - \varepsilon g(B,D)\zeta \\ &- \frac{1}{2(n-1)}[S(B,D)\zeta + (n-1)\varrho(D)B \\ &- \varepsilon(n-1)g(B,D)\zeta - \frac{1}{\varepsilon}\varrho(D)QB], \end{aligned} \tag{3.3}$$

similarly, we have

$$\begin{aligned} W^*(\zeta,B)F &= \varrho(F)B - \varepsilon g(B,F)\zeta \\ &- \frac{1}{2(n-1)}[S(B,F)\zeta + (n-1)\varrho(F)B \\ &- \varepsilon(n-1)g(B,F)\zeta - \frac{1}{\varepsilon}\varrho(F)QB]. \end{aligned} \tag{3.4}$$

We believe that a tensor-derivative of S by $W^*(\zeta, B)$ is zero i.e., $W^*(\zeta, B).S=0$. Then the (ε)-Kenmotsu manifold M admitting cr soliton is M-projective Ricci symmetric. It provides

$$S(W^*(\zeta,B)D,F) + S(D,W^*(\zeta,B)F) = 0. \tag{3.5}$$

Using (3.3) and (3.4) in (3.5), we have

$$\begin{aligned} S(\varrho(D)B - \varepsilon g(B,D)\zeta - \frac{1}{2(n-1)}&[S(B,D)\zeta + (n-1)\varrho(D)B \\ -\varepsilon(n-1)g(B,D)\zeta &- \frac{1}{\varepsilon}\varrho(D)QB], F) \\ +S(D, \varrho(F)B - \varepsilon g(B,F)\zeta - \frac{1}{2(n-1)}&[S(B,F)\zeta + (n-1)\varrho(F)B \\ -\varepsilon(n-1)g(B,F)\zeta &- \frac{1}{\varepsilon}\varrho(F)QB]) = 0. \end{aligned} \tag{3.6}$$

Put $F = \zeta$ and using (2.14) in (3.6), we've obtained

$$S(B,D) + \frac{\varepsilon}{2}(n-1)g(B,D) + \frac{1}{2\varepsilon(n-1)}S(D,QB) = 0, \tag{3.7}$$

which implies

$$S(B,D) = -a_1 S(QB,D) - b_1 g(B,D), \tag{3.8}$$

where $a_1 = \frac{1}{2\varepsilon(n-1)}$ and $b_1 = \frac{\varepsilon}{2}(n-1)$ which implies

$$QB = -a_1 Q^2 B - b_1 B \tag{3.9}$$

i.e.,

$$a_1Q^2 + Q + b_1 = 0.$$

As a result, we might conclude that if $W^*(\zeta, B).S = 0$ then the Ricci-operator Q satisfied with the quadratic equation.

4. (ε)-Kenmotsu manifold admitting cr soliton $R(\zeta, B).W^* = 0$

Put $F = \zeta$ in (3.1), we've got

$$\begin{aligned} W^*(B, D)\zeta = R(B, D)\zeta - \frac{1}{2(n-1)}[&S(D, \zeta)B - S(B, \zeta)D \\ &+g(D, \zeta)QB - g(B, \zeta)QD], \end{aligned} \tag{4.1}$$

using (2.11), (2.14) and (2.15) in (4.1), we get

$$W^*(B, D)\zeta = \alpha[\varrho(B)D - \varrho(D)B], \tag{4.2}$$

where $\alpha = \frac{1}{2}(1 - \varepsilon)$.
The aforementioned equation can be expressed as

$$g(W^*(B, D)\zeta, F) = \alpha[\varrho(B)g(D, F) - \varrho(D)g(B, F)], \tag{4.3}$$

this implies

$$\varrho(W^*(B, D)F) = \varepsilon\alpha[\varrho(D)g(B, F) - \varrho(B)g(D, F)]. \tag{4.4}$$

Now suppose an (ε)-Kenmotsu manifold did admit cr soliton is M-projective semi symmetric i.e., $R(\zeta, B).W^* = 0$ satisfies, that suggests

$$\begin{aligned} &R(\zeta, B)(W^*(D, F)L) - W^*(R(\zeta, B)D, F)L \\ &-W^*(D, R(\zeta, B)F)L - W^*(D, F)R(\zeta, B)L. \end{aligned} \tag{4.5}$$

Using (2.12) in (4.5) and taking $L = \zeta$, then we are able to write

$$\begin{aligned} &\varrho(W^*(D, F)\zeta)B - \varepsilon g(B, W^*(D, F)\zeta)\zeta \\ &-\varrho(D)W^*(B, F)\zeta + \varepsilon g(B, D)W^*(\zeta, F)\zeta \\ &-\varrho(F)W^*(D, B)\zeta + \varepsilon g(B, F)W^*(D, \zeta)\zeta \\ &-W^*(D, F)B + \varrho(B)W^*(D, F)\zeta = 0, \end{aligned} \tag{4.6}$$

which implies

$$\begin{aligned} &-\varepsilon\alpha\varrho(D)g(B, F)\zeta + \varepsilon\alpha\varrho(D)g(B, D)\zeta \\ &-\varrho(D)W^*(B, F)\zeta + \varepsilon g(B, D)W^*(\zeta, F)\zeta \\ &-\varrho(F)W^*(D, B)\zeta + \varepsilon g(B, F)W^*(D, \zeta)\zeta \\ &-W^*(D, F)B + \varrho(B)W^*(D, F)\zeta = 0. \end{aligned} \tag{4.7}$$

Using ζ as an inner product in (4.7) and taking (4.2), we've got

$$\begin{aligned} \alpha[(2\varepsilon^2 - 1)\varrho(F)g(B, D) - (2\varepsilon^2 - 1)\varrho(D)g(B, F)] \\ -\frac{1}{\varepsilon}\varrho(W^*(D, F)B) = 0. \end{aligned} \tag{4.8}$$

Putting $F=\zeta$ in (4.8), we've got

$$\alpha[(2\varepsilon^2-1)g(B,D)-\frac{(2\varepsilon^2-1)}{\varepsilon}\varrho(B)\varrho(D)] \\ -\frac{1}{\varepsilon}\varrho(W^*(D,\zeta)B)=0. \tag{4.9}$$

From (3.1), we've got

$$\begin{aligned}\varrho(W^*(D,\zeta)B) &= \varepsilon^2 g(B,D)-\frac{1}{\varepsilon}\varrho(B)\varrho(D)\\ &- \frac{1}{2(n-1)}[-\frac{2(n-1)}{\varepsilon}\varrho(B)\varrho(D)\\ &- \varepsilon S(B,D)+\varepsilon^2(n-1)g(B,D)].\end{aligned} \tag{4.10}$$

Using (4.10) in (4.9), we've got

$$\alpha[(2\varepsilon^2-1)g(B,D)-\frac{(2\varepsilon^2-1)}{\varepsilon}\varrho(B)\varrho(D)]-\varepsilon g(B,D)-\frac{1}{\varepsilon^2}\varrho(B)\varrho(D) \\ +\frac{1}{2(n-1)}[-\frac{2(n-1)}{\varepsilon^2}\varrho(B)\varrho(D)-S(B,D)+\varepsilon(n-1)g(B,D)]=0. \tag{4.11}$$

On simplification, we have

$$S(B,D)=2(n-1)[\alpha(2\varepsilon^2-1)-\frac{\varepsilon}{2}]g(B,D) \\ +2(n-1)[\frac{-\alpha(2\varepsilon^2-1)}{\varepsilon}]\varrho(B)\varrho(D), \tag{4.12}$$

this can be expressed as

$$S(B,D)=a_2 g(B,D)+b_2\varrho(B)\varrho(D), \tag{4.13}$$

where

$$a_2=2(n-1)[\alpha(2\varepsilon^2-1)-\frac{\varepsilon}{2}],$$

and

$$b_2=2(n-1)[\frac{-\alpha(2\varepsilon^2-1)}{\varepsilon}].$$

Hence following theorem can be derived from the preceding considerations:

Theorem 4.1 *If* (ε)*-Kenmotsu manifold admitting cr soliton with* M*-projective semi-symmetric* $R(\zeta,B).W^*=0$*, then manifold is called a* ϱ*-Einstein manifold.*

5. (ε)-Kenmotsu manifold admitting cr soliton $R(\zeta,B).\tilde{P}=0$

In a (ε)-Kenmotsu manifold, Pseudo-projective curvature tensor $\tilde{P}$ is described by

$$\tilde{P}(B,D)F=aR(B,D)F+b[S(D,F)B-S(B,F)D] \\ -\frac{r}{n}(\frac{a}{n-1}+b)[g(D,F)B-g(B,F)D], \tag{5.1}$$

put $F=\zeta$ and using (2.4), (2.11), (2.14) in (5.1)

$$\tilde{P}(B,D)\zeta=[-a-(n-1)b-\frac{r}{\varepsilon n}(\frac{a}{n-1}+b)][\varrho(D)B-\varrho(B)D]. \tag{5.2}$$

The aforementioned equation can be written as

$$\tilde{P}(B,D)\zeta = \gamma[\varrho(D)B - \varrho(B)D], \tag{5.3}$$

where $\gamma = [-a-(n-1)b-\frac{r}{\varepsilon n}(\frac{a}{n-1}+b)]$,
that implies

$$g(\tilde{P}(B,D)\zeta, F) = \gamma[\varrho(D)g(B,F) - \varrho(B)g(D,F)], \tag{5.4}$$

then (5.4) becomes

$$\varrho(\tilde{P}(B,D)F) = \frac{\gamma}{\varepsilon}[\varrho(B)g(D,F) - \varrho(D)g(B,F)]. \tag{5.5}$$

Considering that now (ε)-Kenmotsu manifold admits cr soliton and is the pseudo projective semi symmetric i.e., $R(\zeta,B).\tilde{P}=0$ holds in M, it denotes

$$\begin{gathered} R(\zeta,B)(\tilde{P}(D,F)L) - \tilde{P}(R(\zeta,B)D,F)L \\ -\tilde{P}(D,R(\zeta,B)F)L - \tilde{P}(D,F)R(\zeta,B)L = 0. \end{gathered} \tag{5.6}$$

Taking (2.12) in (5.6) and using $L=\zeta$, we've obtained

$$\begin{gathered} \varrho(\tilde{P}(D,F)\zeta)B - \varepsilon g(B,\tilde{P}(D,F)\zeta)\zeta - \tilde{P}(\varrho(D)B - \varepsilon g(B,D)\zeta, F)\zeta \\ -\tilde{P}(D,\varrho(F)B - \varepsilon g(B,F)\zeta)\zeta - \tilde{P}(D,F)[\varrho(\zeta)B - \varepsilon g(B,\zeta)\zeta] = 0, \end{gathered} \tag{5.7}$$

using (5.4), above equation becomes

$$\begin{aligned} \varepsilon\gamma[-\varrho(F)g(B,D)\zeta + \varrho(D)g(B,F)\zeta] - \varrho(D)\tilde{P}(B,F)\zeta \\ +\varepsilon g(B,D)\tilde{P}(\zeta,F)\zeta - \varrho(F)\tilde{P}(D,B)\zeta + \varepsilon g(B,F)\tilde{P}(D,\zeta)\zeta \\ -\tilde{P}(D,F)B + \varrho(B)\tilde{P}(D,F)\zeta = 0. \end{aligned} \tag{5.8}$$

Using ζ as an inner product in (5.8), we've obtained

$$\begin{aligned} \varepsilon\gamma[-\varepsilon\varrho(F)g(B,D)\zeta + \varepsilon\varrho(D)g(B,F)\zeta] - \frac{1}{\varepsilon}\varrho(D)\varrho(\tilde{P}(B,F)\zeta) \\ +g(B,D)\varrho(\tilde{P}(\zeta,F)\zeta) - \frac{1}{\varepsilon}\varrho(F)\varrho(\tilde{P}(D,B)\zeta) + g(B,F)\varrho(\tilde{P}(D,\zeta)\zeta) \\ -\frac{1}{\varepsilon}\varrho(\tilde{P}(D,F)B) + \frac{1}{\varepsilon}\varrho(B)\varrho(\tilde{P}(D,F)\zeta) = 0, \end{aligned} \tag{5.9}$$

using (5.3) in (5.9), we've obtained

$$\begin{aligned} \gamma[(\varepsilon^2 - \frac{1}{\varepsilon^2}+1)\varrho(F)g(B,D) - (\varepsilon^2 - \frac{1}{\varepsilon^2}+1)\varrho(D)g(B,F)] \\ +\frac{1}{\varepsilon}\varrho(\tilde{P}(D,F)B) = 0. \end{aligned} \tag{5.10}$$

Put $F=\zeta$ in (5.10) and (2.4), we've obtained

$$\begin{aligned} \gamma[(\varepsilon^2 - \frac{1}{\varepsilon^2}+1)g(B,D) - \frac{1}{\varepsilon}(\varepsilon^2 - \frac{1}{\varepsilon^2}+1)\varrho(B)\varrho(D)] \\ +\frac{1}{\varepsilon}\varrho(\tilde{P}(D,\zeta)B) = 0. \end{aligned} \tag{5.11}$$

From (5.1), we've obtained

$$\begin{aligned} \frac{1}{\varepsilon}\varrho(\tilde{P}(D,\zeta)B) &= a[\varepsilon^2 g(D,B) - \frac{1}{\varepsilon}\varrho(B)\varrho(D)] \\ &+ b[-\frac{1}{\varepsilon}(n-1)\varrho(B)\varrho(D) - \varepsilon S(D,B)] \\ &- \frac{r}{n}(\frac{a}{n-1}+b)[\frac{1}{\varepsilon^2}\varrho(B)\varrho(D) - \varepsilon g(D,B)]. \end{aligned} \tag{5.12}$$

Using (5.12) in (5.11) and simplifying, we've obtained

$$\begin{aligned} S(B,D) &= \frac{1}{b\varepsilon}[\gamma(\varepsilon^2-\frac{1}{\varepsilon^2}+1)+a\varepsilon^2+\frac{\varepsilon r}{n}(\frac{a}{n-1}+b)]g(B,D) \\ &- \frac{1}{b\varepsilon^2}[\gamma(\varepsilon^2-\frac{1}{\varepsilon^2}+1)+a+b(n-1)+\frac{r}{\varepsilon n}(\frac{a}{n-1}+b)]\varrho(B)\varrho(D). \end{aligned} \tag{5.13}$$

The aforementioned equation can be written as

$$S(B,D) = a_3 g(B,D) + b_3 \varrho(B)\varrho(D), \tag{5.14}$$

where

$$a_3 = \frac{1}{b\varepsilon}[\gamma(\varepsilon^2-\frac{1}{\varepsilon^2}+1)+a\varepsilon^2+\frac{\varepsilon r}{n}(\frac{a}{n-1}+b)],$$

and

$$b_3 = -\frac{1}{b\varepsilon^2}[\gamma(\varepsilon^2-\frac{1}{\varepsilon^2}+1)+a+b(n-1)+\frac{r}{\varepsilon n}(\frac{a}{n-1}+b)].$$

The following theorem can be derived from the preceding considerations:

Theorem 5.1 *A (ε)-Kenmotsu manifold M is ϱ-Einstein manifold if it admits cr soliton and is pseudo projective semi symmetric* $R(\zeta,B).\tilde{P}=0$.

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References

1. K. H. Arun Kumar, K. T. Pradeep Kumar, Venkatesha and S. V. Vishnuvardhana, *C-bochner φ-symmetric (ε)-kenmotsu manifold admitting semi-symmetric metric connection,* GANITA, 72(1) (2022), 111-119.
2. K. H. Arun Kumar and K. T. Pradeep Kumar, *Conformal Ricci Soliton in Generalized Sasakian-Space-Forms,* International Journal of Mathematics And its Applications, 10(1) (2022), 9-15.
3. C. S. Bagewadi and Gurupadavva Ingalahalli, *Ricci solitons in Lorentzian α-sasakian manifolds,* Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis. 28(2012), 59-68.
4. N. Basu and A. Bhattacharya, *Conformal Ricci soliton in Kenmotsu manifold,* Glob. J. Adv. Res. Class. Mod. Geom. 4,1(2015), 15-21.
5. C. L. Bejan and M. Crasmareanu, *Ricci solitons in manifolds with quasi constant curvature,* Publ. Math. Debrecen. 78,1(2011), 235-243.
6. U. C. De and A. Sarkar, *On (ε)-Kenmotsu manifold,* Hadronic J. 32 (2009), no. 2, 231-242.
7. R. S. Hamilton, *The Ricci fow on surfaces, In Mathematics and General Relativity,* Contemp. Math. Soc., Providence, RI, Vol.71, (1988), 237-262.
8. A. E. Fischer, *An introduction to conformal Ricci flow,* A spacetime safari: essays in honour of Vincent Moncrief. Vol. 21 (2004), S171-S128.
9. G. P. Pokhariyal and R. S. Mishra, *Curvature tensor and their relativistic significance II,* Yokohama Mathematical Journal 19 (1971), 97-103.
10. M. M. Tripathi, *Ricci solitons in contact metric manifolds,* arXiv:0801, 4222v1, [math DG] (2008).

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