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# Rings in Which Every Element is Sum of a Unit and Finitely Many Nonzero Idempotents

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ABSTRACT: We define a ring $R$ to be a $UI$ - ring when each element of $R$ can be represented as the sum of a unit and finitely many nonzero idempotents of $R$. In this article we have shown that semisimple rings, artinian rings and semiprimary rings are $UI$ - ring. Also we have proved if $R$ is a $UI$ - ring then for every $n > 1$, $M_n(R)$ is a $UI$ - ring and for each $n > 1$, $R$ is a $UI$ - ring if and only if $T_n(R)$ is a $UI$ - ring.

Key Words: Matrix ring, Jacobson radical, Artinian rings.

## Contents



## 1. Introduction

In ring theory, several classes of rings have been studied based on the additive generation of their elements by idempotents and units. Clean rings and AGI rings are two such important classes. Nicholson [4] initially introduced clean rings. In a clean ring, every element is expressible as the combination of an invertible element and an idempotent element. While in AGI ring, it is known that every element is expressible as a finite combination of idempotents [1] and in [2] AGEI ring is studied. Motivated by these concepts, we have proposed the concept of a UI-ring as a generalization of both. In the present work, a ring $R$ defined to be a $UI$ - ring if every member of $R$ can be represented as the addition of an invertible element and a finite set of idempotent elements. In the present article it is proved that semisimple rings, artinian rings and semiprimary rings are $UI$ - ring. In this paper, it is established that if $R$ is a $UI$ - ring, then $\forall n > 1$, The ring of $(n \times n)$ matrices with elements from the ring $R$ denoted $M_n(R)$ is also a $UI$ - ring. Moreover, $R$ is a $UI$ - ring if and only if $T_n(R)$, the ring of $(n \times n)$ upper triangular matrices with elements from the ring $R$ is a $UI$ - ring.

## 2. Preliminaries

All rings examined in this study are considered to be associative with a identity element. For a given ring $R$, $M_n(R)$ refers to the full matrix ring and $T_n(R)$ refers to the upper triangular matrix ring with elements from the ring $R$. For any undefined terms and notations, we refer [3] and [5]

## 3. Results

**Definition 1** *A ring $R$ is defined as a $UI$ - ring if every elements of $R$ can be represented as the addition of a unit and a finite number of nonzero idempotent elements belonging to $R$. i.e., Let $a \in R$, $a = e_1 + e_2 + \cdots + e_n + u$ ,for some $e_i^2 = e_i \neq 0 \in R$ and unit $u \in R$.*

**Example 3.1** 1. *All Boolean ring, all division rings, all AGI rings are $UI$ - ring.*

2. *For every $n > 1$, $\mathbb{Z}_n$ is $UI$ - ring.*

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**Proposition 3.2** *If $R$ is a $UI$ - ring and $I$ is an idempotent free ideal of $R$ then $R/I$ is an $UI$ - ring.*

**Proof:** The element $\bar{a}$ belongs to the quotient ring $R/I$. Given that $R$ is a $UI$ - ring, $a = e_1 + e_2 + \cdots + e_n + u$ for some $e_i^2 = e_i \in R$ $(1 \leq i \leq n)$ and $u$ is unit in $R$. Therefore, $\bar{a} = \bar{e_1} + \bar{e_2} + \cdots + \bar{e_n} + \bar{u}$. Since $I$ is an idempotent free ideal, $e_i$ doesnot belong to $I$ i.e., the elements $\bar{e_i}$ for each indices $i$ such that $(1 \leq i \leq n)$, are nonzero idempotents in $R/I$. Consequently, $\bar{a}$ can be represented as the sum of an invertible element and a finite collection of non zero idempotents of $R/I$. Therefore, $R/I$ possesses the structure of $UI$ - ring. □

**Theorem 3.1** *Let $R$ be a ring in which idempotent elements can be lifted modulo its Jacobson radical $J(R)$ and $R/J$ is an AGI ring then it follows that $R$ is a $UI$ - ring.*

**Proof:** Assume that quotient ring $R/J$ is an AGI ring and let $\bar{a}$ be an element of $R/J$, then $\bar{a} = \bar{e_1} + \bar{e_2} + \cdots + \bar{e_n}$, for some $\bar{e_i^2} = \bar{e_i} \in R/J$ $(1 \leq i \leq n)$
$a - (e_1 + e_2 + \cdots + e_n) \in J$
$1 - (a - (e_1 + e_2 + \cdots + e_n)) = u$ , for some invertible element $u \in U(R)$
$a = 1 + e_1 + e_2 + \cdots + e_n + u$. Hence, the element $a$ can be represented as the sum of an invertible element and a finite collection of idempotent elements of $R$. Therefore $R$ qualifies as a $UI$ - ring. □

**Proposition 3.3** *If $R$ is a ring in which 1 is the only unit in $R$ then, the subsequent statements hold equivalence among themselves*

1. *The ring $R$ possesses the structure of a $UI$- ring.*
2. *The structure of a ring $R$ serve as an additively generated by idempotents (AGI) ring.*

**Theorem 3.2** *Consider an ideal $I$ in a ring $R$ with $I$ lies inside the jacobson radical $J(R)$ and idempotent lift modulo $I$, the statements below are equivalent to each other.*

1. *$R$ belongs to the class of $UI$- ring.*
2. *The quotient ring $R/I$ is $UI$- ring.*

**Proof:** (1) $\implies$ (2) *Obvious*
(2) $\implies$ (1) *Suppose the quotient ring $R/I$ is a $UI$- ring and that idempotents lift modulo $I$, let $\bar{a} \in R/I$,$\bar{a} = \bar{e_1} + \bar{e_2} + \cdots + \bar{e_n} + \bar{u}$, for some $\bar{e_i^2} = \bar{e_i}$ belongs to quotient ring $R/I$ indexed by $i$ with $(1 \leq i \leq n)$ and the element $\bar{u}$ is invertible in $R/I$. Given that $I \subseteq J(R)$, we obatined units lifts modulo $I$. $\bar{a} - (\bar{e_1} + \bar{e_2} + \cdots + \bar{e_n}) = \bar{u}$, $a - (e_1 + e_2 + \cdots + e_n) + I = u + I \in U(R/I)$, thus $a - (e_1 + e_2 + \cdots + e_n)$ is invertible element in $R$ i.e., $a - (e_1 + e_2 + \cdots + e_n) = v$ which implies, $a = (e_1 + e_2 + \cdots + e_n) + v$. Thus $R$ is $UI$-ring.*

□

**Corollary 3.1** *Suppose $N$ denote a nil ideal of a ring $R$. The ring $R$ becomes a $UI$- ring if and only if the quotient ring $R/N$ becomes a $UI$- ring.*

**Proof:** Given $R$ be a $UI$- ring then by Theorem 3.2 we have quotient ring $R/N$ is UI ring.
Conversely, Let $R/N$ is a $UI$- ring, It is well-known that every nil ideal is contained in the Jacobson radical, and idempotents can be lifted modulo every nil ideal. By Theorem 3.2 we can write $R$ is a $UI$-ring. □

**Proposition 3.4** *Let $R$ be the direct product of $R_1, R_2, \cdots, R_k$ then the statements below are equivalent to each other.*

1. *The ring $R$ is a $UI$-ring*

2. *the rings $R_1, R_2, ..., R_k$ are $UI$-rings*

**Proof:** (1) $\implies$ (2) Given that a ring $R$ is a $UI$-ring and $R = R_1 \times R_2 \times \cdots \times R_k$. Let $a \in R$ , $a = (a_1, a_2, \cdots, a_k)$. Given $R$ is a $UI$-ring,

$$\begin{aligned} a = (a_1, a_2, \cdots, a_k) = \; & (e_1, 0, \cdots, 0) + (e_2, 0, \cdots, 0) + \cdots + (e_p, 0, \cdots, 0) \\ & + (0, f_1, \cdots, 0) + (0, f_2, \; \cdots, 0) + \cdots + (0, f_q, \cdots, 0) \\ & + \cdots + (0, 0, \cdots, g_1) + (0, 0, \cdots, g_2) + \cdots + (0, 0, \cdots, g_r) + (u_1, u_2, \cdots, u_k) \end{aligned}$$

$(e_i, 0, \cdots, 0)$, $(1 \le i \le p)$, $(0, f_j, \cdots, 0)$, $(1 \le j \le q)$, $\cdots$, $(0, 0, \cdots, g_k)$ $(1 \le k \le r)$ are idempotents of $R$. Therefore $a_1 = e_1 + e_2 + \cdots + e_p + u_1, a_2 = f_1 + f_2 + \cdots + f_q + u_2, \cdots, a_n = g_1 + g_2 + \cdots + g_r + u_k$. Thus $R_1, R_2, \cdots, R_n$ are $UI$-rings.
(2) $\implies$ (1) Let $a \in R$ , $a = (a_1, a_2, \cdots, a_k)$. Given $R_1, R_2, ..., R_k$ are $UI$-rings

$$a = (a_1, a_2, \cdots, a_n) = \quad (e_1 + e_2 + \cdots + e_p + u_1, f_1 + f_2 + \cdots + f_q + u_2, \cdots, g_1 + g_2 + \cdots + g_r + u_k)$$

Thus the element $a$ can be represented as the sum containing a unit and a finite set of idempotents of $R$. It follows that $R$ is a $UI$-ring. □

**Proposition 3.5** *If $R$ is a $UI$-ring, then diagonal matrix over $R$ can be represented as the sum of a unit together with finitely many idempotent matrices over $R$.*

**Proof:** $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$.
Since $R$ is an UI ring we have

$$\begin{aligned} a_{11} &= e_1 + e_2 + \cdots + e_p + u_1, \\ a_{22} &= f_1 + f_2 + \cdots + f_q + u_2, \\ &\vdots \qquad \vdots \qquad \vdots \\ a_{nn} &= g_1 + g_2 + \cdots + g_r + u_n, \end{aligned}$$

for some $e_j^2 = e_j (1 \le j \le p)$, $f_l^2 = f_l$ $(1 \le l \le q)$ , $g_m^2 = g_m$ $(1 \le m \le r)$and $u_1, u_2 \cdots u_n$ are units of $R$. Therefore,

$$A = \begin{bmatrix} e_1 + e_2 + \cdots + e_p & 0 & \cdots & 0 \\ 0 & f_1 + f_2 + \cdots + f_q & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g_1 + g_2 + \cdots + g_r \end{bmatrix} + \begin{bmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_n \end{bmatrix}$$

It follows that every diagonal matrix in $M_n(R)$ representable as the sum of a unit together with finitely many number of idempotent matrices of $M_n(R)$. □

**Theorem 3.3** *If a ring $R$ is a $UI$- ring, then the corresponding matrix ring $M_n(R)$ is also satisfies the $UI$- ring condition.*

**Proof:** Suppose a ring $R$ is a UI ring and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ element of $M_n(R)$. Then,

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & 1 & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & 1 \end{bmatrix} + \begin{bmatrix} a_{11}-1 & 0 & \cdots & 0 \\ 0 & a_{22}-1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}-1 \end{bmatrix}.$$

$$A = B_1 + B_2$$

$B_1$ can be represented as sum of idempotents of $M_n(R)$ and by Proposition 3.5 we have diagonal matrix element can be represented as a combination of a unit and a finite number of nonzero idempotent elements from $M_n(R)$. Hence, the matrix ring $M_n(R)$ also possesses the $UI$- ring property. □

**Proposition 3.6** *A ring $R$ is a $UI$- ring if and only if the ring of upper triangular $(n \times n)$ matrices over $R$ denoted $T_n(R)$, is also possesses the $UI$- ring property.*

**Proof:** Direct part immediately immediately follows from Theorem 3.3.

Conversely, Let $a \in R$ then $A = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in T_n(R)$. As $T_n(R)$ possesses the UI ring property, $A = E_1 + E_2 + \cdots + E_n + U$, $E_i = (e_{ij}) \in Id(T_n(R))$ and $U = (u_{ij})$ is a unit in $T_n(R)$. In an idempotent matrix, each diagonal element is an idempotent of $R$ and in a unit matrix, diagonal elements are units of $R$. Thus the element $a$ can be represented as a combination of a unit and finite number of nonzero idempotents of $R$. Hence, the ring $R$ also possesses the $UI$-ring property. □

**Proposition 3.7** *If a ring $R$ is a semisimple ring then it is a $UI$-ring.*

**Proof:** Let $R$ be a semisimple ring, then according to Wedderburn Artin theorem $R$ can be decomposed into completely reducible simple rings. i.e.,$R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times M_{n_3}(D_3) \times \cdots \times M_{n_k}(D_k)$. Since division ring is UI ring which implies $M_{n_i}(D_i), 1 \leq i \leq k$ is UI ring. Thus $R$ is UI ring.

□

**Proposition 3.8** *If a ring $R$ is Artinian ring, then it is a $UI$-ring.*

**Proof:** Given that a ring $R$ is Artinian ring then the quotient ring $R/J$ is semisimple then by Proposition3.7 we can write the quotient ring $R/J$ is an UI ring. Given that a ring $R$ is an artinian ring then its jacobson radical is a nilpotent, implying that it is nil ideal. Thus by Corollary 3.1 we can write a ring $R$ possesses the $UI$-ring property. □

**Proposition 3.9** *A semiprimary ring $R$ necessarily satisfies the condition of being a $UI$-ring.*

**Proof:** Given that $R$ is a semiprimary ring which implies its factor ring $R/J(R)$ is semisimple and its jacobson radical $J(R)$ is nilpotent. Hence, we get $R/J(R)$ is UI ring and $J(R)$ is nil ideal. Hence, by applying Corollary 3.1 it follows that a ring $R$ is a $UI$-ring. □

**Proposition 3.10** *If a ring $R$ is a local ring in which every idempotent element lift modulo $J(R)$, then $R$ is a $UI$-ring.*

**Proof:** When $R$ is a local ring, the corresponding quotient ring $R/J(R)$ is necessarily a division ring. Since all division rings are $UI$-ring, we obtain $R/J(R)$ is a $UI$-ring. Thus, by applying Theorem 3.2 it follows that a ring $R$ is a $UI$-ring. □

**Proposition 3.11** *Let $R$ be a ring. Then the skew power series ring $R[[x,\alpha]]$ is a $UI$-ring if and only if the basering $R$ is a $UI$-ring.*

**Proof:** Let the ring $R[[x,\alpha]]$ is UI-ring and we have $R \cong R[[x,\alpha]]/(x)$. Then by 3.2 we get $R$ is UI-ring. Conversely, given that $R$ be a UI-ring and let $f(x) = b_0 + b_1x + b_2x^2 + \cdots \in R[[x,\alpha]]$, $b_0, b_1, b_2, \cdots \in R$. Since $R$ is UI-ring, $b_0 = e_1 + e_2 + \cdots + e_n + u$ ,for some $e_i^2 = e_i \neq 0$ and unit $u$ in $R$. Thus $f(x) = e_1 + e_2 + \cdots + e_n + (u + a_1x + a_2x^2 + \cdots)$, where $e_i^2 = e_i \neq 0$ and $(u + a_1x + a_2x^2 + \cdots)$ unit in $R[[x,\alpha]]$. Thus, $R$ is a $UI$-ring. □

## 4. Conclusion

In this work, we have explored the structure of rings in which elements can be decomposed additively in terms of idempotents and units. Starting from AGI rings, where elements are sums of finitely many idempotents, and clean rings, where elements are sums of an idempotent and a invertible element, we introduced the notion of UI-rings as a natural generalization. UI-rings allow each element to be expressed as a sum of finitely many nonzero idempotents along with a unit, thereby combining the additive flexibility of AGI rings with the invertibility feature of clean rings. It is proved that semisimple rings, artinian rings and semiprimary rings are $UI$ - ring. Also, it is proved if $R$ is $UI$ - ring then for each $n > 1$, $M_n(R)$ is $UI$ - ring and for each $n > 1$, $R$ is $UI$ - ring if and only if $T_n(R)$ is $UI$ - ring. It is shown that, if $I$ is an idempotent free ideal of a ring $R$ then the property of being a $UI$ - ring pass to the quotient ring $R/I$ , it is also shown that $I \subseteq J$, a ring $R$ is $UI$-ring and if and only if $R/I$ is $UI$-ring. This study provides a foundation for further exploration of additive structures in rings and their applications in ring theory and related algebraic areas.

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