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Certain Variants of Neighborhood Number of Wheel Related Graphs

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ABSTRACT: The concept of neighborhood number in a graph has broad applications across various fields. It quantifies how effectively a subset of vertices can collectively cover the entire network. For instance, in social network system the neighborhood number can be used to determine the sets of influential users whose immediate connections facilitate efficient information dissemination throughout the network. In a graph G, a subset S of $V(G)$ is said to be neighborhood set (n-set), if the union of sub graphs of G induced by the closed neighbors of elements in S produces graph G. The minimum cardinality of a minimal neighborhood set of G is called the neighborhood number of a graph G, denoted by $n(G)$. The structure of wheel graph is instrumental for designing, managing and analyzing systems where a central entity directly connects to all others, optimizing various practical tasks in real life. This paper aims to explore some variants of neighborhood sets and their dimensions of certain wheel related graphs.

Key Words: Neighborhood set, neighborhood dimension.

Contents

1. Introduction

In the complex network, security is a major issue which can be addressed by a concept of graph theory called domination. In 1862, de Jaenisch initiated the idea of domination. In a graph $G(V, E)$, domination number $\gamma(G)$, is the minimum cardinality of a set S, such that $S \subseteq V(G)$, where every $v \in V(G)$ is either in S or adjacent to a vertex of S. Then S is called a dominating set. Each of dominating vertices acts as a monitoring or control point of all the neighboring vertices. In network search, domination contributes to the problems such as intrusion detection, key management, secure routing and monitoring, developing routing algorithms, wireless network system, transport and communication networks etc. Depending upon the requirement of the network applicability models, there arose a many variation in the domination concept such as independent domination, connected domination [7], total domination [8], secure domination [10] and restrained domination [13]. Chessboard problems inspired the notion of independent domination, and its basic characterisations were given by Berge [3]. The goal

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was to identify minimum number of chess pieces which dominates various squares of a chessboard. In 1996, E Sampathkumar et.al defined a variation called strong and weak domination number [6]. In 2015 S.K. Vaidya et.al [16], determined the Steiner domination number for some wheel related graphs.

The neighborhood provides the local view for each node in the network, plays a vital role for several domination related parameter. One such variation or modification of domination is its neighborhood number of a graph. A vertex $u \in V(G)$, is said to be a neighbor of $v \in V(G)$ if v adjacent to u. An open neighborhood of a vertex $v \in V(G)$, is given by $N(v) = \{u \in V : d(u,v) = 1\}$. Further, a closed neighborhood of $v \in V(G)$, is the set $N[v] = N(v) \cup \{v\}$, where $d(u,v)$ is the distance between $u, v \in V(G)$. A subset S of $V(G)$ is called a *neighborhood set* or n-set of G, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by the set $N[v]$. The minimum cardinality of such minimal subset S is called *neighborhood number* of graph denoted as $n(G)$. Neighborhood sets plays a vital role in the network where, it is required to ensure the full coverage of the network with minimum active vertices in a network application problem such as network design, resource allocation, information dissemination, social networks, biological systems, and infrastructure security. Also, in several networks, the vertices in n-set represents key stations or sensors capable of monitoring or influencing all other in the system. In 1985, E. Sampathkumar and P. S. Neeralagi [4] introduced neighborhood sets and neighborhood number of a graph and obtained the results of some standard graphs. In 2001, Sonar. N. D and B. Cheluvaraju [14] defined the maximal neighborhood set $S \subseteq V(G)$, if $\bar{S} = V - S$ does not contain a neighborhood set and found minimum cardinality of a maximal neighborhood set,denoted as $nm(G)$. In 2018, B. Sooryanarayana et.al [1]. studied a special class of dominating set called the neighborhood set along with its complements and resolving property and in 2021 they extended the results to determine a few variations of resolving sets of path, cycle and wheel graphs by excluding neighborhood property and also deduced certain relations on super-hereditary property of a graph [2]. Results on hereditary and super-hereditary properties related to dominating sets in graphs are found in [17]. In this paper, we are extending the results of [1], [2] and [14] to find certain types of neighborhood sets. For more works and notations not defined here on domination number and neighborhood number we refer [5] [9] [11] [12] [15] [18]. The purpose of considering wheel related graphs in this paper is because of its important role in the design and analysis of network systems, especially due to its unique structure involving a leader - centric group of vertices. Wheel like structures are used in embedded processor networks, VLSI circuits, chip layouts, routing and distance minimization. The central node in a wheel graph provides a single point through which all vertices can communicate efficiently. If one rim connection fails the central hub still connects all devices and the peripheral cycle allows limited alternate routes supporting fault tolerance to some extent.

The following results are recalled.

Theorem 1.1 ([4]) *A subset S of $V(G)$ is an n-set of G if and only if every edge of $\langle V(G) - S \rangle$ belongs to a triangle one of whose vertices is in S.*

Corollary 1.2 ([4]) *Let S be an n-set of G. Then $\overline{S}$ is totally disconnected if and only if G is triangular free.*

Theorem 1.3 ([4]) *For any $m \in \mathbb{Z}^+$ with $m \geq 4$, the lower neighborhood number $n(C_m) = \lceil \frac{m}{2} \rceil$.*

Remark 1.4 ([4]) *Every $(m-1)$-element subset S of a connected graph of order m is always an n-set.*

Remark 1.5 ([4]) *For a graph G, $n(G) = 1$ if and only if G has a vertex v such that $N[v] = V(G)$.*

Theorem 1.6 ([4]) *Let $e = ab$ be an edge of a graph G such that e is not an edge of a cyle C_3 in G and S be an n-set of G. Then $a, b \in N[v]$ for some $v \in S$ if and only if $a = v$ or $b = v$.*

Theorem 1.7 ([4]) *For each edge $e = uv$ in G if there exists a vertex $w \in S$ such that both $u, v \in N[w]$ then S is an n-set of G.*

Theorem 1.8 ([1]) *For any $m \in Z^+$, $nmd(P_m) = \begin{cases} \lceil \frac{m}{2} \rceil, & \text{if } m \leq 3 \\ \lfloor \frac{m}{2} \rfloor, & \text{if } m \geq 4 \end{cases}$*

Theorem 1.9 ([2]) *For any super-hereditary property $\mathcal{P}$ of the graph G either $l_p(G)$=$l^*_p(G)$ or $\mathcal{P}^* = \emptyset$.*

2. Neighborhood sets and its variants

Every neighborhood set is a dominating set. Throughout this paper we call neighborhood set defined above as an nd-set. An nd-set S of the graph G is called a $\hat{nd}$-set if $\overline{S}$ is not an nd-set of G. An nd-set S of the graph G is called an nd^*-set if $\overline{S}$ is also an nd-set of G. A set S of vertices of the graph G is called an $\overline{nd}$-set if neither S nor $\overline{S}$ is an nd-set of G. The minimum cardinality of an nd-set, nd^*-set, $\hat{nd}$-set and $\overline{nd}$-set are respectively, called nd-number, nd^*-number, $\hat{nd}$-number and $\overline{nd}$-number of G, denoted by $nd(G)$, $nd^*(G)$, $\hat{nd}(G)$, and $\overline{nd}(G)$.

Remark 2.1 *Based on the preceding definitions it follows that* $nd^*(G) \geq nd(G)$ *and* $\hat{nd}(G) \geq nd(G)$ *for every graph* G.

Remark 2.2 *Let* S *be a subset of a connected graph* G *with* $|V(G)| = m$. *Then* S *is always a nd-set whenever* $|S| \geq m - 1$.

Lemma 2.3 *If a graph* G *with* $|V(G)| > 4$ *contains an edge* $e = ab$ *such that* e *is not an edge of a cycle* C_3 *in* G *and* $e \in \langle S \rangle$ *then* S *is an* $\overline{nd}$*-set of* G *with* $|S| = 2$.

Proof: Let S be an $\overline{nd}$-set of a graph G with $|V(G)| > 4$. Then neither S nor $\overline{S}$ an nd-set of G. Let $e = ab$ be an edge of G not contained in any triangle of G. If $S = \{a, b\}$, then S is not an nd-set of G as $O(G) > 4$, at least one edge or a vertex does not belong to $\langle S \rangle$ such that $\langle S \rangle \neq G$. On the other hand, an edge $ab \notin \langle V - S \rangle$ and hence $G \neq \bigcup_{v \in \overline{S}} \langle N[v] \rangle$ implies $\overline{S}$ not an nd-set of G. Therefore, S is an $\overline{nd}$-set with $|S| = 2$. □

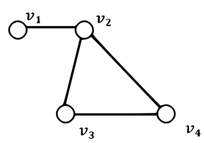

Figure 1: Example to nd-sets of a graph

Example: For the graph G of Figure 1, the possible nd-sets are $\{v_2\}$, $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_4\}$, $\{v_2, v_3\}$, $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_3, v_4\}$, $\{v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_4\}$ among these, minimal set is $\{v_2\}$ with cardinality 1. Hence, $nd(G) = 1$. Let $\{v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_4\}$, $\{v_2, v_3\}$, $\{v_1, v_3, v_4\}$ be nd^* - sets of G and among them minimal set is $\{v_2\}$ with cardinality 1. Hence, $nd^*(G) = 1$. Let $\{v_1, v_2\}$, $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_2, v_3, v_4\}$ be $\hat{nd}$ -sets and the minimal set is $\{v_1, v_2\}$ with cardinality 2. Hence, $\hat{nd}(G) = 2$. And $\overline{nd}$-set does not exist as either S or $\overline{S}$ is an nd-set(from Remark 1.5).

In the next sections, all the four variants of neighborhood sets of wheel related graphs- Helm, Closed Helm, Gear, Flower, closed Flower, Double wheel and Sunflower Graphs are discussed and hence obtained the respective lower neighborhood numbers.

3. Neighborhood Numbers of Helm Graph

Let $H_m(V, E)$ be a Helm graph obtained by Wheel Graph W_{m+1} by adding a pendant vertex to each rim vertices of W_{m+1}. Let $V(H_m) = \{v_0, v_i, u_i : 1 \leq i \leq m\}$ be the vertex set such that $E(H_m) = \{v_0 v_i, v_i u_i : 1 \leq i \leq m\} \cup \{v_i v_{i+1} : 1 \leq i \leq m - 1\} \cup \{v_m v_1\}$ be its edge set, with v_0 as its central vertex, v_i's as rim vertices and u_i's as pendant vertices adjacent to rim vertices. Let $v_0 v_i (1 \leq i \leq m)$ are edges between central vertex and rim vertices, $v_i v_{i+1}, v_m v_1$, $1 \leq i \leq m - 1$ are edges on the rim and $v_i u_i$, $1 \leq i \leq m$ are edges between rim and pendant vertices, for all $1 \leq i \leq m$.

Theorem 3.1 *For any integer* $m \in Z^+$ *and* $m \geq 3$,

- **(i)** $nd^*(H_m) = nd(H_m) = m.$
- **(ii)** $\hat{nd}(H_m) = nd(H_m) + 1.$

(iii) $\overline{nd}(H_m) = 2$.

Proof: For (i): Let S_1 be an $nd-$set of Helm graph H_m. Since H_m contains m pendant edges $u_iv_i(1 \leq i \leq m)$ with u_i as pendant vertex and v_i as rim vertex of wheel in Helm, by Theorem 1.6 an nd-set S_1 should contain either u_i or v_i for each i. Thus $|S_1| \geq m$. If $u_i \in S_1$ then edges $u_iv_i \in \langle S_1\rangle$ but edges $v_0v_i, v_iv_{i+1}, v_mv_1 \notin \langle S_1\rangle$ so, to cover edges $v_0v_i, v_iv_{i+1}, v_mv_1$ S_1 should contain a vertex v_0 implies $|S_1| = m+1$. But for the minimal S_1 set we consider vertices v_i's in S_1 so that each v_i induces a pendant edge v_iu_i and also from Theorem 1.1 $v_0v_i, v_iv_{i+1}, v_mv_1 \in \langle S_1\rangle$ as these vertices are in triangles of sub graph wheel of H_m formed due to spokes and rim edges, which induces all the edges of wheel. Therefore, $G = \bigcup_{v\in S_1} \langle N[v]\rangle$. Hence, $|S_1| \leq m$ and $nd(H_m) = |S_1| = m$.

On the other hand, the set $\overline{S}_1 = V(H_m) - S_1 = \{v_0, u_i : 1 \leq i \leq m\}$. Since, $v_0 \in \overline{S}_1$ adjacent to all rim vertices and edges $v_iv_{i+1}, v_mv_1, 1 \leq i \leq m-1$ lies in a triangle in which v_0 is one of the vertex and $u_i \in V - S_1$ incident with edges $v_iu_i, 1 \leq i \leq m$ implies $G = \bigcup_{v\in \overline{S}_1} \langle N[v]\rangle$. Hence, $\overline{S}_1$ is also an nd-set. Therefore, $nd^*(H_m) = m$.

For (ii): Let S_2 be an $\hat{nd}$-set. Then, S_2 is an nd - set and $\overline{S}_2$ is not an nd-set. From the proof of (i), an nd-set S_1 should contain m vertices but then $\overline{S}_1$ is also an nd-set. Let us suppose if S_1 contain one more vertex say, u_i an apex vertex adjacent to v_i then an edge $v_iu_i \notin \bigcup_{v\in \overline{S}_2} \langle N[v]\rangle$, hence $\overline{S}_2$ is not an nd-set. Therefore, $|S_2| = \hat{nd}(H_m) = m+1$.

For (iii): Let S_3 be an minimal $\overline{nd}$ - set. Then both S_3 and $\overline{S}_3$ are not nd-sets. The result is true from Lemma 2.3(since, H_m contain an edge v_iu_i, for all $1 \leq i \leq m$ not in a triangle of H_m). □

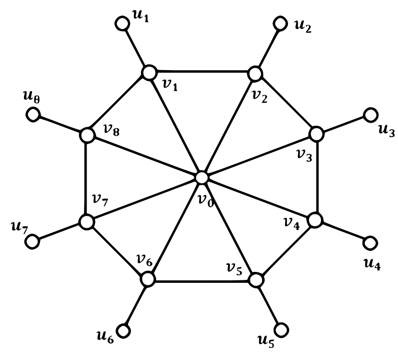

Figure 2: Helm graph H_8

Example: For the graph $G = H_8$ of Figure 2, $nd(H_8) = 8 = nd^*(H_8)$ with an nd-set (also an nd^*-set) $S = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$; $\hat{nd}(H_8) = 9$ with a $\hat{nd}$-set $S = \{u_1, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$; and $\overline{nd}(H_8) = 2$ with an $\overline{nd}$-set $S = \{v_1, u_1\}$.

4. Neighborhood Numbers of Closed Helm Graph

Let $CH_m(V, E)$ be a Closed Helm graph obtained by taking a Helm graph H_m and by adding edges between the pendant vertices. Let $V(CH_m) = \{v_0, v_i, u_i : 1 \leq i \leq m\}$ be the vertex set and $E(CH_m) = \{v_0v_i, v_iu_i, : 1 \leq i \leq m\} \cup \{v_iv_{i+1}, u_iu_{i+1} : 1 \leq i \leq m-1\} \cup \{v_mv_1, u_mu_1\}$ be its edge set, with v_0 as its central vertex, v_i's as rim vertices and u_i's as pendant vertices (apex vertices) adjacent to rim vertices. Let v_0v_i are edges between central vertex and rim vertices, v_iv_{i+1}, v_mv_1 are edges on the rim,u_iu_{i+1}, u_mu_1 are edges between apex vertices and v_iu_i are edges between rim and pendant vertices.

Theorem 4.1 *For any integer $m \in Z^+$ and $m \geq 3$,*

$$nd(CH_m) = \begin{cases} m, & \text{if } m = 3 \text{ or } m \text{ is even} \\ m+1, & \text{if } m \geq 5 \text{ and } m \text{ is odd} \end{cases}$$

Proof: Let S be an nd-set of $G = CH_m$. Let $S = S_1 \cup S_2$, where S_1 is an nd-set of outer cycle C_m of apex vertices and S_2 be an nd-set of a graph $CH_m - \langle S_1\rangle$.

Case 1: When m is even.

For the case of even m, the set $S_1 = \{u_i : 1 \leq i \leq m-1$ and i is odd$\}$ is an nd-set of a outer cycle C_m of apex vertices (from Theorem 1.3) with $|S_1| = \frac{m}{2}$. And also, edges $u_iu_{i+1}, u_iu_{i-1}, u_iv_i, u_1u_m \in \langle S_1\rangle$, when i is odd. Let S_2 be an nd-set of $\langle V(CH_m) - S_1\rangle$, to cover edges u_iv_i when i is even, from Theorem 1.6 either $u_i \in S_2$ or $v_i \in S_2$ (since u_iv_i does not belong to a triangle). Since, $u_i \in \langle S_1\rangle$ for minimality condition let $S_2 = \{v_i$: $1 \leq i \leq m$ and i is even$\}$ then edges $\{v_iu_i, v_iv_{i+1}, v_iv_{i-1}, v_mv_1, v_0v_i\} \in \langle S_2\rangle$ when i is even, with $|S_2| = \frac{m}{2}$. Therefore, $CH_m = \langle S_1 \cup S_2\rangle$. Hence, $|S| = |S_1| + |S_2| = \frac{m}{2} + \frac{m}{2} = m$.

Case 2: When m is odd.

For the case m is odd, from Theorem 1.3 nd-set S_1 of outer cycle C_m should contain $\frac{m+1}{2}$ vertices with $S_1 = \{u_i : 1 \leq i \leq m$ and i is odd$\}$. And $S_2 = \{v_i : 1 \leq i \leq m$, i is even$\} \cup\{v_m\}$ is an nd-set of a graph $H = \bigcup_{v\in S_2}\langle N[v]\rangle$ with $|S_2| = \frac{m-1}{2} + 1 = \frac{m+1}{2}$. Hence, $|S| = |S_1| + |S_2| = \frac{m+1}{2} + \frac{m+1}{2} = m + 1$. When $m = 3$, the set $S = \{u_1, u_2, v_3\}$ is an nd-set with $|S| = 3$. Hence, $nd(CH_3) = 3$. □

Theorem 4.2 *For any integer $m \in Z^+$ and $m \geq 3$,*

$$nd^*(CH_m) = \begin{cases} m, & \text{if } m \text{ is even} \\ \text{does not exist}, & \text{if } m \text{ is odd} \end{cases}$$

Proof: For the case m even, S is an nd-set(from Theorem 4.1). And also $\overline{S}$ is an nd-set (since $CH_m = \bigcup_{v\in\overline{S}}\langle N[v]\rangle$). Hence, S is an nd^*-set. Therefore, $nd^*(CH_m) = m$, for even m. For the case m is odd, the set S_1 is an nd -set of outer cycle C_m with $|S_1| = \lceil\frac{m+1}{2}\rceil$ and $|\overline{S}_1| = \lceil\frac{m-1}{2}\rceil < \lceil\frac{m}{2}\rceil$. From Theorem 1.3 at least one pair of adjacent vertices $u_i, u_j \in S_1$ and edge $u_iu_j \notin \langle V(CH_m) - S_1\rangle$ results $\overline{S}_1$ not an nd-set. Hence, $\overline{S}$ is not an nd-set. Therefore, nd^*- set does not exist for the graph CH_m when m is odd. □

Theorem 4.3 *For any integer $m \in Z^+$ and $m \geq 3$, $\hat{nd}(CH_m) = m + 1$.*

Proof: For the case of odd m the result is true from Theorem 4.2 with $|S| = m + 1$. For the case of even m, from Theorem 4.1 $|S| = m$ for S to be an nd- set but then $\overline{S}$ is also an nd-set, a contradiction. Hence,$|S| > m$. Let us suppose S contains one more vertex v_j or $u_j \in \overline{S}$ adjacent to $u_i \in S$ then an edge u_iu_j or $u_iv_j \notin \bigcup_{v\in\overline{S}}\langle N[v]\rangle$. Hence $\overline{S}$ is not an nd-set and $|S| < m + 1$. Therefore, $\hat{nd}(CH_m) = m + 1$. □

Theorem 4.4 *For any integer $m \in Z^+$ and $m \geq 3$, $\overline{nd}(CH_m) = 2$.*

Proof: Since CH_m contain edges not in C_3 of CH_m the result follows from Lemma 2.3 that $|S| = 2$ and hence, $\overline{nd}(CH_m) = 2$. □

Example: For the graph $G = CH_8$ (when m is even)of Figure 3a, $nd(CH_8)$= 8 = $nd^*(CH_8)$ with an nd-set (also an nd^*-set) $S = \{ u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8 \}$; $\hat{nd}(CH_8) = 9$ with $\hat{nd}$-set S= $\{ u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8, u_8, \}$; and $\overline{nd}(CH_8) = 2$ with a $\overline{nd}$-set S =$\{ v_1, u_1\}$.

For the graph $G = CH_9$ (when m is odd)of Figure 3b, $nd(CH_9) = 10$ with an nd-set S= $\{ u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8, u_9, v_9\}$; nd^*-set does not exist for the graph CH_9; $\hat{nd}(CH_9) = 10$ with $\hat{nd}$-set $S = \{ u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_8, u_9, v_9\}$; and $\overline{nd}(CH_9) = 2$ with a $\overline{nd}$-set S =$\{ v_1, u_1\}$.

(a) The Graph CH_8 (b) The Graph CH_9

5. Neighborhood Numbers of Gear Graph

Let $G_m(V,E)$ be a Gear graph obtained by taking a wheel graph $W_{1,m}$ by adding a vertex in between every pair of rim vertices. Let $V(G_m) = \{v_0, v_i, u_i : 1 \leq i \leq m\}$ be the vertex set with $|V(G_m)| = 2m+1$ and $E(G_m)$=$\{v_0v_i,\ v_iu_i,\ u_mv_1$:$1 \leq i \leq m\} \cup \{u_iv_{i+1}$:$1 \leq i \leq m-1$ $\}$ be its edge set with $|E(G_m)| = 3m$, where v_0 is central vertex, v_i is a rim vertex and u_i is a vertex between every pair of rim vertices. Let v_0v_i are edges between central vertex and rim vertices, v_iu_i, u_iv_{i+1}, u_mv_1 are edges on the rim.

Theorem 5.1 *For any integer $m \in Z^+$ and $m \geq 3$,*

(i) $nd^*(G_m) = nd(G_m) = m$.

(ii) $\hat{nd}(G_m) = nd(G_m) + 1$.

(iii) $\overline{nd}(G_m) = 2$.

Proof: For (i): Let S_1 be an nd-set of Gear graph G_m. Then $\overline{S}_1$ is independent(since G_m is triangle free graph and by Remark 1.2). But then, $|\overline{S}_1| \geq m$ (since $|\overline{S}_1 \cap \{u_i, v_0 : 1 \leq i \leq m\}| \geq m$). Hence, $nd(G_m) \leq |S_1| = |V(G_m)| - |\overline{S}_1| \leq 2m+1-(m+1) = m$. On the other hand, let $S_1 = \{v_i : 1 \leq i \leq m\}$. Then $\overline{S}_1 = \{v_0, u_i : 1 \leq i \leq m\}$. Also, S_1 and $\overline{S}_1$ are independent and hence S_1 as well as $\overline{S}_1$ are nd-sets. Therefore, S_1 is an nd^*-set. So, $nd(G_m) = nd^*(G_m) = m$.

For (ii): Let S_2 be an $\hat{nd}$-set. Then S_2 is an nd-set and $\overline{S}_2$ is not an nd-set. From the proof of (i), an nd-set S_2 should contain m vertices but then $\overline{S}_2$ is also an nd-set. Let us suppose if S_2 contain one more vertex u_i adjacent to v_i from the set $\overline{S}_2$ then an edge $v_iu_i \notin \bigcup_{v\in \overline{S}_2} \langle N[v] \rangle$ (since G_m is triangular free graph), hence $\overline{S}_2$ is not an nd-set. Therefore, $|S_2| = nd(G_m) + 1$.

For (iii): Let S_3 be an $\overline{nd}$-set. Since G_m is triangle free graph the result follows from Lemma 2.3 that $|S_3| = 2$. Therefore, $\overline{nd}(G_m) = 2$. □

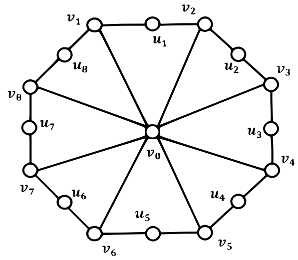

Figure 4: Gear graph G_8

Example: For the graph $G = G_8$ of Figure 4, $nd(G_8) = 8 = nd^*(G_8)$ with an nd-set (also an nd^*-set) $S_1 = \{v_1, v_2, v_3, v_4,\ v_5,\ v_6,\ v_7,\ v_8\}$; $\hat{nd}(G_8) = 9$ with $\hat{nd}$-set S_2= $\{\ v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6,\ v_7,\ v_8,\ u_1,\}$; and $\overline{nd}(G_8) = 2$ with an $\overline{nd}$-set S_3 =$\{\ v_1,\ u_1\}$.

6. Neighborhood Numbers of Flower Graph

Let $F_m(V, E)$ be a Flower graph obtained from a Helm graph by joining each pendent vertex to the central vertex v_0. Let $V(F_m)$=$\{v_0,\ v_i,\ u_i$:$1 \leq i \leq m\}$ be the vertex set with $|V(F_m)| = 2m + 1$ and $E(F_m)$=$\{v_0v_i,\ v_iu_i,\ v_0u_i$:$1 \leq i \leq m\} \cup \{v_mv_1\} \cup \{v_iv_{i+1}$:$1 \leq i \leq m-1\}$ be its edge set with $|E(G_m)| = 4m$, where v_i is a rim vertex and u_i is an apex vertex. Let v_0v_i are edges between central vertex and rim vertices, v_iu_i are edges between rim and apex vertices v_iv_{i+1}, v_mv_1 are edges on the rim and v_0u_i are edges between central and apex vertices.

Theorem 6.1 *For any integer $m \in Z^+$ and $m \geq 3$,*

- **(i)** $nd^*(F_m) = nd(F_m) = 1$.
- **(ii)** $\hat{nd}(F_m) = 3$.
- **(iii)** $\overline{nd}(F_m)$ *doesn't exist .*

Proof: For (i): Let S_1 be an nd-set of a graph F_m. Since $deg(v_0) = |V(F_m) - 1|$ and $N[v_0] = V(F_m)$, the result is true from Remark 1.5 that $|S_1| = 1$. Hence, $nd(F_m) = 1$. On the other side, if $|S_1| = 1$ then $|\overline{S}_1| = |V(F_m) - 1|$ and from Remark 1.4 $\overline{S}_1$ is also an nd-set. Hence, $nd^*(F_m) = |S_1| = 1$.

For (ii): Let S_2 be an $\hat{nd}$-set of a graph F_m.Then S_2 is an nd-set and $\overline{S}_2$ is not an nd-set. From the proof of (i), an nd-set S_2 should contain a central vertex v_0 but then $\overline{S}_2$ is also an nd-set. Therefore, $|S_2| > 1$. Let us suppose if $|S_2| = 2$ with $S_2 = \{v_0, v_i\}$ or $S_2 = \{v_0, u_i\}$ for any i and $1 \leq i \leq m$ be an nd-set then $F_m = \bigcup_{v \in \overline{S}_2} \langle N[v] \rangle$ results $\overline{S}_2$ an nd-set(since all the vertices lies in a triangle where v_0 is one of the vertex with $deg(v_0) = |V(F_m) - 1|$ and from Remark 1.1). Therefore, $|S_2| \geq 3$. Let $S_2 = \{v_0, v_i, u_i$: $i \in \{1, 2, ..., m\}\}$, for one $i\}$ then $F_m = \bigcup_{v \in S_2} \langle N[v] \rangle$ and $F_m \neq \bigcup_{v \in \overline{S}_2} \langle N[v] \rangle$ as edge $v_iu_j \notin \langle \overline{S}_2 \rangle$ for $i = j$, $1 \leq i, j \leq m$ resulting S_2 an nd-set and $\overline{S}_2$ not an nd-set. Hence, $|S_2| \leq 3$. Therefore, $\hat{nd}(F_m) = 3$.

For (iii): Let S_3 be an $\overline{nd}$-set. Since F_m has a central vertex v_0 with $deg(v_0) = |V(F_m) - 1|$ from Remark 1.5 either S_3 or $\overline{S}_3$ is an nd-set. Hence, $\overline{nd}$-set does not exist for the graph F_m. □

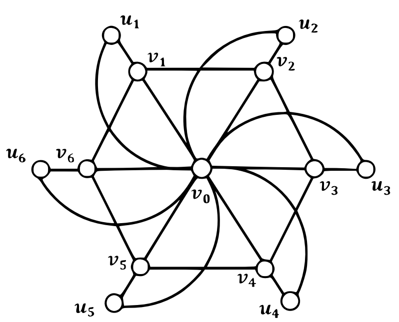

Figure 5: Flower Graph F_6

Example: For the graph $G = F_6$ of Figure 5, $nd(F_6) = 1 = nd^*(F_6)$ with an nd-set (also an nd^*-set) $S_1 = \{v_0\}$; $\hat{nd}(F_6) = 3$ with $\hat{nd}$-set S_2= $\{\ v_0,\ v_1,\ u_1\}$; and $\overline{nd}(F_6) = 0$ as $\overline{nd}$-set doesnot exist.

7. Neighborhood Numbers of Closed Flower Graph

Let $CF_m(V, E)$ be a closed Flower graph obtained from a Flower graph by adding edges between apex vertices. Let $V(CF_m)$=$\{v_0,\ v_i,\ u_i$:$1 \leq i \leq m\}$ be the vertex set with $|V(CF_m)| = 2m + 1$ and $E(CF_m)$=$\{v_0v_i,\ v_iu_i,\ v_0u_i$:$1 \leq i \leq m\} \cup \{v_mv_1,\ u_mu_1\} \cup \{v_iv_{i+1}$,$u_iu_{i+1}$:$1 \leq i \leq m-1\}$ be its edge set with $|E(CF_m)| = 5m$, where v_i is a rim vertex and u_i is an apex vertex. Let v_0v_i are edges between central vertex and rim vertices, v_iu_i are edges between rim and apex vertices v_iv_{i+1}, v_mv_1 are edges on the rim, u_iu_{i+1}, u_mu_1 are edges between the apex vertices and v_0u_i are edges between central and apex vertices.

Theorem 7.1 *For any integer $m \in Z^+$ and $m \geq 3$,*

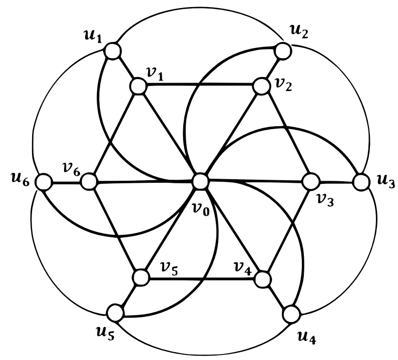

Figure 6: Closed Flower Graph CF_6

(i) $nd^*(CF_m) = nd(CF_m) = 1.$

(ii) $\hat{nd}(CF_m) = 3.$

(iii) $\overline{nd}(CF_m)$ *doesn't exist .*

Proof: Since CF_m contain a vertex v_0 of degree $V(CF_m) - 1$, new edges added between apex vertices belongs to a triangle in which v_0 is one of the vertex. So, the result follows from Theorem 6.1. □

8. Neighborhood Numbers of Double Wheel Graph

Let $DW_m(V, E)$ be a Double Wheel graph consisting 2 cycles C_m where the vertices of both the cycles are connected to a common vertex called central vertex. Let $V(DW_m)$=$\{v_0,\ v_i,\ u_i$:$1 \leq i \leq m\}$ be the vertex set with $|V(DW_m)| = 2m + 1$ and $E(DW_m)$=$\{v_0v_i,\ v_0u_i$:$1 \leq i \leq m\} \cup \{v_mv_1,\ u_mu_1\} \cup \{v_iv_{i+1}$,$u_iu_{i+1}$:$1 \leq i \leq m-1\}$ be its edge set with $|E(DW_m)| = 4m$, where v_i and u_i are rim vertices of cycle 1 and cycle 2 respectively. Let v_0v_i are edges between central vertex and vertices of cycle 1 and v_0u_i are edges between central vertex and vertices of cycle 2, v_iv_{i+1}, v_mv_1 are edges on the cycle 1, u_iu_{i+1}, u_mu_1 are edges on the cycle 2.

Theorem 8.1 *For any integer* $m \in Z^+$ *and* $m \geq 3$,

(i) $nd^*(DW_m) = nd(DW_m) = 1.$

(ii) $\hat{nd}(DW_m) = \begin{cases} 4, & \text{if } m = 3 \\ 3, & \text{if } m \geq 4 \end{cases}$

(iii) $\overline{nd}(DW_m)$ *doesn't exist .*

Proof: For (i): Let S_1 be an nd-set of a graph DW_m. Since $deg(v_0) = |V(DW_m) - 1|$ and $N[v_0] = DW_m$ (as v_0 is adjacent to the vertices of both the cycle), the result is true from Remark 1.5 that $|S_1| = 1$. Hence, $nd(DW_m) = 1$. On the other side, if $|S_1| = 1$ then $|\overline{S}_1| = |V(DW_m) - 1|$ and from Remark 1.4 $\overline{S}_1$ is also an nd-set. Hence, $nd^*(DW_m) = |S_1| = 1$.

For (ii): Let S_2 be an $\hat{nd}$-set of a graph DW_m when $m \geq 4$. Then S_2 is an nd-set and $\overline{S}_2$ is not an nd-set. From the proof of (i), an nd-set S_2 should contain a central vertex v_0 but then $\overline{S}_2$ is also an nd-set. Therefore, $|S_2| \geq 2$. Let us suppose if $|S_2| = 2$ then

Case 1: S_2=$\{v_i,\ v_j\}$ $(0 \neq i < j)$

Subcase(1.1): v_i is adjacent to v_j.

In this case, S_2 is not an nd-set (Since $m \geq 4$ and the edge $v_{j+1}v_{j+2} \notin \bigcup_{v \in S_2} \langle N[v] \rangle$), a contradiction.

Subcase(1.2): v_i is not adjacent to v_j.

When $m \geq 5$, the set S_2 is not an nd-set as there exists atleast one edge e such that none of the end vertices of e lies in a triangle of DW_m where one of the vertex of triangle belongs to S_2. When $m = 4$, both S_2 and $\overline{S}_2$ are nd-sets, a contradiction.

Case 2: S_2={v_0, v_i}

In this case, a vertex $v_j \in \overline{S}_2$ which is adjacent to v_i dominates v_i as well as v_0 and also an edge v_0v_i lie in a triangle one of whose vertex $v_j \in \overline{S}_2$. Hence, by Theorem 1.1, $\overline{S}_2$ is an nd-set, a contradiction.

Case 3: S_2={v_i, u_j} (where, v_i is vertex of Cycle 1 and u_j is a vertex of Cycle 2)

In this case, S_2 is not an nd-set as $DW_m \neq \bigcup_{v\in S_2}\langle N[v]\rangle$ and also, both $v_i, u_j \in S_2$ lie in a triangle in which one of the vertex belong to $\overline{S}_2$ implies $\overline{S}_2$ is an nd-set, a contradiction.

From all the above cases we see that $|S_2| \geq 3$. On the other hand, a set $S_2 = \{v_0, v_i, v_{i+1} : 1 \leq i \leq m-1\}$ or $\{v_0, u_i, u_{i+1} : 1 \leq i \leq m-1\}$, where v_0 is a central vertex is an $\hat{nd}$-set as $S_2 = \{v_0\}$ is an nd-set from Result(i) and $\overline{S}_2$ is not an nd-set (since $u_iu_{i+1} \notin \bigcup_{u\in\overline{S}_2}\langle N[u]\rangle$ or $v_iv_{i+1} \notin \bigcup_{v\in\overline{S}_2}\langle N[v]\rangle$). Hence, $|S_2| \leq 3$. Therefore, $\hat{nd}(DW_m) = 3$.

When $m = 3$, a set $S_2 = \{v_0, u_1, u_2, u_3\}$ or $S_2 = \{v_0, v_1, v_2, v_3\}$ is an $\hat{nd}$-set. Hence, $\hat{nd}(DW_3) = 4$.

For (iii): Let S_3 be an $\overline{nd}$-set. Since DW_m has a central vertex v_0 with $deg(v_0) = |V(DW_m) - 1|$ from Remark 1.5 either S_3 or $\overline{S}_3$ is an nd-set. Hence, $\overline{nd}$-set does not exist for the graph DW_m. □

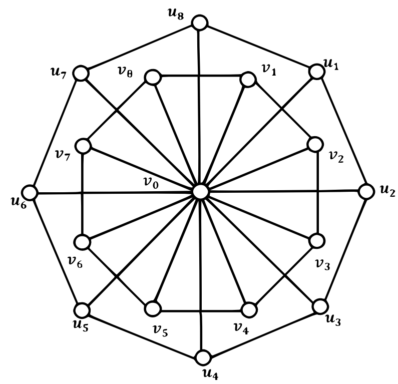

Figure 7: Double Wheel Graph DW_8

Example: For the graph $G = DW_8$ of Figure 7, $nd(DW_8) = 1 = nd^*(DW_8)$ with an nd-set (also an nd^*-set) $S_1 = \{v_0\}$; $\hat{nd}(DW_8) = 3$ with $\hat{nd}$-set S_2= { v_0, v_1, v_2}; and $\overline{nd}(DW_8) = 0$ as $\overline{nd}$-set does not exist.

9. Neighborhood Numbers of Sun Flower Graph

Let $S_m(V, E)$ be a sunflower graph obtained by W_m by taking a vertex u_i for each rim edge of W_m and joining both the end vertices to a vertex u_i. Let $V(S_m)$={v_0, v_i, u_i:$1 \leq i \leq m$} be the vertex set with $|V(S_m)| = 2m + 1$ and $E(S_m)$={v_0v_i, v_iu_i:$1 \leq i \leq m$} ∪ {v_mv_1, u_mv_1} ∪ {v_iv_{i+1},u_iv_{i+1}:$1 \leq i \leq m-1$} be its edge set with $|E(S_m)| = 4m$, where v_i's are rim vertices of a wheel and u_i's are vertices adjacent to rim vertices. Let v_0v_i are edges between central vertex and rim vertices and v_iu_i, $v_{i+1}u_i$ u_mv_1 are edges between rim vertex and vertices adjacent to rim vertices. v_iv_{i+1}, v_mv_1 are edges on the rim.

Theorem 9.1 *For any integer $m \in Z^+$ and $m \geq 3$,*

- **(i)** $nd^*(S_m) = nd(S_m) = \lfloor \frac{m+1}{2} \rfloor$.
- **(ii)** $\hat{nd}(S_m) = \lfloor \frac{m+4}{2} \rfloor$.
- **(iii)** $\overline{nd}(S_m) = 3$.

Proof: For (i): Let S_1 be an nd-set of a graph S_m. Every vertex of a graph lie in atleast one triangle so from Theorem 1.1 one vertex in S_1 is sufficient to cover all the vertices of a triangle. If $u_i, 1 \leq i \leq m$ belongs to S_1 then it covers vertices v_i and edges v_iv_{i+1}, $v_{i+1}u_i$ when $1 \leq i \leq m-1$ and v_mv_1,v_iu_i, u_mv_1 when $1 \leq i \leq m$. But vertex $v_0 \notin \langle S_1\rangle$ and edges $v_0v_i \notin \langle S_1\rangle$ $1 \leq i \leq m$, hence v_0 should be in S_1 resulting $|S_1| = m + 1 > \lfloor \frac{m+1}{2} \rfloor$. For the optimal condition, let S_1={v_i: $1 \leq i \leq m$, i is odd} as each v_i covers four triangles of a graph and there are $2m$ triangles in a graph, $|S_1| = \frac{2m}{4} = \frac{m}{2}$, when m is even. Also, when m is odd, from Theorem 1.3 an nd-set S_1 should contain $\frac{m+1}{2}$ vertices (since vertices of nd-set lies on outer cycle of wheel). Hence, $|S_1| = \frac{2m}{4} + 1 = \lfloor \frac{m+1}{2} \rfloor$. On the other hand, if S_1 $=\{v_i : 1 \leq i \leq m\}$ then the set $\overline{S}_1 = \{u_i,\ v_0 : 1 \leq i \leq m\}$ is an nd-set as $S_m = \bigcup_{v \in \overline{S}_1} N[v]$. Therefore, $nd(S_m) = nd^*(S_m) = \lfloor \frac{m+1}{2} \rfloor$.

For (ii): Let S_2 be an $\hat{nd}$-set of a graph S_m. Then S_2 is an nd-set and $\overline{S}_2$ is not an nd-set. From the proof of (i), an nd-set S_2 should contain a $\lfloor \frac{m+1}{2} \rfloor$ but then $\overline{S}_2$ is also an nd-set (as every vertex of S_1 lie in a triangle in which one of the vertex belong to $\overline{S}_1$), a contradiction. Therefore, $|S_2| > \lfloor \frac{m+1}{2} \rfloor$. Let us suppose if S_2 contain one more vertex v_{i+1} or u_i adjacent to $v_i \in S_2$ then $\overline{S}_2$ in an nd-set, a contradiction. Hence, a set $S_2 = \{v_i : 1 \leq i \leq m,\ i \text{ is odd}\} \cup \{v_{i+1},\ u_i :\ i \in \{1, 2, ..., m\}\}$ is an $\hat{nd}$-set. Therefore, $|S_2| = \frac{m}{2} + 2 = \frac{m+4}{2}$. When m is odd, from the result of (i) S_2 contain one pair of adjacent vertices and hence one more vertex $u_i \in S_2$ is sufficient to be an $\hat{nd}$-set. So, $|S_2| = \frac{m+1}{2} + 1 = \lfloor \frac{m+4}{2} \rfloor$

For (iii): Let S_3 be an $\overline{nd}$-set. Since all the vertices of S_m lie in a triangle of S_m a set $S_3 = \{v_i,\ v_{i+1},\ u_i :\ i \in \{1, 2, ..., m\}\}$ is an $\overline{nd}$-set of S_m (as edges $v_i u_i,\ v_{i+1} u_i \notin \langle \overline{S}_3 \rangle$). Hence, $|S_3| = 3$.

□

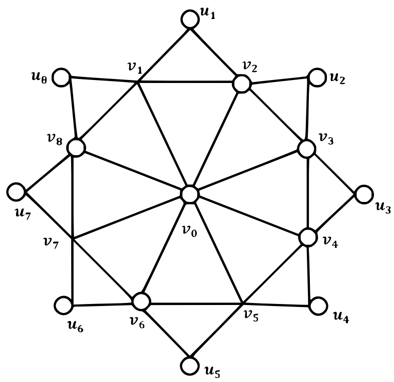

Figure 8: Sun Flower Graph S_8

Example: For the graph $G = S_8$ of Figure 8, $nd(S_8) = 4 = nd^*(S_8)$ with an nd-set (also an nd^*-set) $S_1 = \{\ v_1,\ v_3,\ v_5,\ v_7\}$; $\hat{nd}(S_8) = 6$ with $\hat{nd}$-set $S_2 = \{\ v_1,\ u_1,\ v_2,\ v_3,\ v_5,\ v_7\}$; and $\overline{nd}(S_8) = 3$ with $\overline{nd}$-set $S_3 = \{\ v_1,\ u_1,\ v_2\}$.

10. Conclusion

In this paper, various nd-sets of H_m, CH_m, G_m, F_m, CF_m, DW_m and S_m with their minimum cardinalities are discovered. It is shown that majority of the graphs considered here have $nd(G) = nd^*(G)$ and also if G contains an edge that is not a part of a triangle when $|V(G)| > 4$, then $\overline{nd}(G) = 2$. In network theory, new dimensions introduced here could be used as new parameters. The idea of a neighborhood can be used to solve network security issues since each neighborhood set is a dominating set.

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