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# Exploring Rainbow Coloring and Connectivity Analysis in Hybrid Graph Structures

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ABSTRACT: The study of connectivity in complex networks is a vital area within graph theory and network science. One notable concept that enhances secure and reliable communication is rainbow coloring, here, the edges are colored to ensure that between any two vertices, one can find a path where each edge has a different color. To achieve this condition, the minimum quantity of colors needed is defined as the rainbow connection number.This measure reflects the strength and fault tolerance of network structures and has practical relevance in areas such as secure data transmission and efficient routing protocols. The research focuses on understanding the role of structural characteristics of graphs on their rainbow connectivity. We provide new insights and results regarding hybrid graph structures like Tribun graph $T_n$, Chain graph is a point shackle $K_4P_n$, Diamond ladder graph $Dl_n$. The results contribute to a deeper understanding of rainbow connection in graph models and contribute to the development of resilient and well-structured communication networks.

Key Words: Diameter, rainbow connection number, edge coloring.

## Contents



## 1. Introduction

In graph theory the first and most well-known problems is graph coloring. In 1852, De Morgan wrote a letter to his friend Hamilton, mentioning that one of his students had noticed an interesting fact: only four colors were needed to color the counties on a map of England, such that no neighboring counties do not shared the same color. This observation marked the beginning of the concept known as graph coloring [1].

Graph coloring refers to the process of assigning colors to certain elements of a graph. Several forms of graph coloring exist, such as total coloring, vertex coloring, b-coloring, interval coloring, edge coloring, and rainbow coloring. In rainbow coloring, colors are allocated to the edges do not share the same color to the adjacent two edges — each edge in a path must be uniquely colored to maintain the rainbow property.

Rainbow connectivity, introduced by Chartrand and co-authors in 2008, offers an interesting extension of classical graph connectivity. In this framework, the goal is to assign color the graph edges of a connected graph so that joined the two vertices is a path in which all edges carry distinct colors [2].For graph $G$ is a nontrivial connected, an edge coloring can be represented as a function $c : E(G) \to \{1, 2, \ldots, k\}$, where k takes the values natural number indicating the total colors employed. Unlike proper edge coloring, this approach allows adjacent edges to share the same color. Every edge in every path has a distinct color is called as a rainbow path. If each pair of distinct $u$ and $v$ vertices in the graph is connected by exists at least one rainbow path, is termed rainbow connected. The least number of colors required to ensure the property of rainbow connection number, denoted by $rc(G)$ [8]. Li and Sun, Provided an extensive survey covering various forms, algorithms, and open problems in rainbow connectivity, including edge vs. vertex

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versions and strong rainbow connectivity [3].

L. Sunil Chandran et.al investigate the relationship of rainbow connection number relates to both vertex and edge connectivity like Strengthened bounds and Verified bounds are tight for certain graph families and proved conjectured limits for chordal and girth 7 graphs [4]. Annika et.al studies the rainbow connection number $rc(G)$ in Erdos–Renyi random graphs. The authors focus on the threshold behaviour of the property $rc(G) \leq r$, where $r =$ fixed integer. They show that for $r = 2$, the graph achieving diameter is equal to 2 and rainbow connection 2 occurs nearly simultaneously. However, for $r$ is greater then equal to 3, this does not hold. They introduce a new threshold and demonstrate that it serves as an upper bound for the rainbow connection number [5].

Srinivasa Rao and Murali, in [6] and [7], studied the properties of the rainbow connection number $rc(G)$ as well as $src(G)$ the strong rainbow connection number for various for various types of graphs, including the stacked book graph, grid graph, and prism graph. Their study additionally examined how these graphs behave under rainbow coloring and analysed their critical structural properties related to rainbow connectivity. Magfiroh et al. [9] investigated the concept of b-coloring, a refined form of proper vertex coloring, each color group includes at least one vertex that is adjacent to vertices in all other color classes. The study focused on determining the exact b-chromatic number for several special graphs, including the king's tour graph $K_{n,m}$, tribun graph $T_n$, diamond ladder graph $Dl_n$, three-cycle ladder graph $T_n$, and chain graph $K_4P_n$. The authors established the values of the b-chromatic number for these graphs using pattern recognition and axiomatic deductive methods, contributing novel results to the ongoing exploration of graph coloring parameters.

In recent years, several new graph-coloring measures have been studied. Chandran et al. [10] introduced the concept of the very strong rainbow connection number, denoted as $vsrc(G)$ and derived tight bounds and complexity results for various graph families. Bai et al. [11] developed an efficient method for determining the rainbow disconnection number in 2-trees, confirming prior conjectures. Bushaw, Johnston and Rombach [12] formalized the rainbow saturation number and analyzed its behavior across different graph classes, contrasting it with known saturation parameters.

Modern communication systems often rely on robust and fault-tolerant architectures. Graph models such as the Tribun graph, chain-shackle $K_4P_n$ graph, and diamond ladder graph provide structural analogs for:
- Multi-tier service architectures (Tribun graphs)
- Linear modular networks or block-chain consensus structures (Chain-Shackle $K_4P_n$)
- Layered redundant topologies like in sensor grids and neural interconnects (Diamond ladder graphs)

Studying the rainbow connectivity of these structures ensures optimal path diversity for resilience, helps in estimating security margins in routing protocols, and assists in developing efficient coloring algorithms for online and offline applications.

In this research manuscript, we gain the rainbow connection number of the hybrid graph structures like Tribun graph, Diamond ladder graph, and Chain graph, which presents a challenging problem. We focus on designing edge coloring that guarantee rainbow connectivity between all pairs of vertices. The results illustrate how these structures influence the least number of colors needed. This work enhances the current understanding of rainbow coloring in complex networks and may serve as a foundation for future research in this area.

## 2. Definitions

**Definition 2.1.** For a connected graph $G$, the rainbow connection number $rc(G)$ is bounded both below and above by fundamental graph parameters. Specifically, it satisfies the inequality:

$$diam(G) \leq rc(G) \leq E(G)$$

Here, $diam(G)$ represents the diameter of the graph, while $E(G)$ denotes the number of edges in $G$

**Definition 2.2. Tribun graph Tn**
Tribun graph $Tn$ is a connected graph, consisting of three groups of vertices: $x_j$ for $1 \leq j \leq n$, $y_j$ for $1 \leq j \leq n+1$ and $z_j$ for $1 \leq j \leq 2n+1$. The edges in this graph are given by: each $z_j$ is connected to $z_(j+1)$ forming a continuous path, each vertex $x_j$ is linked to its corresponding vertex $z_j$ and each vertex $y_j$ is also connected to its corresponding vertex $z_j$. This structure results in a layered graph with between the $x_j$, $y_j$ and $z_j$ vertex graph.

**Definition 2.3. Chain graph is a point shackle** $K_4P_n$
A chain graph also knows as a point shackle is a graph formed by connecting multiple duplicates of the complete graph $K_4$ in a linear sequence. It is denoted by shack $K_4P_n$. when $n$ counts the $K_4$ graphs are used to connect them. Each $K_4$ in the sequence shares a vertex with the adjacent $K_4$ blocks through shared vertices.

**Definition 2.4. Diamond ladder graph** $Dl_n$
A diamond ladder graph $Dl_n$ is a structured graph formed by combing a traditional ladder graph with additional diagonal and cross connections. It consists of three set of vertices: $x_j$ and $z_j$ for $1 \leq j \leq n$ representing the two vertical sides of the ladder and $y_j$ for $1 \leq j \leq 2n$ serving as internal connections. The edges include connections between consecutive $x_j$ and $z_j$ vertices to form the ladder sides, edges between each $x_j$ and its corresponding $z_j$ and edges from each $x_j$ and $z_j$ to two corresponding $y_j$ vertices. Additionally certain $y_j$ vertices are connected in sequence, giving the graph its distinctive diamond shaped internal structure.

## 3. Figures

Figure 1, 2 & 3 represents Tribun graph, Chain graph is a point of shackle and Diamond ladder graph.

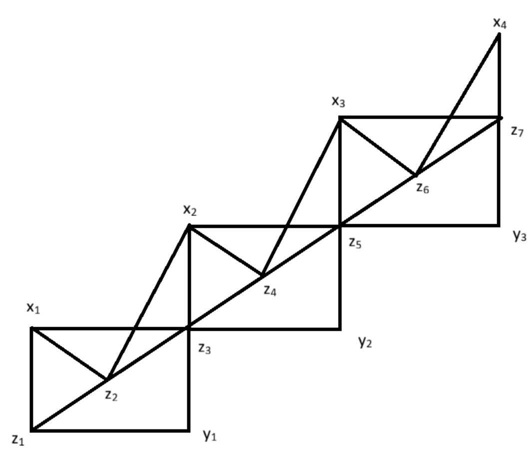


Figure 1: Tribun graph $T_3$

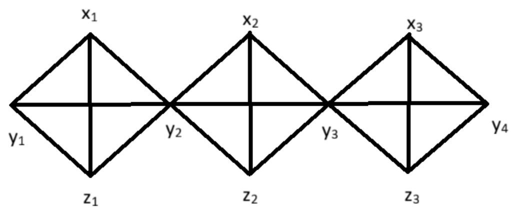


Figure 2: Chain graph is a point of shackle $K_4P_3$

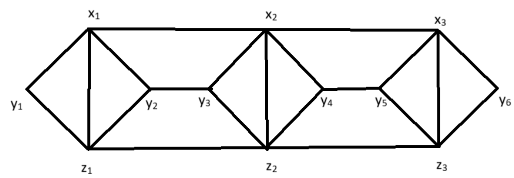


Figure 3: Diamond ladder graph $Dl_3$

## 4. Some Results of Standard Graphs

**Theorem 4.1.** When graph $G$ can be defined as Tribun graph $T_n$, then

$$rc(G) = \begin{cases} \frac{3n}{2} + 1 & if \text{ n } is\ even \\ \frac{3n+1}{2} & if \text{ n } is\ odd \end{cases}$$

**Proof:** The Tribun graph Tn, defined for $n \geq 2$, represents a connected graph whose vertices are specified by $V(T_n) = \{x_j : 1 \leq j \leq n+1\} \cup \{y_j : 1 \leq j \leq n\} \cup \{z_j : 1 \leq j \leq 2n+1\}$ and its edges set is described by $E(Tn) = \{z_j z_{j+1} : 1 \leq j \leq 2n\} \cup \{x_j z_j : 1 \leq j \leq 4n+1\} \cup \{y_j z_j : 1 \leq j \leq 2n\}$

The graph $G$ assumed to be connected. Which means that, for every pair of distinct vertices $u, v \in V(G)$, there exists a always one path connecting $u$ and $v$. This property guarantees that the rainbow connection number is both finite and properly defined.

To demonstrate the result, we consider it in two separate cases.

**Case (i):** Consider the situation when $n$ is an even number.

**Step (i):** We determine the diameter of a graph $G$, denoted $diam(G)$, is the maximum distance between any pair of vertices, and it provide a lower bound for the rainbow connection number $rc(G) \geq diam(G)$.

In this graph, we observe that

$$diam(G) = \frac{3n}{2} + 1$$

Hence, it follows that

$$rc(G) \geq \frac{3n}{2} + 1$$

**Step (ii):** In order to determine an upper bound, we aim to construct an edge coloring for $G$ using $\frac{3n}{2}+1$ colors, such that the graph remains rainbow connected.

We define a coloring function

$$c : E(G) \rightarrow \{1, 2, 3, \ldots, \frac{3n}{2} + 1\}$$

Colors are allocated to edges in such a way that in graph every pair of vertices is connected by a path where two edges are don't assigned the same color.

$$x_i z_i = i \; for \; i = 1$$

$$y_i z_i = i + 1 \; for \; i = 1$$

$$z_i z_{i+1} = i \; for \; i = 1 \; and \; 2$$

$$x_i z_{i+1} = i \; for \; i = 1$$

$$x_i z_i = i \; for \; i = 2$$

$$z_i z_{i+1} = i \; for \; 3 \leq i \leq n$$

$$z_i z_{i+1} = \frac{i+4}{2} \; for \; 4 \leq i \leq n$$

$$x_i z_{i+1} = \frac{3i-2}{2}, \; for \; 3 \leq i \leq n$$

$$x_i z_{2i-1} = \frac{3i}{2} - 1 \; for \; 2 \leq i \leq n$$

$$y_i z_{2i+1} = \frac{3i}{2} - 1 \; for \; 1 \leq i \leq n$$

$$y_i z_{2i-1} = x_i z_{2i+1} = \frac{3i}{2} + 1 \; for \; 2 \leq i \leq n$$

$$x_i z_{2i} = \frac{3i}{2} \; for \; 2 \leq i \leq n$$

Thus we conclude that

$$rc(G) \leq \frac{3n}{2} + 1$$

From step (i) and step (ii), we have

$$\frac{3n}{2} + 1 \leq rc(G) \leq \frac{3n}{2} + 1$$

Thus, the Tribun graph has a rainbow connection number equal to $\frac{3n}{2} + 1$.

**Case (ii):** Consider when $n$ is an odd number.

**Step (i):** We define the diameter of a graph $G$, denoted by $diam(G)$, as the maximum distance between any two vertices in $G$. In other words, it is the largest value of the shortest-path distance taken over all pairs of vertices $u, v \in V(G)$. This quantity is significant because it establishes a fundamental lower bound for the rainbow connection number of the graph; that is, no edge-coloring can achieve a rainbow connection number $rc(G) \geq diam(G)$.

In this graph, we find that

$$diam(G) = \frac{3n+1}{2}$$

Hence, it abbreviate that

$$rc(G) \geq \frac{3n+1}{2}$$

**Step (ii):** To establish an upper bound, we attempt to construct an edge coloring of $G$ using $\frac{3n+1}{2}$ colors, so the graph remains rainbow connected.

Next, we define a coloring function

$$c : E(G) \to \{1, 2, 3, \ldots \frac{3n+1}{2}\}$$

We will construct an edge-coloring of the graph $G$ such that for any pair of vertices $u, v \in V(G)$, there exists at least one path connecting $u$ and $v$ in which all edges are assigned distinct colors; in other words, no two edges along this path receive the same color.

$$x_i z_i = i \ for \ i = 1$$
$$y_i z_i = i + 1 \ for \ i = 1$$
$$z_i z_{i+1} = i \ for \ i = 1 \ and\,, 2$$
$$x_i z_{i+1} = i \ for \ i = 1$$
$$x_i z_i = i \ for \ i = 2$$
$$z_i z_{i+1} = i \ for \ 3 \leq i \leq n$$
$$z_i z_{i+1} = \frac{i+4}{2} \ for \ 4 \leq i \leq n$$
$$x_i z_{i+1} = \frac{3i-1}{2} \ for \ 3 \leq i \leq n$$
$$x_i z_{2i-1} = \frac{3i-1}{2} \ for \ 2 \leq i \leq n$$
$$y_i z_{2i+1} = \frac{3i-1}{2} \ for \ 1 \leq i \leq n$$
$$y_i z_{2i-1} = x_i z_{2i+1} = \frac{3i+1}{2} \ for \ 2 \leq i \leq n$$
$$x_i z_{2i} = \frac{3i-3}{2} \ for \ 2 \leq i \leq n$$

Thus we conclude that

$$rc(G) \leq \frac{3n+1}{2}$$

From step (i) and step (ii), we have

$$\frac{3n+1}{2} \leq rc(G) \leq \frac{3n+1}{2}$$

Thus, Tribun graph has a rainbow connection number equal to $\frac{3n+1}{2}$.

□

**Theorem 4.2.** When graph $G$ is defined as Chain graph is a point of shackle $K_4P_n$ then $rc(G) = n$.

**Proof:** A chain graph $K_4P_n$ is a connected graph whose vertex set is defined as $V(K_4P_n) = \{x_j : 1 \leq j \leq n\} \cup \{y_j : 1 \leq j \leq n+1\} \cup \{z_j : 1 \leq j \leq n\}$ and its edge defined as $E(K_4P_n) = x_jy_{j+1} : 1 \leq j \leq n\} \cup \{z_jy_j : 1 \leq j \leq n\} \cup \{y_jy_{j+1} : 1 \leq j \leq n\} \cup \{x_iz_j : 1 \leq j \leq n$.

This graph forms a chain like connection of sub graphs where each unit resembles a $K_4$ complete graph connected through common vertices in sequences.

Upon examination, it is clear that the graph $G$ possesses the property of connectivity. It means, given any pair of distinct vertices within the graph, one can always possible to identify at least one path that links them together. This characteristic guarantees that no vertex is isolated and that every vertex can be reached from any other vertex. Such a feature is essential, as it provides the foundation for implementing a rainbow coloring on the graph.

**Step (i):** To determine the diameter of the graph, which is the largest distance between the any two vertices measured along the shortest possible paths. The diameter is fundamental in determining the least number of colors necessary to achieve a proper rainbow coloring. Thus, it provide a natural lower limit for $rc(G)$.

Since the diameter of $G$ be is $n$, denoted by

$$diam(G) = n$$

From this we infer that

$$rc(G) \geq n$$

**Step (ii):** In order to identify the sufficient number of colors, we proceed by constructing an edge coloring that utilizes n different colors. The objective is to the edges to apply colors so that, for all the pair of vertices, there exists possibly at least one path in so all edges receive different colors, thereby forming a rainbow path.

Coloring function is defined as

$$c : E(G) \to \{1, 2, 3, \ldots, n\}$$

$$x_jz_j = y_jy_{j+1} = y_jz_j = z_jz_{j+1} = x_jy_{j+1} = j \; for \; 1 \leq j \leq n$$

From the above assignment of colors to the edges of $G$,

it is clear that

$$rc(G) \leq n$$

By combing the step (i) and step (ii), we have

$$n \leq rc(G) \leq n$$

If both bounds are coincide. We establish with certainty that

$$rc(G) = n$$

□

**Theorem 4.3.** When the graph $G$ is considered as a Diamond Ladder Graph $Dl_n$ then $rc(G) = n+2$.

**Proof:** A diamond ladder graph $Dl_n$ is family of connected graph. Diamond ladder graph has a vertex set $V(Dl_n) = \{x_j : 1 \leq j \leq n\} \cup \{y_j : 1 \leq j \leq 2n\} \cup \{z_j : 1 \leq j \leq n\}$ and the edge set $E(Dl_n) = \{x_j x_{j+1} : 1 \leq j \leq n-1\} \cup \{z_j z_{j+1} : 1 \leq j \leq n-1\} \cup \{x_j z_j : 1 \leq j \leq n\} \cup \{x_j y_j : 1 \leq j \leq 2n\} \cup \{y_j z_j : 1 \leq j \leq 2n\} \cup \{y_{2j} y_{2j+1} : 1 \leq j \leq n-1\}$.

Let graph $G$ is connected. It means, for every pair of distinct vertices $u, v \in V(G)$, there exist a path that link $u$ to $v$. As a result, the rainbow connection number $rc(G)$ is defined and finite. Since all the vertices are connected with in the graph.

**Step (i):** we aim to calculate the diameter of the graph, which we denote by $diam(G)$. This represents the greatest distance among all shortest paths connecting any two vertices in the graph.

In this case of this graph the diameter is

$$diam(G) = n + 2$$

The value for the rainbow connection number, leading the inequality

$$rc(G) \geq diam(G)$$

Therefore, we can state that

$$rc(G) \geq n + 2$$

**Step (ii):** TWe construct an edge coloring using precisely $n + 2$ distinct colors to rainbow connection number $rc(G)$ for upper bound. The coloring is defined by the following function

$$c : E(G) \to \{1, 2, 3, \ldots, n + 2\}$$

Colors are systematically assigned to the edges of the graph $G$ in such a way that for each pair of distinct vertices $u, v \in V(G)$, there exist at least one path connecting $u$ and $v$ in which every edge along the path is colored differently. This ensures that no two edges on such a path share the same color, thereby creating a rainbow path between the vertices and satisfying the conditions required for a rainbow-connected graph..

$$z_j y_{2j-1} = 1 \ for \ 1 \leq j \leq n$$
$$x_j y_{2j} = 1 \ for \ 1 \leq j \leq n$$
$$z_j y_{2j} = 2 \ for \ 1 \leq j \leq n$$
$$x_j y_{2j-1} = 2 \ for \ 1 \leq j \leq n$$
$$x_j x_{j+1} = j + 2 \ for \ 1 \leq j \leq n$$
$$y_j y_{j+1} = j + 1 \ for \ 2 \leq j \leq n$$
$$z_j z_{j+1} = j + 2 \ for \ 1 \leq j \leq n$$
$$x_j z_j = j + 1 \ for \ 1 \leq j \leq n$$

Based on this construction we conclude that

$$rc(G) \leq n + 2$$

From the bounds established in step (i) and step (ii)
We now have

$$n + 2 \leq rc(G) \leq n + 2$$

When both bounds coincide,
Therefore we deduce that rainbow connection number of Diamond ladder graph is

$$rc(G) = n + 2$$

□

## 5. Conclusion

The study examines the role and impact of rainbow coloring in the context of network analysis, emphasizing its role in ensuring distinct and reliable communication paths between nodes.

The results obtained for

**(i)** When graph $G$ can be defined as Tribun graph $T_n$, then

$$rc(G) = \begin{cases} \frac{3n}{2} + 1 & if \text{ n } is\ even \\ \frac{3n+1}{2} & if \text{ n } is\ odd \end{cases}$$

**(ii)** When graph $G$ is defined as Chain graph is a point of shackle $K_4 P_n$ then $rc(G) = n$.

**(iii)** When the graph $G$ is considered as a Diamond Ladder Graph $Dl_n$ then $rc(G) = n + 2$.

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