Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) v. 2026 (44) 4 : 1–19.
ISSN-0037-8712
doi:10.5269/bspm.80120

Triangular Norm-Based Interval Valued L-Fuzzy Soft Ideals in Nearrings

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ABSTRACT: This study explores interval-valued L-fuzzy soft ideals within nearrings, where this structure is established over a complete bounded lattice. The approach employs interval-valued triangular norms and conorms as tools for handling graded membership and uncertainty. The algebraic characteristics of these ideals are examined, together with their behavior under nearring homomorphisms and the corresponding coset structures. We also analyze the relationship between such ideals and their associated level sets, thereby extending the scope of fuzzy soft algebraic theory. The framework not only brings together earlier notions of fuzzy and soft ideals but also introduces threshold-based flexibility, which broadens its range of applicability. Possible applications include decision-making, reasoning under uncertainty, and computational intelligence, particularly in contexts where algebraic precision and soft set-based modeling need to be combined.

Key Words: nearring, lattice, t-norm, soft ideal, level set

Contents

1. Introduction

In the current era of rapid digitalization, the collection of massive volumes of data has become both common and convenient. However, such accessibility often comes with an inherent challenge: the data obtained is frequently imprecise, vague, or uncertain. Uncertainty naturally arises in real-world problems from areas such as social sciences, medical diagnostics, engineering, and economics.Traditional crisp mathematical models that rely on binary classifications are often inadequate for effectively capturing and processing such uncertainty.

To address these challenges, various uncertainty-handling frameworks have been proposed, including fuzzy sets, rough sets, and probabilistic models. While each approach provides unique insights, none alone is sufficient to tackle the wide spectrum of practical problems involving uncertainty. To bridge this gap, Molodtsov [22] introduced soft set theory in 1999, which offers a parameterized approach well-suited for decision-making and computational tasks. The adaptability of soft sets arises from their ability to encode multiple parameters, thereby making them more flexible in modeling vague and complex data.

The fusion of soft and fuzzy set concepts led to the development of fuzzy soft sets, first introduced by Maji, Biswas, and Roy [20]. This framework enhanced modeling capabilities by incorporating graded membership levels. Subsequent research extended these ideas in various directions. Mujumdar and Samata [24] explored the benefits of this theory in decision analysis, whereas Yang et al. [26] developed an interval-valued formulation to handle intricate uncertainty scenarios. Collectively, these works demonstrate the growing importance of combining soft and fuzzy methodologies to achieve more comprehensive uncertainty representation.

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2020 *Mathematics Subject Classification:* 16Y30, 06B10, 03E72.
Submitted November 13, 2025. Published February 03, 2026

Significant progress has also been made in analyzing the algebraic aspects of soft and fuzzy frameworks. Aktas and Cagman [2] initiated research on soft groups, which was later generalized to fuzzy soft groups by Aygunoglu and H. Aygun [3]. Subsequent developments include the introduction of soft rings and the study of fuzzy soft ideals in nearrings by Acar *et al.* [1] and Inan and Ozturk [12], respectively. Ozturk and Inan [25] later examined soft subnearrings and their fuzzy analogues. These contributions highlight the adaptability of soft and fuzzy algebraic ideas within traditional algebraic frameworks.

In parallel, lattice-based fuzzy frameworks have drawn significant attention. For instance, Kedukodi et al. [15] studied L-fuzzy prime ideals in nearrings, while Kuncham et al. [18] explored interval-valued L-fuzzy cosets. Additional interconnections between fuzzy, rough, and soft sets have been investigated in several studies [8,9,23,16]. Jagadeesha et al. [14] further enriched this direction by developing fuzzy implication operators on lattices. Such works indicate the importance of hybrid models where lattice theory, fuzzy structures, and algebraic systems complement one another.

Motivated by these works, the present paper proposes a new algebraic framework called *Interval Valued L-Fuzzy Soft Ideals* (IVLF soft ideals) in nearrings. Unlike earlier approaches, our construction is defined over a complete bounded lattice L, which need not be distributive or totally ordered. The framework is established using interval-valued t-norm and conorms, thereby extending the scope of soft ideals to capture graded membership within interval uncertainty.

This generalized setting unifies several earlier notions of fuzzy, soft, and interval-valued ideals, while also introducing new algebraic properties. We analyze their stability under nearring homomorphisms, define corresponding soft cosets, and examine the role of level sets in this context. Beyond theoretical generalization, these results pave the way for potential applications in decision-making, approximate reasoning, and computational intelligence, where both algebraic rigor and uncertainty-handling flexibility are essential. Moreover, the study offers a unifying perspective that connects interval-valued fuzzy methods with soft set parameterization, and thereby provides a foundation for further applications in machine learning, information sciences, and algebraic cryptographic systems [6,7,10].

Furthermore, this investigation bridges the conceptual gap between algebraic and logical representations of uncertainty by extending soft ideals to an interval-valued L-fuzzy environment. The integration of lattice operations with triangular norms ensures that the proposed framework supports both algebraic closure and flexible uncertainty aggregation. Unlike conventional fuzzy or soft ideals, IVLF soft ideals accommodate partial order and interval membership simultaneously, enabling deeper structural analysis. The study not only generalizes earlier fuzzy ideal constructions but also enhances interpretability in computational systems that rely on graded truth values. Finally, the methodological approach adopted here lays a theoretical foundation for future extensions in multi-parameter algebraic systems and hybrid intelligent models.

2. Preliminaries

We recall only the essential definitions required for our work. For a more details on lattices we refer to [10], on triangular norms to [17], on fuzzy ideals in nearrings to [6,7], and on soft sets to [22,20,21]

Let $(L, \wedge_L, \vee_L)$ denote a complete bounded lattice, with its minimal and maximal elements m and M respectively. The ordering relation on L is expressed by $\leq_L$.

Definition 2.1 [11]

A *t-norm* on L is a mapping $T : L \times L \to L$ that satisfies commutativity, associativity, and monotonicity in both arguments. The element M serves as the identity of T. The dual concept, called a *t-conorm* $\zeta : L \times L \to L$, takes m as its identity. A t-norm T is said to be *idempotent* if $T(p, p) = p \ \forall \ p \in L$.

Definition 2.2 [13] Let T_f and T_g be the t-norms defined on L such that $T_f(a, b) \leq_L T_g(a, b), \forall a, b \in L$. Then,

$$T_I([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = [\, T_f(\underline{a}, \underline{b}),\ T_g(\overline{a}, \overline{b})\,].$$

defines an interval-valued triangular norm (IVTN) on $C(L)$. The parallel construction with t-conorms leads to an interval-valued triangular conorm (IVTCN).

Definition 2.3 [7] An *interval-valued L-fuzzy set* (IVLF set) on a universe E is a mapping $\hat{\nu} : E \to C(L)$ given by

$$\hat{\nu}(u) = [\underline{\nu}(u), \overline{\nu}(u)], \quad \underline{\nu}(u) \leq_L \overline{\nu}(u),\ u \in E.$$

Definition 2.4 Let X be a set of parameters and $Y \subseteq X$. A pair $(\hat{g}, Y)$ is called an *IVLF soft set* over E if $\hat{g} : Y \to \mathcal{P}(E)$ assigns to each $y \in Y$ an IVLF subset $\hat{g}(y)$, denoted by $\hat{g}_y$.

3. List of Notations and Abbreviations

Symbol / Abbreviation	Description
$\mathbb{N}$	Nearring under consideration
$\mathcal{I}$	An ideal of the nearring $\mathbb{N}$
$\mathcal{A}$	Parameter set of a soft set / soft ideal
E	Universal set
$P(E)$	Power set of E, i.e., the collection of all subsets of E
L	Complete bounded lattice
$\mathcal{C}(L)$	Collection of all closed intervals in L, i.e., $\mathcal{C}(L) = \{[a, b] : a, b \in L,\ a \leq b\}$
$[a, b] \in \mathcal{C}(L)$	Interval element in L with $0 \leq a \leq b \leq 1$
$\mathcal{T}$	A triangular norm (t-norm) on lattice L; a binary operation $\mathcal{T} : L \times L \to L$ satisfying commutativity, associativity, monotonicity, and having M as the identity element
Ξ	A triangular conorm (t-conorm) on lattice L; a binary operation $\Xi : L \times L \to L$ satisfying commutativity, associativity, monotonicity, and having m as the identity element
$F(\mathbb{N})$	Family of all interval-valued L-fuzzy soft sets defined over the nearring $\mathbb{N}$
M	Greatest (maximum) element of the lattice L
m	Least (minimum) element of the lattice L
μ	Interval-valued L-fuzzy soft mapping
IVF	Interval Valued Fuzzy
IVLF	Interval Valued L-Fuzzy
IVTN	Interval Valued Triangular Norm
IVTCN	Interval Valued Triangular Conorm
T	A specific IVTN on L
ζ	A specific IVTCN on L
$x_{\hat{k}}$	IVF point with support x and interval membership $\hat{k}$
$x_{\hat{k}} q(\hat{f}_a)$	IVF point $x_{\hat{k}}$ quasi-coincident with $\hat{f}_a$
$(\hat{f}_a)_{\hat{k}}$	$\hat{k}$-level set of IVF soft mapping $\hat{f}_a$
$(\hat{f}_a)_{\hat{k}q}$	Set of elements quasi-coincident with $(\hat{f}_a)$ at threshold $\hat{k}$
$(\hat{f}_a)_{\hat{k}\vee q}$	Generalized set combining membership and quasi-coincidence
IFP	Insertion of Factors Property
$\mathbb{N}/(\hat{f}, A)$	Collection of IVLF soft cosets of $(\hat{f}, A)$ in $\mathbb{N}$
$e_{\mathbb{N}}$	Identity element of $\mathbb{N}$

4. IVLF Soft Ideals in Nearrings

Definition 4.1

Let E denote the universal set, X the collection of parameters and $Y \subseteq X$. If $\mathcal{P}(E)$ represent the IVLF power set of E and $\hat{g} : Y \to \mathcal{P}(E)$ then pair $(\hat{g}, Y)$ is called as IVLF soft set over E. For every $y \in Y$, we write $\hat{g}(y)$ as $\hat{g}_y$.

Remark 4.1 The concept of an IVLF soft ideal extends the idea of an interval-valued L-fuzzy ideal to soft structures, where each parameter $a \in A$ induces a distinct fuzzy interval representation on N.

Definition 4.2 Let $(\hat{g}, Y)$ be an IVLF soft set over E and $\hat{\lambda}, \hat{\gamma} \in C(L)$ with $\hat{\lambda} < \hat{\gamma}$. In this case, $(\hat{g}, Y)$ is called IVLF soft ideal with thresholds $\hat{\lambda}, \hat{\gamma}$ if $\forall\, y \in Y$, $\forall\, p, q, j \in E$.
1. $\zeta_I(\hat{\lambda}, \hat{g_y}(p+q)) \geq T_I(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{g_y}(p)), \zeta_I(\hat{\lambda}, \hat{g_y}(q))))$,
2. $\zeta_I(\hat{\lambda}, \hat{g_y}(-p)) \geq T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{g_y}(p)))$.
3. $\zeta_I(\hat{\lambda}, \hat{g_y}(q+p-q)) \geq T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{g_y}(p)))$,
4. $\zeta_I(\hat{\lambda}, \hat{g_y}(pq)) \geq T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{g_y}(p)))$,
5. $\zeta_I(\hat{\lambda}, \hat{g_y}(p(q+j)-pq)) \geq T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{g_y}(j)))$.

Here T_I and ζ_I are the associated IVTN and IVTCN of IVLF soft ideal $(\hat{g}, Y)$ respectively.

Remark 4.2 (i) If we take $\hat{\lambda} = [k, k]$, $\hat{\gamma} = [K, K]$ and the associated IVTN as $T_I(\hat{a}, \hat{b}) = min(\hat{a}, \hat{b})$ then $\zeta_I(\hat{\lambda}, \hat{g_y}(\hat{a}+\hat{b})) = \hat{g_y}(\hat{a}+\hat{b})$ and $T_I(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{g_y}(\hat{a})), \zeta_I(\hat{\lambda}, \hat{g_y}(\hat{b}))))= T_I(\hat{g_y}(\hat{a})), \hat{g_y}(\hat{b})) = min(\hat{g_y}(\hat{a})), \hat{g_y}(\hat{b})$. Here the condition (i) of Definition 4.2 reduces to
$\hat{g_y}(\hat{a}+\hat{b}) \geq min(\hat{g_y}(\hat{a}), \hat{g_y}(\hat{b}))$ this corresponds to condition (i) stated in Definition 6 in Ozturk and Inan [25]. Similarly we can show that other condition is equivalent.

Now we provide examples for IVLF soft ideal with thresholds over N.

Example 4.1 Consider the nearring $\{0, u, v, w\}$ as defined in Table 1.

Table 1: Nearing operation for Example 4.1

$+$	0	u	v	w
0	0	u	v	w
u	u	0	w	v
v	v	w	0	u
w	w	v	u	0

$\cdot$	0	u	v	w
0	0	0	0	0
u	0	0	u	u
v	0	u	w	v
w	0	u	v	w

The lattice L relevant to this construction is presented in Figure 1.

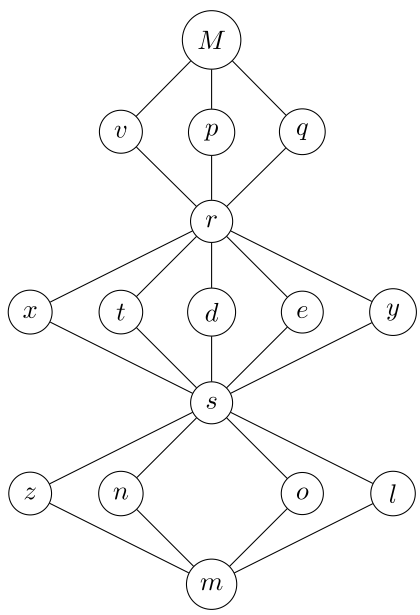

Figure 1: Lattice $L = \{m, z, n, o, l, s, x, t, d, e, y, r, v, p, q, M\}$

Let $B = \{p_1, p_2, p_3, p_4, p_5, p_6\}$, $A = \{p_1, p_2, p_3, p_4\} \subset B$.
For each i with $1 \leq i \leq 4$ we define $\hat{f}_{p_i} : A \to \mathcal{F}(N)$ as follows:

$$\hat{f}_{p_1} = \{(0, [x, 1]), (u, [a, q]), (v, [0, r]), (w, [0, r])\},$$

$$\hat{f}_{p_2} = \{(0,[x,1]),(u,[b,q]),(v,[0,s]),(w,[0,t])\},$$
$$\hat{f}_{p_3} = \{(0,[x,1]),(u,[c,q]),(v,[0,u]),(w,[0,v])\},$$
$$\hat{f}_{p_4} = \{(0,[x,1]),(u,[d,q]),(v,[0,w]),(w,[0,y])\}.$$

(i) For $g,h \in L$, define

$$\mathcal{T}_1(g,h) = \begin{cases} g, & \text{if } h = M, \\ h, & \text{if } g = M, \\ m, & \text{otherwise,} \end{cases} \qquad \mathcal{T}_2(g,h) = g \wedge_L h,$$

$$\Xi_1(g,h) = g \vee_L h, \quad \Xi_2(g,h) = \begin{cases} g, & \text{if } h = m, \\ h, & \text{if } g = m, \\ M, & \text{otherwise.} \end{cases}$$

Let $\hat{\lambda} = [m,s]$ and $\hat{\gamma} = [v,M]$. Then $\hat{f}_{p_i}$ is an IVLF soft ideal of N with thresholds $\hat{\lambda},\hat{\gamma}$.
(ii) For $g,h \in L$, define

$$\mathcal{T}_1(g,h) = \mathcal{T}_2(g,h) = g \wedge_L h, \quad \Xi_1(g,h) = \Xi_2(g,h) = g \vee_L h.$$

Let $\hat{\lambda} = [s,q]$ and $\hat{\gamma} = [v,M]$. Then $(\hat{f},A)$ is an IVLF soft ideal of N with thresholds $\hat{\lambda},\hat{\gamma}$.

Example 4.2 Consider the nearring $N = \{0,1,2,3,4,5,6,7\}$ as defined in Table 2.

Table 2: Nearring operations for Example 4.2

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	6	0	4	3	7	3	5
3	3	7	1	5	2	6	0	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	2	4	0	7	3	5	1
7	7	3	5	1	6	2	4	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	1	1	0	0	1	1
3	0	0	1	1	0	0	1	1
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
6	0	0	1	1	0	0	1	1
7	0	0	1	1	0	0	1	1

The corresponding lattice is shown in Figure 2.
Let $B = \{p_1,p_2,p_3,p_4,p_5\}$, and $A = \{p_1,p_2,p_3\} \subset B$. For each i with $1 \le i \le 3$, we define $\hat{f}_{p_i} : A \to \mathcal{F}(N)$ as follows:

$$\hat{f}_{p_1}(x) = \begin{cases} [u,w], & \text{if } x \in \{0,1\}, \\ [s,t], & \text{if } x \in \{2,4\}, \\ [m,p], & \text{if } x \in \{3,5,6,7\}, \end{cases}$$

$$\hat{f}_{p_2}(x) = \begin{cases} [u,v], & \text{if } x \in \{0,1\}, \\ [r,s], & \text{if } x \in \{2,4\}, \\ [m,r], & \text{if } x \in \{3,5,6,7\}, \end{cases}$$

$$\hat{f}_{p_3}(x) = \begin{cases} [w,M], & \text{if } x \in \{0,1\}, \\ [s,t], & \text{if } x \in \{2,4\}, \\ [m,q], & \text{if } x \in \{3,5,6,7\}. \end{cases}$$

For $g,h \in L$, define

$$\mathcal{T}_1(g,h) = \begin{cases} g, & \text{if } h = M, \\ h, & \text{if } g = M, \\ m, & \text{otherwise,} \end{cases} \qquad \mathcal{T}_2(g,h) = g \wedge_L h,$$

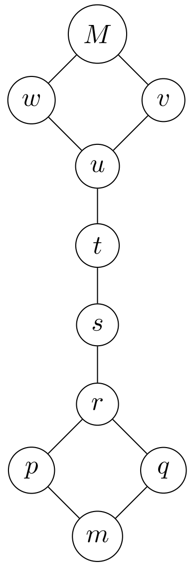

Figure 2: Lattice $L = \{m, p, q, r, s, t, u, v, w, M\}$

$$\Xi_1(g,h) = \Xi_2(g,h) = g \vee_L h.$$

Let $\hat{\lambda} = [s,t]$ and $\hat{\gamma} = [w,M]$. Then $(\hat{f}, A)$ is an IVLF soft ideal of N with thresholds $\hat{\lambda}, \hat{\gamma}$.

Remark 4.3 In Examples 4.1 and 4.2, the nearrings N considered are of orders 4 and 8, respectively. The operation tables are constructed so that N satisfies all the axioms of a nearring. It is observed that N is a nearring but not a ring, since it does not satisfy the left distributive law. Specifically, for certain elements $u, v, w \in N$, we have

$$u \cdot (v + w) = u \cdot u = u, \quad \text{but} \quad u \cdot v + u \cdot w = u + u = 0,$$

hence $u \cdot (v + w) \neq u \cdot v + u \cdot w$.

In earlier works, authors commonly used the lattice $[0,1]$, which forms a chain. In contrast, in our study, we employ a general lattice, as illustrated in Figures 1 and 2. An important property of a general lattice is the presence of incomparable elements—a feature evident in the lattices shown in Figures 1 and 2. Furthermore, while previous studies employed the idempotent t-norm and t-conorm operations $\wedge$ and $\vee$, we introduce more general (non-idempotent) triangular norms and conorms defined as follows:

$$\mathcal{T}_1(g,h) = \begin{cases} g, & \text{if } h = M, \\ h, & \text{if } g = M, \\ m, & \text{otherwise,} \end{cases} \qquad \Xi_2(g,h) = \begin{cases} g, & \text{if } h = m, \\ h, & \text{if } g = m, \\ M, & \text{otherwise.} \end{cases}$$

Both $\mathcal{T}_1$ and Ξ_2 are not idempotent, yet they satisfy the axioms of a t-norm and t-conorm, respectively. With these constructions, we generalize the soft-ideal theory from the standard lattice $[0,1]$ (a distributive lattice) to a general lattice that need not be distributive, and from idempotent to non-idempotent triangular norms and conorms. Hence, these examples establish that neither distributivity of the lattice nor idempotency of the triangular norms is a necessary condition in the definition of soft ideals over nearrings.

Definition 4.3 Let $(\hat{f}, A)$ be an IVLF soft set of N and $\hat{\lambda}, \hat{k} \in C(L)$. Then for each $a \in A$ the set $(\hat{f}_a)_{\hat{k}} = \{x \in N \mid \zeta_I(\hat{\lambda}, \hat{f}_a(x)) \geq \hat{k}\ \}$ is called *level set of* $(\hat{f}, A)$.

Proposition 4.1 *Let $(\hat{f}, A)$ be an IVLF soft ideal of N with thresholds $\hat{\lambda}, \hat{\gamma}$ and $a \in A$. If $\zeta_I(\hat{\lambda}, (\hat{f}_a)(0)) \geq \hat{\gamma}$ then $(\hat{f}_a)_{\hat{\gamma}} \neq \emptyset$. If the associated IVTN of IVLF soft ideal is idempotent and $(\hat{f}_a)_{\hat{\gamma}} \neq \emptyset$ then $0 \in (\hat{f}_a)_{\hat{\gamma}}$.*

Proof:
Fix $a \in A$. If $\zeta_I(\hat{\lambda}, \hat{f}_a(0)) \geq \hat{\gamma}$, then $0 \in (\hat{f}_a)_{\hat{\gamma}}$, hence $(\hat{f}_a)_{\hat{\gamma}} \neq \varnothing$.

Conversely, assume the associated IVTN T_I of $(\hat{f}, A)$ is idempotent and $(\hat{f}_a)_{\hat{\gamma}} \neq \varnothing$. Pick $x \in (\hat{f}_a)_{\hat{\gamma}}$, so $\zeta_I(\hat{\lambda}, \hat{f}_a(x)) \geq \hat{\gamma}$. Using the ideal property and monotonicity of the IVTN, we obtain

$$
\begin{aligned}
\zeta_I(\hat{\lambda}, \hat{f}_a(0)) &= \zeta_I(\hat{\lambda}, \hat{f}_a(x-x)) \\
&\geq T_I\big(\hat{\gamma},\, T_I\big(\zeta_I(\hat{\lambda}, \hat{f}_a(x)),\, \zeta_I(\hat{\lambda}, \hat{f}_a(-x))\big)\big) \\
&\geq T_I\big(\hat{\gamma},\, T_I\big(\zeta_I(\hat{\lambda}, \hat{f}_a(x)),\, T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x)))\big)\big) \quad \text{(by condition (ii) and monotonicity)} \\
&= T_I\big(\hat{\gamma},\, T_I\big(\zeta_I(\hat{\lambda}, \hat{f}_a(x)),\, \hat{\gamma}\big)\big) \quad \text{(since } \zeta_I(\hat{\lambda}, \hat{f}_a(x)) \geq \hat{\gamma}\text{)} \\
&\geq T_I\big(\hat{\gamma},\, T_I(\hat{\gamma}, \hat{\gamma})\big) = T_I(\hat{\gamma}, \hat{\gamma}) = \hat{\gamma}, \quad \text{(idempotence of } T_I\text{)}.
\end{aligned}
$$

Thus $\zeta_I(\hat{\lambda}, \hat{f}_a(0)) \geq \hat{\gamma}$, so $0 \in (\hat{f}_a)_{\hat{\gamma}}$ and $(\hat{f}_a)_{\hat{\gamma}} \neq \varnothing$.

□

Proposition 4.2 *Let $(\hat{f}, A)$ be an IVLF soft ideal of N with thresholds $\hat{\lambda}, \hat{\gamma}$ and $a \in A$. If the associated IVTN T_I is idempotent then $\zeta_I(\hat{\lambda}, \hat{f}_a(0)) \geq T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x)))\ \forall\ x \in N$.*

Proof:
Fix $a \in A$. Then, for any $x \in N$,

$$
\begin{aligned}
\zeta_I(\hat{\lambda}, \hat{f}_a(0)) &= \zeta_I(\hat{\lambda}, \hat{f}_a(x-x)) \\
&\geq T_I\Big(\hat{\gamma},\, T_I\big(\zeta_I(\hat{\lambda}, \hat{f}_a(x)),\, \zeta_I(\hat{\lambda}, \hat{f}_a(-x))\big)\Big) && \text{(property of interval-valued ideal)} \\
&\geq T_I\Big(\hat{\gamma},\, T_I\big(\zeta_I(\hat{\lambda}, \hat{f}_a(x)),\, T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x)))\big)\Big) && \text{(property of interval-valued ideal)} \\
&= T_I\Big(\hat{\gamma},\, T_I\big(\hat{\gamma},\, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)),\, \zeta_I(\hat{\lambda}, \hat{f}_a(x)))\big)\Big) && \text{(associativity of } T_I\text{)} \\
&= T_I\Big(\hat{\gamma},\, T_I\big(\hat{\gamma},\, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big)\Big) && \text{(idempotent property of } T_I\text{)} \\
&= T_I\Big(T_I(\hat{\gamma}, \hat{\gamma}),\, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\Big) && \text{(associativity of } T_I\text{)} \\
&= T_I\big(\hat{\gamma},\, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big) && \text{(idempotent property of } T_I\text{).}
\end{aligned}
$$

Therefore, $\forall\ x \in N$, $\zeta_I(\hat{\lambda}, \hat{f}_a(0)) \ \geq\ T_I\big(\hat{\gamma},\, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big)$.

□

Remark 4.4 In this paper we consider IVLF soft ideals $(\hat{f}, A)$ of N which satisfy the condition $\zeta_I(\hat{\lambda}, (\hat{f}_a)(0)) \geq \hat{\gamma}\ \forall\ a \in A$.

Proposition 4.3 *Let $(\hat{f}, A)$ be an IVLF soft ideal with thresholds $\hat{\lambda}, \hat{\gamma}$ and let the associated IVTN T_I be idempotent. Then $\forall\ x, y \in N$, the following are equivalent:*

(1) *(i)* $\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) \geq T_I\big(\hat{\gamma},\, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)),\, \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big)$,

(ii) $\zeta_I(\hat{\lambda}, \hat{f}_a(-x)) \geq T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x)))$.

(2) $\zeta_I(\hat{\lambda}, \hat{f}_a(x-y)) \geq T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big)$.

Proof: Suppose $(\hat{f}, A)$ is an IVLF soft ideal with thresholds $\hat{\lambda}, \hat{\gamma}$ and let T_I be idempotent.
(1) ⇒ (2):

$$\begin{aligned}
\zeta_I(\hat{\lambda}, \hat{f}_a(x-y)) &\geq T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(-y)))\big) \\
&\geq T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))))\big) && \text{(by (ii) and monotonicity)} \\
&= T_I\big(\hat{\gamma}, T_I(T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \hat{\gamma}), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) && \text{(associativity)} \\
&= T_I\big(\hat{\gamma}, T_I(T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) && \text{(commutativity)} \\
&= T_I\big(T_I(\hat{\gamma}, \hat{\gamma}), T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) && \text{(associativity)} \\
&= T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) && \text{(idempotence).}
\end{aligned}$$

(2) ⇒ (1):
Taking $x = 0$ in (2),

$$\begin{aligned}
\zeta_I(\hat{\lambda}, \hat{f}_a(-y)) &= \zeta_I(\hat{\lambda}, \hat{f}_a(0-y)) \\
&\geq T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(0)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) \\
&\geq T_I\big(\hat{\gamma}, T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) && \text{(by Remark 4.4)} \\
&= T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))) && \text{(idempotence).}
\end{aligned}$$

Thus condition (ii) holds.
Now for (i), note that

$$\begin{aligned}
\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) &= \zeta_I(\hat{\lambda}, \hat{f}_a(x-(-y))) \\
&\geq T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(-y)))\big) \\
&\geq T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))))\big) && \text{(from above)} \\
&= T_I\big(\hat{\gamma}, T_I(T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \hat{\gamma}), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) \\
&= T_I\big(T_I(\hat{\gamma}, \hat{\gamma}), T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) \\
&= T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big).
\end{aligned}$$

Thus (i) is also satisfied. Hence, (1) and (2) are equivalent. □

Remark 4.5 Proposition 4.3 establishes a practical equivalence that is frequently used in later results. In particular, this characterization simplifies the verification of IVLF soft ideal conditions for specific threshold pairs $(\hat{\lambda}, \hat{\gamma})$.

Definition 4.4 Let $(\hat{f}, A)$ be an IVLF soft set over a nearring N. We call $(\hat{f}, A)$ a $\hat{\theta}$*-identity IVLF soft set* over N if for every $a \in A$ and $x \in N$,

$$\hat{f}_a(x) = \begin{cases} \hat{\theta}, & \text{when } x = e, \\ [m, m], & \text{otherwise,} \end{cases}$$

here e is the multiplicative identity of N.

Proposition 4.4 *If $(\hat{f}, A)$ is a $\hat{\theta}$-identity IVLF soft set over N, and T_1 is an idempotent IVTN, then $(\hat{f}, A)$ forms an IVLF soft ideal of N.*

Proof: Fix $a \in A$. We consider several cases.
Case (i): $p = e$, $q = e$. Then $p + q = e$.

$$\zeta_I(\hat{\lambda}, \hat{f}_a(p+q)) = \zeta_I(\hat{\lambda}, \hat{f}_a(e)) = \zeta_I(\hat{\lambda}, \hat{\theta}),$$

$$T_I(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(p)), \zeta_I(\hat{\lambda}, \hat{f}_a(q)))) = T_I(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{\theta}), \zeta_I(\hat{\lambda}, \hat{\theta}))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{\theta})).$$

By Remark 4.2(i), $\zeta_I(\hat{\lambda}, \hat{\theta}) \geq T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{\theta}))$. Hence the inequality is satisfied.
Case (ii): $p = e$, $q \neq e$. Then $p + q = q$.

$$\zeta_I(\hat{\lambda}, \hat{f}_a(p+q)) = \zeta_I(\hat{\lambda}, \hat{f}_a(q)) = \zeta_I(\hat{\lambda}, [m, m]) = \hat{\lambda},$$

$$T_I(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(p)), \zeta_I(\hat{\lambda}, \hat{f}_a(q)))) = T_I(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{\theta}), \hat{\lambda})) \leq T_I(\hat{\gamma}, \hat{\lambda}) \leq \hat{\lambda}.$$

Thus the condition holds.
Case (iii): $p \neq e$, $q = e$. This case is symmetric to Case (ii), so the result follows directly.
Case (iv): $p \neq e$, $q \neq e$, and $p + q \neq e$. Then

$$\zeta_I(\hat{\lambda}, \hat{f}_a(p+q)) = \hat{\lambda},$$

$$T_I(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(p)), \zeta_I(\hat{\lambda}, \hat{f}_a(q)))) = T_I(\hat{\gamma}, T_I(\hat{\lambda}, \hat{\lambda})) \leq T_I(\hat{\gamma}, \hat{\lambda}) \leq \hat{\lambda}.$$

Hence the inequality holds.
Case (v): $p \neq e$, $q \neq e$, and $p + q = e$. Then

$$\zeta_I(\hat{\lambda}, \hat{f}_a(p+q)) = \zeta_I(\hat{\lambda}, \hat{\theta}) \geq \hat{\lambda},$$

$$T_I(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(p)), \zeta_I(\hat{\lambda}, \hat{f}_a(q)))) = T_I(\hat{\gamma}, T_I(\hat{\lambda}, \hat{\lambda})) \leq T_I(\hat{\gamma}, \hat{\lambda}) \leq \hat{\lambda}.$$

Therefore $T_I(\hat{\gamma}, \dots) \leq \zeta_I(\hat{\lambda}, \hat{f}_a(p+q))$.

The remaining conditions of an IVLF soft ideal can be verified in the same manner. Thus $(\hat{f}, A)$ is indeed an IVLF soft ideal of N.

□

Definition 4.5 An IVLF soft set $(\hat{f}, A)$ is called a $\hat{\theta}$-absolute IVLF soft set over N if $\hat{f}_a(x) = \hat{\theta}$ $\forall$ $a \in A$ and $x \in N$.

Proposition 4.5 *Let $(\hat{f}, A)$ be a $\hat{\theta}$-absolute IVLF soft set over N. If the associated IVTN T_I is idempotent, then $(\hat{f}, A)$ is an IVLF soft ideal of N.*

Proof: Let $a \in A$, and T_I be an idempotent IVTN. For any $x, y \in N$, $\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) = \zeta_I(\hat{\lambda}, \hat{\theta})$. On the other hand,

$$T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) = T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{\theta}), \zeta_I(\hat{\lambda}, \hat{\theta}))\big) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{\theta})) \qquad \text{(idempotence of } T_I\text{)}.$$

By Remark 4.2(i), we have $\zeta_I(\hat{\lambda}, \hat{\theta}) \geq T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{\theta}))$. Hence

$$\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) \ \geq \ T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big).$$

The remaining conditions of an IVLF soft ideal can be verified in a similar manner. Therefore $(\hat{f}, A)$ is an IVLF soft ideal of N. □

Proposition 4.6 *Let $(\hat{f}, A)$ be an IVLF soft ideal of N with associated IVTN T_I idempotent, and let $a \in A$. If $\zeta_I(\hat{\lambda}, \hat{f}_a(x-y)) = \zeta_I(\hat{\lambda}, \hat{f}_a(0)) \quad \forall\ x, y \in N$, then $T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y)))$.*

Proof: Let $a \in A$. Assume that $\zeta_I(\hat{\lambda}, \hat{f}_a(x-y)) = \zeta_I(\hat{\lambda}, \hat{f}_a(0))$.
Now,

$$\begin{aligned}
\zeta_I(\hat{\lambda}, \hat{f}_a(x)) &= \zeta_I(\hat{\lambda}, \hat{f}_a((x-y)+y)) \\
&\geq T_I\big(\hat{\gamma},\ T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x-y)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) \\
&\geq T_I\big(\hat{\gamma},\ T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(0)), T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))))\big) \\
&\geq T_I\big(\hat{\gamma},\ T_I(\hat{\gamma}, T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))))\big) \qquad \text{(by Remark 4.4)} \\
&= T_I\big(\hat{\gamma}, T_I(T_I(\hat{\gamma}, \hat{\gamma}), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) \qquad \text{(associativity of } T_I) \\
&= T_I(\hat{\gamma}, T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y)))) \\
&= T_I(T_I(\hat{\gamma}, \hat{\gamma}), \zeta_I(\hat{\lambda}, \hat{f}_a(y))) \qquad \text{(associativity)} \\
&= T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))) \qquad \text{(idempotence of } T_I).
\end{aligned}$$

Thus, $T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))) \ \geq \ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y)))$.
By symmetry, the reverse inequality also holds: $T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))) \ \geq \ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x)))$.
Therefore, $T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y)))$. □

Proposition 4.7 *Let $(\hat{f}, A)$ be an IVLF soft ideal of N, $x \in N$, and $a \in A$.*
(i) If $\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) = \zeta_I(\hat{\lambda}, \hat{f}_a(y+x)) = \zeta_I(\hat{\lambda}, \hat{f}_a(y)), \quad \forall\, y \in N$, then $\zeta_I(\hat{\lambda}, \hat{f}_a(x)) = \zeta_I(\hat{\lambda}, \hat{f}_a(0))$.
(ii) If the associated IVTN T_I is idempotent and $\zeta_I(\hat{\lambda}, \hat{f}_a(x)) = \zeta_I(\hat{\lambda}, \hat{f}_a(0))$, then $\forall\ y \in N$, $T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y+x)))$.

Proof: (i) Assume $\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) = \zeta_I(\hat{\lambda}, \hat{f}_a(y+x)) = \zeta_I(\hat{\lambda}, \hat{f}_a(y)), \quad \forall\, y \in N$. Taking $y = 0$, we get $\zeta_I(\hat{\lambda}, \hat{f}_a(x)) = \zeta_I(\hat{\lambda}, \hat{f}_a(0))$.
(ii) Suppose $\zeta_I(\hat{\lambda}, \hat{f}_a(x)) = \zeta_I(\hat{\lambda}, \hat{f}_a(0))$ and T_I is idempotent. By Proposition 4.2, $\forall\ y \in N$,

$$\zeta_I(\hat{\lambda}, \hat{f}_a(0)) = \zeta_I(\hat{\lambda}, \hat{f}_a(x)) \ \geq \ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))). \tag{4.1}$$

Since $(\hat{f}, A)$ is an IVLF soft ideal,

$$\begin{aligned}
\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) &\geq T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) \\
&\geq T_I\big(\hat{\gamma}, T_I(T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big) \quad \text{(by (4.1))} \\
&= T_I(\hat{\gamma}, T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y)))) \\
&= T_I(T_I(\hat{\gamma}, \hat{\gamma}), \zeta_I(\hat{\lambda}, \hat{f}_a(y))) \\
&= T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))).
\end{aligned}$$

Hence

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y))) \;\geq\; T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))). \tag{4.2}$$

On the other hand,

$$\begin{aligned}
\zeta_I(\hat{\lambda}, \hat{f}_a(y)) &= \zeta_I(\hat{\lambda}, \hat{f}_a(-x+(x+y))) \\
&\geq T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(-x)), \zeta_I(\hat{\lambda}, \hat{f}_a(x+y)))\big) \\
&\geq T_I\big(\hat{\gamma}, T_I(T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y))), \zeta_I(\hat{\lambda}, \hat{f}_a(x+y)))\big) \quad \text{(by (4.1))} \\
&= T_I(\hat{\gamma}, T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y)))) \\
&= T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y))).
\end{aligned}$$

Thus,

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))) \;\geq\; T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y))). \tag{4.3}$$

From (4.2) and (4.3), we conclude $T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y)))$.

A symmetric argument shows that $T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y+x)))$.

Therefore, $\forall\, y \in N$, $T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(y+x)))$. □

Proposition 4.8 *Let $(\hat{f}, A)$ be an IVLF soft subset of N. If $\forall\, \hat{k} \in (\hat{\lambda}, \hat{\gamma}]$, $a \in A$ the level set $(\hat{f}_a)_{\hat{k}}$ is an ideal of N, then $(\hat{f}, A)$ is an IVLF soft ideal of N.*

Conversely, if $(\hat{f}, A)$ is an IVLF soft ideal of N with the associated IVTN T_I idempotent, then $(\hat{f}_a)_{\hat{k}}$ is an ideal of N $\forall\, \hat{k} \in (\hat{\lambda}, \hat{\gamma}]$, $a \in A$.

Proof: Assume $(\hat{f}, A)$ is an IVLF soft subset of N, and that for every $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$ and $a \in A$, the corresponding level set $(\hat{f}_a)_{\hat{k}}$ is an ideal of N. We show that $(\hat{f}, A)$ satisfies the conditions of an IVLF soft ideal.

Assume, for contradiction, the existence of $a \in A$ and $x, y \in N$ such that

$$\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) < T_I\big(\hat{\gamma},\, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)),\, \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big).$$

Let

$$\hat{k} = T_I\big(\hat{\gamma},\, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)),\, \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big).$$

Then $\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) < \hat{k}$ implies that $x + y \notin (\hat{f}_a)_{\hat{k}}$, even though $x, y \in (\hat{f}_a)_{\hat{k}}$, contradicting the ideal property of $(\hat{f}_a)_{\hat{k}}$. Hence, $(\hat{f}, A)$ must be an IVLF soft ideal of N.

Conversely, assume $(\hat{f}, A)$ is an IVLF soft ideal of N with idempotent T_I. Then for any $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$ and $a \in A$, the defining properties of IVLF soft ideals guarantee that $(\hat{f}_a)_{\hat{k}}$ forms an ideal of N. □

Definition 4.6 An IVLF soft ideal $(\hat{f}, A)$ of N is said to have the *insertion of factors property* (IFP) if $\forall\, x, y \in N$, $a \in A$, and $n \in N$, we have

$$\zeta_I(\hat{\lambda}, \hat{f}_a(xny)) \;\geq\; T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(xy))\big).$$

Proposition 4.9 *Let $(\hat{f}, A)$ be an IVLF soft ideal of N. If $\forall\, \hat{k} \in (\hat{\lambda}, \hat{\gamma}]$, $a \in A$ the level set $(\hat{f}_a)_{\hat{k}}$ has IFP, then $(\hat{f}, A)$ has IFP.*

Conversely, if the associated IVTN T_I of $(\hat{f}, A)$ is idempotent and $(\hat{f}, A)$ has IFP, then $(\hat{f}_a)_{\hat{k}}$ has IFP $\forall\, \hat{k} \in (\hat{\lambda}, \hat{\gamma}]$, $a \in A$.

Proof: ($\Rightarrow$) Suppose $(\hat{f}, A)$ does not have IFP. Then for some $a \in A$, there exist $x, n, y \in N$ such that

$$\zeta_I(\hat{\lambda}, \hat{f}_a(xny)) \;<\; T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(xy))\big).$$

Choose

$$\hat{k} = T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(xy))\big).$$

Then $\hat{k} \leq \hat{\gamma} \wedge \zeta_I(\hat{\lambda}, \hat{f}_a(xy))$. Thus $\hat{\gamma} \geq \hat{k}$ and $\zeta_I(\hat{\lambda}, \hat{f}_a(xy)) \geq \hat{k}$, implying $xy \in (\hat{f}_a)_{\hat{k}}$. But since $\zeta_I(\hat{\lambda}, \hat{f}_a(xny)) < \hat{k}$, we have $xny \notin (\hat{f}_a)_{\hat{k}}$.

Hence, for $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$, we get $xy \in (\hat{f}_a)_{\hat{k}}$ but $xny \notin (\hat{f}_a)_{\hat{k}}$, contradicting the assumption that $(\hat{f}_a)_{\hat{k}}$ has IFP $\forall\ \hat{k} \in (\hat{\lambda}, \hat{\gamma}]$. Therefore, $(\hat{f}, A)$ must have IFP.

($\Leftarrow$) Conversely, assume $(\hat{f}, A)$ has IFP and let $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$, $a \in A$. Suppose $xy \in (\hat{f}_a)_{\hat{k}}$.
Then $\zeta_I(\hat{\lambda}, \hat{f}_a(xy)) \ \geq\ \hat{k}$.
Since $(\hat{f}, A)$ has IFP, we have $\zeta_I(\hat{\lambda}, \hat{f}_a(xny)) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(xy))\big)$.
Thus, $\zeta_I(\hat{\lambda}, \hat{f}_a(xny)) \ \geq\ T_I(\hat{\gamma}, \hat{k}) \ \geq\ T_I(\hat{k}, \hat{k}) = \hat{k}$. Hence $xny \in (\hat{f}_a)_{\hat{k}}$. Therefore, $(\hat{f}_a)_{\hat{k}}$ has IFP. □

Proposition 4.10 *Let $(\hat{f}, A)$ be an IVLF soft ideal of a nearfield N with thresholds $\hat{\lambda}, \hat{\gamma}$ and $a \in A$. If the associated IVTN T_I is idempotent, then $\forall\ x \in N$ we have*

$$\zeta_I(\hat{\lambda}, \hat{f}_a(0)) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big) \ =\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(1_N))\big).$$

Proof: By Proposition 4.2, we have

$$\zeta_I(\hat{\lambda}, \hat{f}_a(0)) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big) \quad \forall x \in N.$$

In particular,

$$\zeta_I(\hat{\lambda}, \hat{f}_a(0)) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(1_N))\big).$$

Now take $0 \neq x \in N$. Then

$$\zeta_I(\hat{\lambda}, \hat{f}_a(x)) = \zeta_I(\hat{\lambda}, \hat{f}_a(1_N x)) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(1_N))\big).$$

Hence

$$T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big) \ \geq\ T_I\big(\hat{\gamma}, T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(1_N)))\big) = T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(1_N))\big),$$

where the last equality follows from the associativity and idempotence of T_I. Thus

$$T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(1_N))\big). \tag{4.4}$$

Also, since $1_N = xx^{-1}$, we have

$$\zeta_I(\hat{\lambda}, \hat{f}_a(1_N)) = \zeta_I(\hat{\lambda}, \hat{f}_a(xx^{-1})) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big).$$

Thus

$$T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(1_N))\big) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big). \tag{4.5}$$

From (4.4) and (4.5), it follows that

$$T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big) = T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(1_N))\big).$$

Therefore,

$$\zeta_I(\hat{\lambda}, \hat{f}_a(0)) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))\big) = T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(1_N))\big),$$

$\forall\ x \in N$. □

Proposition 4.11 *Let $(\hat{f}, A)$ be an IVLF soft ideal of a zero-symmetric nearring N with thresholds $\hat{\lambda}, \hat{\gamma}$ and $a \in A$. Then $\forall\ x, y, i \in N$ we have*

$$\zeta_I\big(\hat{\lambda}, \hat{f}_a(x(y+i) - xy)\big) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(i))\big) \ \Rightarrow\ \zeta_I(\hat{\lambda}, \hat{f}_a(xi)) \ \geq\ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(i))\big).$$

Proof: Let $x, y, i \in N$ and $a \in A$. By condition (5) of an IVLF soft ideal we have

$$\zeta_I(\hat{\lambda}, \hat{f}_a(x(y+i) - xy)) \ \geq \ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(i))\big).$$

Choosing $y = 0$, we obtain

$$\zeta_I(\hat{\lambda}, \hat{f}_a(x(0+i) - x0)) = \zeta_I(\hat{\lambda}, \hat{f}_a(xi)) \ \geq \ T_I\big(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(i))\big).$$

Hence the result follows. □

Proposition 4.12 *Let $(\hat{f}, A)$ be an IVLF soft ideal of N with thresholds $\hat{\lambda}, \hat{\gamma}$ and $a \in A$. Define*

$$N_{\hat{f}_a} = \{\, x \in N \mid T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(0))) \,\}.$$

If the associated IVTN T_I is idempotent, then $N_{\hat{f}_a}$ is an ideal of N.

Proof: Let $x, y \in N_{\hat{f}_a}$. Then

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(0))),$$

and similarly for y.

By Proposition 4.2, we know that $\forall\ z \in N$,

$$\zeta_I(\hat{\lambda}, \hat{f}_a(0)) \ \geq \ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(z))).$$

Hence, in particular,

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(0))) \ \geq \ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y))). \tag{4.6}$$

Since $(\hat{f}, A)$ is an IVLF soft ideal,

$$\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) \ \geq \ T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))\big).$$

Substituting $x, y \in N_{\hat{f}_a}$ gives

$$\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) \ \geq \ T_I\big(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(0)), \zeta_I(\hat{\lambda}, \hat{f}_a(0)))\big).$$

By associativity and idempotence of T_I, this reduces to

$$\zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) \ \geq \ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(0))).$$

Thus

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y))) \ \geq \ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(0))). \tag{4.7}$$

From (4.6) and (4.7), we conclude

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(x+y))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(0))).$$

Hence $x + y \in N_{\hat{f}_a}$. Other ideal conditions can be verified similarly. Therefore $N_{\hat{f}_a}$ is an ideal of N. □

Proposition 4.13 *Let $g : N_1 \to N_2$ be an onto homomorphism. Let $(\hat{f}, A)$ be an IVLF soft subset of N_2. If $\forall\ \hat{k} \in (\hat{\lambda}, \hat{\gamma}]$ and $a \in A$ the level set $(\hat{f}_a)_{\hat{k}}$ is an ideal of N_2, then $g^{-1}((\hat{f}, A))$ is an IVLF soft ideal of N_1.*

Proof: Take $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$ and $a \in A$. Suppose $x \in N_1$ with $x \in g^{-1}((\hat{f}_a)_{\hat{k}})$. Then $g(x) \in (\hat{f}_a)_{\hat{k}}$, i.e.

$$\zeta_I(\hat{\lambda}, \hat{f}_a(g(x))) \ \geq \ \hat{k}.$$

Equivalently,

$$\zeta_I(\hat{\lambda}, (g^{-1}\hat{f}_a)(x)) \ \geq \ \hat{k},$$

so $x \in (g^{-1}\hat{f}_a)_{\hat{k}}$.

Now let $x, y \in g^{-1}((\hat{f}_a)_{\hat{k}})$. Then $g(x), g(y) \in (\hat{f}_a)_{\hat{k}}$. Since $(\hat{f}_a)_{\hat{k}}$ is an ideal of N_2, we have $g(x)+g(y) \in (\hat{f}_a)_{\hat{k}}$. Hence $g(x+y) \in (\hat{f}_a)_{\hat{k}}$, which means

$$x + y \in (g^{-1}\hat{f}_a)_{\hat{k}}.$$

Other ideal properties follow similarly. Therefore $(g^{-1}\hat{f}_a)_{\hat{k}}$ is an ideal of N_1 $\forall$ $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$ and $a \in A$. By proposition 4.8, it follows that $g^{-1}((\hat{f}, A))$ is an IVLF soft ideal of N_1. □

Proposition 4.14 *Let $g : N_1 \to N_2$ be an onto homomorphism. If $(\hat{f}, A)$ is an IVLF soft ideal of N_2, then $g^{-1}((\hat{f}, A))$ is an IVLF soft ideal of N_1 with the same thresholds as $(\hat{f}, A)$.*

Proof: Let $(\hat{f}, A)$ be an IVLF soft ideal of N_2 and $a \in A$. For $x, y \in N_1$, consider

$$\begin{aligned}
\zeta_I(\hat{\lambda}, g^{-1}(\hat{f}_a)(x+y)) &= \zeta_I(\hat{\lambda}, \hat{f}_a(g(x+y))) \\
&= \zeta_I(\hat{\lambda}, \hat{f}_a(g(x)+g(y))) \\
&\geq T_I\big(\hat{\gamma},\, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(g(x))), \zeta_I(\hat{\lambda}, \hat{f}_a(g(y))))\big) \\
&= T_I\big(\hat{\gamma},\, T_I(\zeta_I(\hat{\lambda}, g^{-1}(\hat{f}_a)(x)), \zeta_I(\hat{\lambda}, g^{-1}(\hat{f}_a)(y)))\big).
\end{aligned}$$

Thus $g^{-1}(\hat{f}_a)$ satisfies the soft ideal condition. Similarly, other conditions can be verified. Therefore $g^{-1}((\hat{f}, A))$ is an IVLF soft ideal of N_1 with the same thresholds. □

Proposition 4.15 *Let $g : N_1 \to N_2$ be an onto map. If $(\hat{f}, A)$ is a g-invariant IVLF soft ideal of N_1, then $\forall$ $a \in A$ and all $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$ we have*

$$g\big((\hat{f}_a)_{\hat{k}}\big) \ = \ \big(g(\hat{f}_a)\big)_{\hat{k}}.$$

Proof: Fix $a \in A$ and $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$. Suppose $y \in g((\hat{f}_a)_{\hat{k}})$. Then $y = g(x)$ for some $x \in (\hat{f}_a)_{\hat{k}}$ which means $\zeta_I(\hat{\lambda}, \hat{f}_a(x)) \geq \hat{k}$. By definition of $g(\hat{f}_a)$,

$$g(\hat{f}_a)(y) = \sup\{\hat{f}_a(w) \mid g(w) = y\}.$$

Since $g(x) = y$, we have $g(\hat{f}_a)(y) = \hat{f}_a(x)$, and so

$$\zeta_I(\hat{\lambda}, g(\hat{f}_a)(y)) = \zeta_I(\hat{\lambda}, \hat{f}_a(x)) \geq \hat{k}.$$

Hence $y \in (g(\hat{f}_a))_{\hat{k}}$, proving $g((\hat{f}_a)_{\hat{k}}) \subseteq (g(\hat{f}_a))_{\hat{k}}$.

Conversely, let $y \in (g(\hat{f}_a))_{\hat{k}}$. Then $\zeta_I(\hat{\lambda}, g(\hat{f}_a)(y)) \geq \hat{k}$, i.e.

$$\zeta_I(\hat{\lambda}, \sup\{\hat{f}_a(w) \mid g(w) = y\}) \geq \hat{k}.$$

Since g is onto, there exists $x \in N_1$ with $g(x) = y$. Thus $\zeta_I(\hat{\lambda}, \hat{f}_a(x)) \geq \hat{k}$, i.e. $x \in (\hat{f}_a)_{\hat{k}}$, and so $y = g(x) \in g((\hat{f}_a)_{\hat{k}})$. Therefore $(g(\hat{f}_a))_{\hat{k}} \subseteq g((\hat{f}_a)_{\hat{k}})$.

Combining both inclusions gives

$$g((\hat{f}_a)_{\hat{k}}) = (g(\hat{f}_a))_{\hat{k}}.$$

□

Proposition 4.16 *Let $g : N_1 \to N_2$ be an onto homomorphism. Suppose $(\hat{f}, A)$ is a g-invariant IVLF soft ideal of N_1 with $a \in A$. If for every $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$ the level set $(\hat{f}_a)_{\hat{k}}$ is an ideal in N_1, then $(g(\hat{f}), A)$ is an IVLF soft ideal of N_2 with the same thresholds as $(\hat{f}, A)$.*

Proof: Let $a \in A$ and fix $\hat{k} \in (\hat{\lambda}, \hat{\gamma}]$. Since $(\hat{f}_a)_{\hat{k}}$ is an ideal in N_1, its homomorphic image $g((\hat{f}_a)_{\hat{k}})$ is an ideal in N_2.

As $(\hat{f}, A)$ is g-invariant, Proposition 4.15 gives

$$g\big((\hat{f}_a)_{\hat{k}}\big) \;=\; (g(\hat{f}_a))_{\hat{k}}.$$

Thus $(g(\hat{f}_a))_{\hat{k}}$ is an ideal of N_2 $\forall\ \hat{k} \in (\hat{\lambda}, \hat{\gamma}]$. By proposition 4.8, $(g(\hat{f}), A)$ is an IVLF soft ideal of N_2 with thresholds $\hat{\lambda}, \hat{\gamma}$. □

Remark 4.6 Homomorphisms preserve the IVLF soft ideal structure, demonstrating that these constructions are stable under algebraic mappings between nearrings. This property ensures compatibility with categorical approaches in fuzzy algebra.

Definition 4.7 Let $(\hat{f}, A)$ be an IVLF soft ideal of N. For $p \in N$ and $a \in A$, define an IVLF soft subset ${}_p(\hat{f}_a)$ by

$${}_p(\hat{f}_a)(n) \;=\; T_I\big(\hat{\gamma},\, \zeta_I(\hat{\lambda}, \hat{f}_a(n-p))\big), \quad \forall n \in N.$$

This subset is called the *IVLF soft coset* determined by p and $(\hat{f}, A)$. The collection of all IVLF soft cosets of $(\hat{f}, A)$ in N is denoted by $N/(\hat{f}, A)$.

Proposition 4.17 *Let $(\hat{f}, A)$ be an IVLF soft ideal of N with associated IVTN T_I idempotent and $a \in A$. Then $N/(\hat{f}, A)$ forms a nearring under the operations*

$${}_x(\hat{f}_a) + {}_y(\hat{f}_a) \;=\; {}_{x+y}(\hat{f}_a), \qquad {}_x(\hat{f}_a)\cdot {}_y(\hat{f}_a) \;=\; {}_{x\cdot y}(\hat{f}_a), \quad \forall\, x, y \in N.$$

Moreover, the mapping

$$[(\hat{f}_a)] : N/(\hat{f}, A) \longrightarrow C(L), \quad [(\hat{f}_a)]({}_x(\hat{f}_a)) = \hat{f}_a(x),$$

is an IVLF soft ideal of $N/(\hat{f}, A)$.

Proof: Let $p, q, r, s \in N$ such that ${}_p(\hat{f}_a) = {}_q(\hat{f}_a)$ and ${}_r(\hat{f}_a) = {}_s(\hat{f}_a)$. By definition, $\forall\ n \in N$ we have

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-p))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-q))), \tag{4.8}$$

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-r))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-s))). \tag{4.9}$$

Putting $n = p$ in (4.8) gives

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(0))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(p-q))) \;\geq\; T_I(\hat{\gamma}, \hat{\gamma}),$$

by monotonicity of T_I. Similarly, substituting $n = r$ in (4.9) gives

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(r-s))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(0))) \;\geq\; T_I(\hat{\gamma}, \hat{\gamma}).$$

Since T_I is idempotent, we conclude

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(p-q))) \;\geq\; \hat{\gamma}, \tag{4.10}$$

$$T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(r-s))) \;\geq\; \hat{\gamma}. \tag{4.11}$$

Next, to show addition is well-defined, consider

$$ {}_{p}(\hat{f}_a)(n) + {}_{r}(\hat{f}_a)(n) = {}_{p+r}(\hat{f}_a)(n). $$

Using (4.10), (4.11), and the properties of IVLF soft ideals, one checks that

$$ {}_{p}(\hat{f}_a) + {}_{r}(\hat{f}_a) \;=\; {}_{q}(\hat{f}_a) + {}_{s}(\hat{f}_a), $$

so addition is well-defined.

For multiplication, a similar argument with (4.10), (4.11), and the ideal conditions shows

$$ {}_{p\cdot r}(\hat{f}_a)(n) = {}_{q\cdot s}(\hat{f}_a)(n), \quad \forall n \in N, $$

so multiplication is well-defined.

Hence $N/(\hat{f}, A)$ is a nearring with zero element ${}_{0}(\hat{f}_a)$ and additive inverse ${}_{-x}(\hat{f}_a)$ for each $x \in N$.

Finally, for $x, y \in N$,

$$ \zeta_I(\hat{\lambda}, [(\hat{f}_a)]({}_{x}(\hat{f}_a) + {}_{y}(\hat{f}_a))) = \zeta_I(\hat{\lambda}, \hat{f}_a(x+y)) \geq T_I(\hat{\gamma}, T_I(\zeta_I(\hat{\lambda}, \hat{f}_a(x)), \zeta_I(\hat{\lambda}, \hat{f}_a(y)))), $$

and the other ideal conditions follow analogously. Thus $[(\hat{f}_a)]$ defines an IVLF soft ideal of $N/(\hat{f}, A)$. □

Proposition 4.18 *Let $(\hat{f}, A)$ be an IVLF soft ideal of N with the associated IVTN T_I idempotent and $a \in A$. Then for every $p, q \in N$,*

$$ {}_{p}(\hat{f}_a) = {}_{q}(\hat{f}_a) \quad \Longleftrightarrow \quad (p-q) \in (\hat{f}_a)_{\hat{\gamma}}. $$

Proof: Suppose ${}_{p}(\hat{f}_a) = {}_{q}(\hat{f}_a)$. Then $\forall\ n \in N$,

$$ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-p))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-q))). \tag{4.12} $$

Putting $n = p$ in (4.12) gives

$$ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(0))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(p-q))). $$

Hence

$$ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(p-q))) \;\geq\; T_I(\hat{\gamma}, \hat{\gamma}) = \hat{\gamma}, $$

by monotonicity and idempotency of T_I. Thus

$$ \zeta_I(\hat{\lambda}, \hat{f}_a(p-q)) \;\geq\; \hat{\gamma}, $$

which means $(p-q) \in (\hat{f}_a)_{\hat{\gamma}}$.

Conversely, assume $(p-q) \in (\hat{f}_a)_{\hat{\gamma}}$, i.e. $\zeta_I(\hat{\lambda}, \hat{f}_a(p-q)) \geq \hat{\gamma}$. For $n \in N$,

$$ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-p))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-q+(q-p)))). $$

Using the IVLF soft ideal property and idempotency of T_I, this yields

$$ T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-p))) = T_I(\hat{\gamma}, \zeta_I(\hat{\lambda}, \hat{f}_a(n-q))). $$

Hence ${}_{p}(\hat{f}_a) = {}_{q}(\hat{f}_a)$. □

Proposition 4.19 *Let $(\hat{f}, A)$ be an IVLF soft ideal of N with idempotent IVTN T_I. Then for each $a \in A$,*

$$ N/(\hat{f}_a)_{\hat{\gamma}} \;\cong\; N/(\hat{f}, A). $$

Proof: Fix $a \in A$. Define $h : N \to N/(\hat{f}, A)$ by

$$h(p) = {}_p(\hat{f}_a), \quad \forall\, p \in N.$$

Clearly h is a homomorphism, since

$$h(p+q) = {}_{p+q}(\hat{f}_a) = {}_p(\hat{f}_a) + {}_q(\hat{f}_a) = h(p) + h(q),$$

$$h(pq) = {}_{pq}(\hat{f}_a) = {}_p(\hat{f}_a) \cdot {}_q(\hat{f}_a) = h(p) \cdot h(q).$$

Now,

$$\ker h = \{p \in N \mid h(p) = h(0)\} = \{p \in N \mid {}_p(\hat{f}_a) = {}_0(\hat{f}_a)\}.$$

By Proposition 4.18, this is equivalent to

$$\ker h = \{p \in N \mid \zeta_I(\hat{\lambda}, \hat{f}_a(p)) \geq \hat{\gamma}\} = (\hat{f}_a)_{\hat{\gamma}}.$$

Therefore, by the First Isomorphism Theorem for nearrings,

$$N/(\hat{f}_a)_{\hat{\gamma}} \;\cong\; N/(\hat{f}, A).$$

□

Remark 4.7 The established results collectively demonstrate that IVLF soft ideals unify interval-valued fuzzy and soft structures within nearring theory. This unified framework may facilitate further generalizations in related algebraic systems.

5. Applications

Although the present work is mainly theoretical, the ideas of triangular norm-based IVLF soft ideals in nearrings are relevant to several applied fields where uncertainty and interval-valued information arise. Some possible directions include:

- **Decision-making systems:** In multi-criteria decision problems, evaluations are often imprecise or interval-based. The structure of IVLF soft ideals allows modeling of tolerance levels in such systems, leading to more flexible and realistic decision processes.

- **Coding theory and cryptography:** Nearrings are closely related to coding and automata theory. By incorporating IVLF soft ideals, one can study coding and cryptographic structures under approximate or uncertain conditions, which may support the design of error-tolerant codes and secure cryptographic protocols.

- **Knowledge representation and intelligent systems:** Artificial intelligence frequently requires handling vague, incomplete, or conflicting information. IVLF soft ideals provide an algebraic tool for representing such uncertainty in pattern recognition, control systems, and reasoning frameworks.

- **Mathematical modeling of uncertain data:** Many real-world data sets involve interval-valued or approximate measurements. The algebraic properties of IVLF soft ideals can be applied to formalize such data and to study stability or robustness of models in uncertain environments.

- **Extensions to other algebraic structures:** While this study focuses on nearrings, the same methodology can be transferred to rings, modules, semigroups, and automata. Extending the approach would provide a broader class of algebraic models capable of incorporating interval-valued fuzzy soft concepts.

Illustrative Example. Consider a supplier selection problem involving three suppliers $U = \{u_1, u_2, u_3\}$ and two evaluation parameters $X = \{\text{Cost}, \text{Reliability}\}$. The evaluations are uncertain, so each supplier is represented by an IVLF soft set $(\hat{f}, X)$ over U, where L is a bounded lattice of performance levels as shown in Figure 1.

Let the interval-valued membership of each supplier with respect to each parameter be given by $\hat{f}_x(u_i) = [l_{i1}, l_{i2}] \in L$. Using the associated interval-valued triangular norm $\mathcal{T}_1$ and conorm Ξ_2 defined earlier, the aggregated evaluation of a supplier can be expressed as

$$\hat{f}(u_i) = \mathcal{T}_1\big(\hat{f}_{\text{Cost}}(u_i), \hat{f}_{\text{Reliability}}(u_i)\big).$$

For a given threshold pair $(\hat{\lambda}, \hat{\gamma})$, the set of suppliers satisfying $\zeta_I(\hat{\lambda}, \hat{f}(u_i)) \geq \hat{\gamma}$ forms an IVLF soft ideal in the nearring N representing acceptable performance levels.

This ideal captures those alternatives whose interval-valued evaluations remain stable under both parameter aggregation and triangular norm composition. Hence, IVLF soft ideals provide a formal algebraic framework to model decision-making in uncertain or approximate environments.

6. Conclusion

This study explored triangular norm-based interval-valued L-fuzzy (IVLF) soft ideals in the setting of nearrings. By employing IVTN and IVTCN, the work extends existing approaches to fuzzy soft ideals and highlights their fundamental algebraic properties. The incorporation of thresholds provides a graded mechanism for membership evaluation, making the framework more adaptable to uncertainty in algebraic environments.

The investigation reinforces the theoretical structure of IVLF soft sets while also indicating their potential applications beyond abstract algebra. In particular, IVLF soft ideals can be applied in decision-making processes, knowledge modeling, and the study of vague or approximate algebraic systems. Since nearrings share close connections with coding theory, automata, and cryptography, the results presented here offer a useful basis for applying fuzzy and soft computing techniques in contexts where imprecision and uncertainty must be managed effectively.

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