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Perfect Bicoloring of the Quintic Graphs of Order at Most 10

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ABSTRACT: This paper studies the problem of finding perfect bicolorings in graphs of degree five and with at most ten vertices. A perfect bicoloring is defined as a partition of the vertex set into two subsets, where each subset induces a regular subgraph. Algebraic techniques are employed to construct parameter matrices that describe the structure of such bicolorings. After constructing these matrices, all possible parameter matrices for graphs of degree five with at most ten vertices are classified, and the cases that correspond to graphs admitting perfect bicolorings are identified.

Key Words: Perfect coloring, equitable partition, quintic graph, parameter matrices.

Contents

1. Introduction

A perfect coloring of a graph $Y = (V, E)$ with t colors is a coloring of the vertex set V that uses all t colors and satisfies the following condition: for any two colors r and s, every vertex of color s has the same number of neighbors of color r, denoted by f_{rs}. The matrix $F = (f_{rs})_{r,s\in\{1,\dots,t\}}$ is called the parameter matrix of the coloring.

Perfect t-colorings form a topic at the intersection of algebraic combinatorics, coding theory, and graph theory. In the literature, such colorings are also referred to as equitable partitions [1]. The study of completely regular codes in graphs has a long and well-established history. In 1973, Delsarte conjectured that Johnson graphs admit no perfect codes, a question that has since attracted considerable attention [2,3,4].

Alaeiyan [5] introduced the bipartite Ala graph $Ala(m; G; k)$ and examined its structural and spectral properties using eigenvalue analysis. Also, it is possible to compute the perfect coloring of the Ala graphs. Moreover, Alaeiyan et al. [6] studies how to find perfect 2- and 3-colorings of these graphs by comparing the eigenvalues of their adjacency matrices with those of parameter matrices. Fon-Der-Flaass classified all perfect bicolorings of hypercubes Q_n for $n < 24$ [7,8,9]. Alaeiyan provided a solution for the perfect 3-coloring of the Heawood graph [10]. In a related direction, the present paper determines all parameter matrices of perfect bicolorings of quintic graphs with at most ten vertices.

In this work, we focus on perfect bicolorings of graphs of degree five with at most ten vertices. A perfect bicoloring partitions the vertex set into two color classes such that each class induces a regular subgraph. We use algebraic methods to construct parameter matrices that encode the structural constraints of such colorings. Based on these matrices, we classify all feasible parameter configurations for quintic graphs of order at most ten and identify the graphs that admit a perfect bicoloring.

The structure of the paper is organized as follows. Section 2 introduced the necessary background on perfect bicolorings, parameter matrices, and eigenvalue conditions. Section 3 presented the main results and identified all parameter matrices that admitted perfect bicolorings for quintic graphs with at most ten vertices. Finally, Section 4 concluded the paper by summarizing the principal findings.

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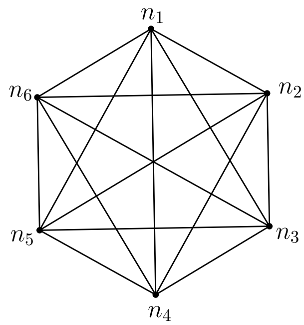
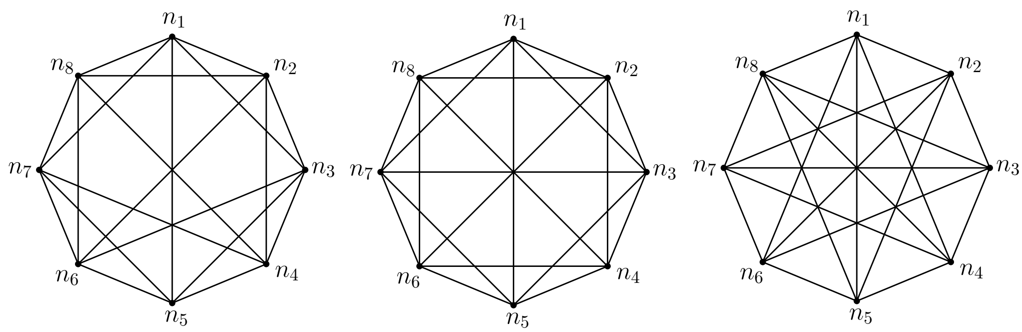


Y_1

Figure 1: Quintic graph of order 6



Y_2 Y_3 Y_4

Figure 2: Quintic graphs of order 8

2. Preliminaries

This section presents the fundamental ideas and definitions related to perfect bicoloring through the use of parameter matrices. These matrices record how vertices of different colors are connected and describe the adjacency structure of the graph. The conditions that help reduce the number of admissible parameter matrices are also outlined, allowing the analysis and the proof of the main results to become more manageable. Quintic graphs are 5-regular graphs in which every vertex has degree 5, and such graphs exist only for even numbers of vertices. The quintic graphs with up to 10 vertices are shown in Figures 1, 2, and 3.

Definition 1 *Let $Y = (V, E)$ be a connected graph and let t be a positive integer. A perfect t-coloring with parameter matrix $F = (f_{rs})_{r,s\in\{1,\ldots,t\}}$ is a surjective mapping*

$$\Gamma : V(Y) \longrightarrow \{1, \ldots, t\},$$

such that for every vertex $v \in V(Y)$ with $\Gamma(v) = r$, the vertex v has exactly f_{rs} neighbors of color s.

When $t = 2$, the colors are red and black in that order. We use R and B to represent the sets of red and black, respectively. We denote the parameter matrix by $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ and consider parameter matrices $\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ and $\begin{bmatrix} f_{22} & f_{21} \\ f_{12} & f_{11} \end{bmatrix}$ up to renaming the colors equal. We start by analyzing the necessary conditions for a perfect bicoloring of quintic graphs with at most 10 vertices, assuming a fixed parameter matrix $F = (f_{rs})_{r,s=1,2}$.

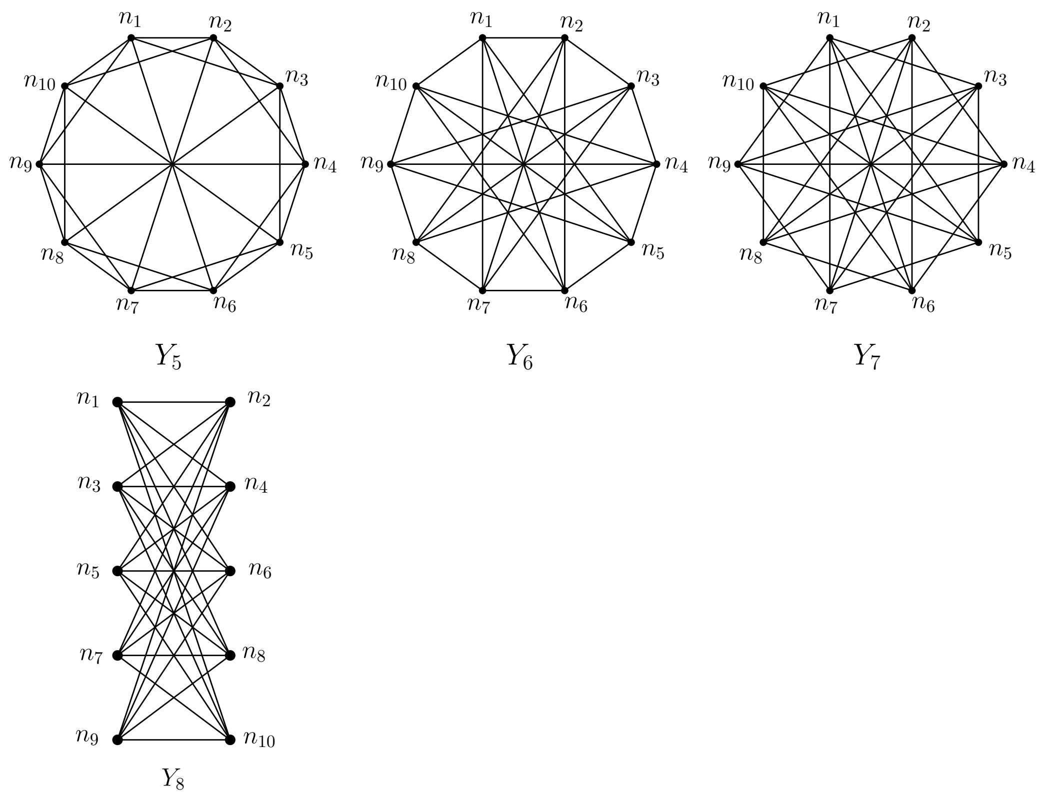


Figure 3: Quintic graphs of order 10

Remark 1 *Suppose that* Y *is a 5-regular graph, and* Γ *is a perfect bicoloring with matrix* $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$. *Then we have* $f_{11} + f_{12} = f_{21} + f_{22} = 5$.

Remark 2 *Suppose that* Γ *is a perfect bicoloring with matrix* $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ *in a connected graph* Y. *Then we have* $f_{12}, f_{21} \neq 0$.

An eigenvalue of the graph Y, denoted by θ, is a scalar satisfying $AX = \theta X$ for some nonzero vector X, where A is the adjacency matrix of Y. Similarly, a number η is called an eigenvalue of a perfect bicoloring with parameter matrix F if η is an eigenvalue of F.
The following theorem describes the relationship between these notions.

Theorem 1 *[2] If* Γ *is a perfect coloring of a graph* Y *with* t *colors, then any eigenvalue of* Γ *is also an eigenvalue of* Y.

We can obtain the eigenvalues of a parameter matrix using the following corollary.

Corollary 1 [11] Let Γ be a perfect bicoloring with parameter matrix $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ of a k-regular graph Y. Then the numbers $f_{11} - f_{21}$ and k are eigenvalues of Γ and hence eigenvalues of Y.

We now apply the lemma to determine the number of red vertices in a perfect bicoloring.

Lemma 1 *[2] Let* R *be the set of all red vertices in a perfect bicoloring of a graph* Y *with matrix* $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$. *Then we have*

$$|R| = \frac{|V(G)f_{21}}{f_{12} + f_{21}}.$$

From the given condition, it follows that any admissible parameter matrix for a perfect bicoloring of a quintic graph must be one of the following:

$$F_1 = \begin{bmatrix} 0 & 5 \\ 1 & 4 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 5 \\ 2 & 3 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 5 \\ 3 & 2 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & 5 \\ 4 & 1 \end{bmatrix},$$
$$F_5 = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}, \quad F_6 = \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}, \quad F_7 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \quad F_8 = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix},$$
$$F_9 = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}, \quad F_{10} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \quad F_{11} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}, \quad F_{12} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix},$$
$$F_{13} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \quad F_{14} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}, \quad F_{15} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

3. Main Results

In this section, we identify the parameter matrices for all perfect bicoloring of the quintic graphs with at most 10 vertices.

Theorem 2 *The graph Y_1 admits a perfect bicoloring only for the matrices F_1, F_7, F_{12}.*

Proof: To construct a perfect bicoloring of the graph Y_1, one of the parameter matrices $F_1, \ldots, F_{15}$ is selected. By applying Theorem 1 and Corollary 1, all matrices except F_1, F_7, F_{10}, and F_{12} are excluded. According to Lemma 1, the matrix F_{10} is also ruled out, since the number of red vertices would not be an integer. We now consider the following three mappings Γ_1, Γ_2, and Γ_3:

$$\Gamma_1(n_1) = R, \quad \Gamma_1(n_2) = \Gamma_1(n_3) = \Gamma_1(n_4) = \Gamma_1(n_5) = \Gamma_1(n_6) = B.$$
$$\Gamma_2(n_1) = \Gamma_2(n_2) = R, \quad \Gamma_2(n_3) = \Gamma_2(n_4) = \Gamma_2(n_5) = \Gamma_2(n_6) = B.$$
$$\Gamma_3(n_1) = \Gamma_3(n_3) = \Gamma_3(n_5) = R, \quad \Gamma_3(n_2) = \Gamma_3(n_4) = \Gamma_3(n_6) = B.$$

It is easy to see that Γ_1, Γ_2, and Γ_3 are perfect bicoloring with the matrices F_1, F_7, and F_{12}, respectively. □

Theorem 3 *The graph Y_2 admits a perfect bicoloring only for the matrix F_3.*

Proof: To construct a perfect bicoloring of the graph Y_2, one of the parameter matrices $F_1, \ldots, F_{15}$ is selected. By Theorem 1 and Corollary 1, all matrices except F_3, F_6, F_9, and F_{11} are excluded. According to Lemma 1, the matrices F_6 and F_{11} cannot occur, since the number of red vertices would not be an integer. The matrix F_9 also fails to yield a perfect bicoloring for the graph Y_2. To see this, assume that Y_2 has a perfect bicoloring with parameter matrix F_9. Since $f_{11} = 1$, two adjacent vertices must both be red, and all remaining vertices must be black. This implies $|B| = 4$, which contradicts the requirement $f_{22} = 3$. We now define the mapping Γ by

$$\Gamma(n_1) = \Gamma(n_4) = \Gamma(n_6) = R, \qquad \Gamma(n_2) = \Gamma(n_3) = \Gamma(n_5) = \Gamma(n_7) = \Gamma(n_8) = B.$$

It is easy to see that the mapping Γ is a perfect bicoloring with the matrix F_3. □

Theorem 4 *The graph Y_3 does not have a perfect bicoloring .*

Proof: To obtain a perfect bicoloring of the graph Y_3, one of the parameter matrices $F_1, \ldots, F_{15}$ is considered. By Theorem 1 and Corollary 1, the admissible matrices are

$$F_1,\ F_7,\ F_{10},\ F_{12},\ F_{14}.$$

According to Lemma 1, the matrices F_1, F_7, and F_{12} are excluded, since the number of red vertices would not be an integer. For the matrix F_{10}, we obtain $|R| = 2$ and $|B| = 6$. Using these values together with the entries of F_{10}, the vertices can be colored and the possible configurations can be examined. This leads to the following cases:

1. If $\Gamma(n_1) = \Gamma(n_2) = R$ and $\Gamma(n_3) = \Gamma(n_4) = \Gamma(n_5) = \Gamma(n_6) = \Gamma(n_7) = B$, then $\Gamma(n_8) = R$, which contradicts the second row of matrix F_{10}.

2. If $\Gamma(n_1) = \Gamma(n_2) = \Gamma(n_3) = \Gamma(n_8) = B$ and $\Gamma(n_4) = \Gamma(n_5) = \Gamma(n_6) = R$, then $\Gamma(n_7) = B$, but this case contradicts the second row of the matrix F_{10}. Thus, there is no perfect bicoloring for the graph Y_3 with matrix F_{10}.

Similarly, we can prove for matrix F_{14} as follows:

For matrix F_{14} we have $|R| = |B| = 4$. Using these numbers and the entries of the matrix F_{14}, we can color the vertices and check different cases. Thus we have the following possibilities:

3. If $\Gamma(n_1) = \Gamma(n_5) = \Gamma(n_7) = R$ and $\Gamma(n_2) = \Gamma(n_3) = \Gamma(n_4) = \Gamma(n_6) = B$, then $\Gamma(n_8) = R$, which contradicts the second row of matrix F_{14}.

4. If $\Gamma(n_1) = \Gamma(n_2) = \Gamma(n_3) = \Gamma(n_5) = B$ and $\Gamma(n_4) = \Gamma(n_6) = \Gamma(n_7) = R$, then $\Gamma(n_8) = R$, but this case contradicts the second row of matrix F_{14}. Hence, there is no perfect bicoloring for the graph Y_3 with matrix F_{14}.

□

Theorem 5 *The graph Y_4 has a perfect bicoloring only for the matrices F_9, F_{12} and F_{14}.*

Proof: To obtain a perfect bicoloring of the graph Y_4, one of the parameter matrices $F_1, \ldots, F_{15}$ is considered. By Theorem 1 and Corollary 1, the admissible matrices are

$$F_1,\ F_3,\ F_7,\ F_9,\ F_{10},\ F_{12},\ F_{14}.$$

According to Lemma 1, the matrices F_1 and F_7 are excluded, since the number of red vertices would not be an integer. We now show that the graph Y_4 admits no perfect bicoloring with parameter matrices F_3 or F_{10}. We first consider the matrix F_3. Suppose that Γ is a perfect bicoloring of Y_4 with parameter matrix F_3, and assume $\Gamma(n_1) = R$. Since $f_{12} = 5$, all vertices must be black except for the neighbors n_3 and n_7. By Lemma 1, the matrix F_3 requires $|R| = 3$, implying $\Gamma(n_3) = \Gamma(n_7) = R$. This contradicts the condition $f_{11} = 0$. It remains to show that Y_4 has no perfect bicoloring with parameter matrix F_{10}. Assume that such a bicoloring exists and let $\Gamma(n_1) = B$. From $f_{22} = 4$, it follows that

$$\Gamma(n_2) = \Gamma(n_4) = \Gamma(n_5) = \Gamma(n_6) = B, \quad \Gamma(n_8) = R.$$

Using $f_{11} = 2$, we obtain $\Gamma(n_3) = \Gamma(n_7) = R$. However, in this situation the black vertex n_4 becomes adjacent to three red vertices, contradicting the condition $f_{21} = 1$. We now consider the following three mappings Γ_1, Γ_2, and Γ_3:

$$\Gamma_1(n_1) = \Gamma_1(n_3) = \Gamma_1(n_5) = \Gamma_1(n_7) = R,$$
$$\Gamma_1(n_2) = \Gamma_1(n_4) = \Gamma_1(n_6) = \Gamma_1(n_8) = B.$$
$$\Gamma_2(n_1) = \Gamma_2(n_6) = \Gamma_2(n_7) = \Gamma_2(n_8) = R,$$
$$\Gamma_2(n_2) = \Gamma_2(n_3) = \Gamma_2(n_4) = \Gamma_2(n_5) = B.$$
$$\Gamma_3(n_1) = \Gamma_3(n_4) = \Gamma_3(n_5) = \Gamma_3(n_8) = R,$$
$$\Gamma_3(n_2) = \Gamma_3(n_3) = \Gamma_3(n_6) = \Gamma_3(n_7) = B.$$

Clearly,the mappings Γ_1, Γ_2 and Γ_3 are perfect bicoloring with the matrices F_9, F_{12} and F_{14}, respectively.

□

Theorem 6 *The graph Y_5 has a perfect bicoloring only for the matrices F_6 and F_{11}.*

Proof: To obtain a perfect bicoloring of the graph Y_5, one of the parameter matrices $F_1, \ldots, F_{15}$ is considered. By Theorem 1 and Corollary 1, the admissible matrices are F_1, F_6, F_7, F_{11}, and F_{12}. According to Lemma 1, the matrices F_1 and F_7 are excluded, since the number of red vertices would not be an integer. The matrix F_{12} also fails to produce a perfect bicoloring for Y_5. Assume that Γ is a perfect bicoloring of Y_5 with parameter matrix F_{12}. Let $\Gamma(n_1) = R$. From the first row of F_{12}, it follows that

$$\Gamma(n_2) = \Gamma(n_3) = R \quad \text{and} \quad \Gamma(n_6) = \Gamma(n_9) = \Gamma(n_{10}) = B.$$

Using the red vertices n_2 and n_3 together with $f_{12} = 3$, the remaining vertices must all be black, which contradicts the condition $|B| = 5$. We now consider the following two mappings, Γ_1 and Γ_2:

$$\begin{aligned}
&\Gamma_1(n_1) = \Gamma_1(n_6) = R,\\
&\Gamma_1(n_2) = \Gamma_1(n_3) = \Gamma_1(n_4) = \Gamma_1(n_5) = \Gamma_1(n_7) = \Gamma_1(n_8)\\
&= \Gamma_1(n_9) = \Gamma_1(n_{10}) = B.\\
&\Gamma_2(n_1) = \Gamma_2(n_2) = \Gamma_2(n_6) = \Gamma_2(n_7) = R,\\
&\Gamma_2(n_3) = \Gamma_2(n_4) = \Gamma_2(n_5) = \Gamma_2(n_8) = \Gamma_2(n_9) = \Gamma 2(n_{10}) = B.
\end{aligned}$$

Clearly, the mappings Γ_1 and Γ_2 are perfect bicoloring with the matrices F_6 and F_{11}, respectively. □

Theorem 7 *The graph Y_6 has a perfect bicoloring only for the matrix F_{12}.*

Proof: To obtain a perfect bicoloring of the graph Y_6, one of the parameter matrices $F_1, \ldots, F_{15}$ is considered. By Theorem 1 and Corollary 1, the admissible matrices are F_1, F_2, F_7, F_8, and F_{12}. According to Lemma 1, the matrices F_1, F_2, F_7, and F_8 cannot occur, since the number of red vertices would not be an integer. The mapping Γ is defined by

$$\Gamma(n_1) = \Gamma(n_2) = \Gamma(n_4) = \Gamma(n_8) = \Gamma(n_{10}) = R, \qquad \Gamma(n_3) = \Gamma(n_5) = \Gamma(n_6) = \Gamma(n_7) = \Gamma(n_9) = B.$$

It is clear that the mapping Γ yields a perfect bicoloring corresponding to the parameter matrix F_{12}. □

Theorem 8 *The graph Y_7 has a perfect bicoloring only for the matrices F_6 and F_{11}.*

Proof: To obtain a perfect bicoloring of the graph Y_7, one of the parameter matrices $F_1, \ldots, F_{15}$ is considered. By Theorem 1 and Corollary 1, the only admissible matrices are F_2, F_6, F_8, F_{11}, and F_{15}. According to Lemma 1, the matrices F_2 and F_8 cannot occur, since the number of red vertices would not be an integer. Moreover, the matrix F_{15} does not produce a perfect bicoloring of Y_7. Assume that Γ is a perfect bicoloring of Y_7 with parameter matrix F_{15}. Let $\Gamma(n_1) = R$. From the first row of F_{15}, it follows that

$$\Gamma(n_3) = \Gamma(n_5) = \Gamma(n_6) = \Gamma(n_7) = R \quad \text{and} \quad \Gamma(n_9) = B.$$

However, the black vertex n_9 would then be adjacent to three red vertices, contradicting the condition $f_{21} = 1$. We now examine the following two mappings:

$$\begin{aligned}
&\Gamma_1(n_1) = \Gamma_1(n_6) = R,\\
&\Gamma_1(n_2) = \Gamma_1(n_3) = \Gamma_1(n_4) = \Gamma_1(n_5) = \Gamma_1(n_7) = \Gamma_1(n_8)\\
&= \Gamma_1(n_9) = \Gamma_1(n_{10}) = B.\\
&\Gamma_2(n_1) = \Gamma_2(n_4) = \Gamma_2(n_6) = \Gamma_2(n_9) = R,\\
&\Gamma_2(n_2) = \Gamma_2(n_3) = \Gamma_2(n_5) = \Gamma_2(n_7) = \Gamma_2(n_8) = \Gamma_2(n_{10}) = B.
\end{aligned}$$

It is clear that the mappings Γ_1 and Γ_2 form perfect bicolorings associated with the parameter matrices F_6 and F_{11}, respectively. □

Theorem 9 *The graph Y_8 admitted a perfect bicoloring only for the matrices F_5 and F_6.*

Proof: To determine a perfect bicoloring of the graph Y_8, one of the parameter matrices $F_1, \ldots, F_{15}$ is considered. According to Theorem 1, Corollary 1, and Lemma 1, the only admissible parameter matrices are F_5 and F_6. The mappings Γ_1 and Γ_2 are defined as follows:

$$\Gamma_1(n_1) = \Gamma_1(n_3) = \Gamma_1(n_5) = \Gamma_1(n_7) = \Gamma_1(n_9) = R,$$
$$\Gamma_1(n_2) = \Gamma_1(n_4) = \Gamma_1(n_6) = \Gamma_1(n_8) = \Gamma_1(n_{10}) = B.$$
$$\Gamma_2(n_1) = \Gamma_2(n_2) = R,$$
$$\Gamma_2(n_3) = \Gamma_2(n_4) = \Gamma_2(n_5) = \Gamma_2(n_6) = \Gamma_2(n_7) = \Gamma_2(n_8)$$
$$= \Gamma_2(n_9) = \Gamma_2(n_{10}) = B.$$

Clearly, the mappings Γ_1 and Γ_2 were P–$2c$ colorings associated with the matrices F_5 and F_6, respectively. □

In conclusion, the main results of this paper were summarized in Table 1.

Table 1: Parameter matrices of the quintic graphs of order at most 10

Graphs	Parameter matrices
Y_1	F_1, F_7, F_{12}
Y_2	F_3
Y_3	no parameter matrix
Y_4	F_9, F_{12}, F_{14}
Y_5	F_6, F_{11}
Y_6	F_{12}
Y_7	F_6, F_{11}
Y_8	F_5, F_6

4. Conclusion

This paper examined the problem of finding perfect bicolorings in graphs of degree five and with at most ten vertices. A perfect bicoloring was defined as a partition of the vertex set into two subsets, where each subset induced a regular subgraph. Algebraic techniques were used to construct parameter matrices that described the structure of such bicolorings. After these matrices were constructed, all possible parameter matrices for graphs of degree five with at most ten vertices were classified, and the cases that corresponded to graphs admitting perfect bicolorings were identified.

Declaration of Competing Interest

The authors confirm that they have no competing interests to disclose.

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Data Availability

No data were generated or analyzed during the course of this research.

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