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On Distance-Based Arithmetic Radio Number of Standard Graph Classes

Ramya Hebbar*, Sooryanarayana B., Vishukumar M. and Sneha G. Kulkarni

ABSTRACT: Let $k \in \mathbb{Z}^+$ and let $\mathrm{G} = (V, E)$ be a connected graph of order n. An *arithmetic k-radio labeling* is a bijection $\eta : V \to \{1, 1+k, 1+2k, \ldots, 1+(n-1)k\}$ such that for any two distinct vertices $u, v \in V$, the condition $|\eta(u) - \eta(v)| > \mathrm{diam(G)} - \mathrm{dist}(u, v)$ is satisfied. The least such k is defined as the *arithmetic radio number* of G, denoted by $\mathcal{R}_a(\mathrm{G})$. In this paper, we establish exact values of $\mathcal{R}_a(\mathrm{G})$ for several families of graphs, including paths, cycles, squares of paths, and the join of graphs. Our results contribute to the broader context of distance-constrained labeling by combining structural graph properties with arithmetic progressions in labeling.

Key Words: Distance labeling, arithmetic radio labeling, radio number, radio graceful.

Contents

1. Introduction

The study of arithmetic radio numbers finds potential applications in frequency allocation problems, wireless sensor networks, and distributed computing systems, where efficient and interference-free assignment of communication channels is essential. Moreover, it extends naturally to task scheduling problems, VLSI design, and resource optimization in parallel processing, where constraints analogous to distance in graphs dictate feasible allocations.

Recent research in related areas, such as radio number, graceful labelings, metric dimension, and broadcast domination, suggests that arithmetic radio labeling could serve as a unifying framework for exploring new extremal properties and optimization strategies in graphs. This motivates a systematic study of arithmetic radio numbers for different families of graphs, with an emphasis on structural characterizations, bounds, and algorithmic aspects.

Let $\mathbb{F}$ be the family of all finite, simple, connected, nontrivial, and un-directed graphs. Let $\mathrm{G} \in \mathbb{F}$ with set of vertices V_G and set of edges E_G. For each pair $u_i, u_j \in V_G$ the distance between them is denoted by $\mathrm{dist}(u_i, u_j)$, *eccentricity* of $v \in V_G$ by $\mathrm{ecc}(v) = \max\{\mathrm{dist}(v, \xi) : \xi \in V_G\}$. The minimum and maximum value of the eccentricities among all the vertices of the graph are called its *radius* and *diameter*,

* Corresponding author.
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respectively, and are denoted by rad(G) and diam(G). A graph $G \in \mathbb{F}$ with $\text{diam}(G) = \text{rad}(G)$ is called *self-centric*. If a graph has a path containing all its vertices, then such a graph is called *semi-Hamiltonian.*

The notion of radio labeling arises from the channel assignment problem, as discussed in [5]. The concept was formally introduced by Chartrand et al. [3] and subsequently explored in various works such as [2,7,9,10,11,13,15,16,8]. A comprehensive survey of radio labeling is available in [4], and related developments are discussed in [2].

In this paper, radio arithmetic graceful labeling is introduce and that naturally extend the classical radio labeling concept by incorporating an arithmetic progression structure into the label set, leading to the notion of *arithmetic radio labeling.* We analyze the arithmetic radio number $R_a(G)$ for various graph classes and highlight its structural implications. A radio labeling (rd-labeling) is a vertex labeling with integers in such a way that the sum of differences of labeling of every pair of two distinct vertices with the distance between then shall exceed one more than its diameter. The largest label assigned by a labeling is known as its span. The minimum value of the span of an rd-labeling of G among all such rd-labelings is called radio number of G, denoted by the symbol $rn(G)$.

The concept *radio graceful* was initiated in [13]. Given $G \in \mathbb{F}$ is identified as radio graceful if it admits a rd-labeling with span is equal to its order. In 2016, Amanda J. N. [1] presented various such graphs and Sooryanarayana B. and Ramya in [6,14] investigated the relationship between the order and diameter of radio graceful graphs, provided necessary conditions for $G \in \mathbb{F}$ to be radio graceful, and characterized radio graceful graphs of the lower orders. Let $\mathbb{R}_g \subseteq \mathbb{F}$ be the family of all radio graceful graphs.

We recall the *generalized distance graphs* introduced in [12]. Let $G \in \mathbb{F}$ be of order n. Let $D = \{\text{dist}(u, v) : u, v \in V_G\}$ and $H \subseteq D$. The distance graph $D_G(H)$ of G associated with the distance subset H is defined on the vertices of G with and edges whenever the distance between then is in H. We simply write G_i to denote the distance graph $D_G(\{i\})$ throughout this paper. Further, a pair $u, v \in V_G$ is said to be a k_i pair in G if $\text{dist}(u, v) = i$.

We refer the following results in next section of this paper:

Theorem 1.1 ([7]) *For any $\ell \in \mathbb{Z}^+$ with $\ell \geq 3$,*

$$rn(C_\ell) = \ell + \left\lfloor \frac{\ell(\ell-2)}{8} \right\rfloor + \xi(\ell),$$

where $$\xi(\ell) = \begin{cases} 0 & \text{if } \ell \equiv 0 \pmod 4. \\ \frac{\ell-3}{2} & \text{if } \ell \equiv 1 \pmod 4. \\ \lfloor \frac{\ell-2}{4} \rfloor & \text{if } \ell \equiv 2, 3 \pmod 4. \end{cases}$$

Theorem 1.2 ([6]) *Let $G \in \mathbb{F}$ be of order n. Then $G \in \mathbb{R}_g \iff D_G(\{\text{diam}(G)\})$ is semi-Hamiltonian.*

In the next section, we introduce and compute a new invariant named as the arithmetic radio number. Arithmetic k-radio labeling is a generalization of radio graceful labeling.

2. Arithmetic k-Radio labeling

Throughout this paper we take $\mathbb{A}(a, k, n)$ be the set of first n terms an arithmetic progression whose first number is a and common difference k.

For any $G \in \mathbb{F}$ of order n and $k \in \mathbb{Z}^+$, a bijection $\phi : V_G \to \mathbb{A}(1, k, n)$ is called an arithmetic k-radio labeling (in short, ar_k-labeling) if $|\phi(\ell_1) - \phi(\ell_2)| + \text{dist}(\ell_1, \ell_2) \geq 1 + \text{diam}(G)$, for any two distinct $\ell_1, \ell_2 \in V_G$. We see that every graph admits an ar_k-labeling for every $k \geq \text{diam}(G)$. Here on wards, for each $k \in \mathbb{Z}^+$, we take $\mathbb{R}_g(k)$ be the set of all graphs in $\mathbb{F}$ that admits an ar_k-labeling. Then, $\mathbb{R}_g(1) = \mathbb{R}_g$ and $\mathbb{R}_g(k) \subseteq \mathbb{R}_g(k + 1)$ for every $k \in \mathbb{Z}^+$. Hence, we intend to find the least possible integer k for which $G \in \mathbb{R}_g(k)$ and call such k as arithmetic radio number of G, denoted by $R_a(G)$. An ar_k-labeling with $k = R_a(G)$ is simply called an arithmetic radio labeling of G.

Remark 2.1 *For any $G \in \mathbb{F}$, $1 \leq R_a(G) \leq \text{diam}(G)$.*

Remark 2.2 *A graph $G \in \mathbb{R}_g$ if and only if $R_a(G) = 1$.*

3. Bounds for Arithmetic radio number of a graph

Let $\mathrm{G} \in \mathbb{F}$ and $|V_G| = n$. Let $\eta : V_G \to \mathbb{A}(1,k,n)$ be a function. Then for any induced subgraph $H \subseteq \mathrm{G}$, by the η-chain of H, we mean an arrangement or a sequence $\omega_1, \omega_2, \ldots, \omega_{|V_H|}$ of the vertices of H such that $\eta(\omega_i) \leq \eta(\omega_j)$ whenever $i \leq j$.

Further, if η is an ar_k-labeling of G, then for the η-chain of $H \subseteq \mathrm{G}$, we see that

$$\begin{aligned} \eta(\omega_n) - \eta(\omega_1) &= \sum_{i=1}^{|V_H|-1} [\eta(\omega_{i+1}) - \eta(\omega_i)] \\ &= \sum_{i=1}^{|V_H|-1} [(\mathrm{diam(G)}+1) - \mathrm{dist}(\omega_{i+1}, \omega_i)] \\ &= (|V_H|-1)(\mathrm{diam(G)}+1) - \sum_{i=1}^{|V_H|-1} \mathrm{dist}(\omega_{i+1}, \omega_i). \end{aligned}$$

Thus, as $\eta(\omega_1) \geq 1$, we have;

$$\eta(\omega_n) \geq 1 + (|V(H)|-1)(\mathrm{diam(G)}+1) - \sum_{i=1}^{|V(H)|-1} \mathrm{dist}(\omega_{i+1}, \omega_i) \tag{3.1}$$

The R.H.S. of the inequality (3.1) is independent of every labeling but it depends only the arrangement or choice of the sequence of vertices in G, that is, the value of $S = \sum_{i=1}^{|V(H)|-1} \mathrm{dist}(\omega_{i+1}, \omega_i)$ for every induced subgraph H of G. Therefore, the optimality of the value of $\eta(\omega_n)$ among all rd-labeling η of G purely depends on the value of S and is minimum if and only if the value of S is maximum.

Lemma 3.1 *For any* $\mathrm{G} \in \mathbb{F}$ *of order* n,

$$\mathrm{R}_a(\mathrm{G}) \geq \left\lceil \frac{rn(\mathrm{G})-1}{n-1} \right\rceil.$$

Proof: Let $\mathrm{R}_a(\mathrm{G}) = k$ and η be any ar_k-labeling of G. Then η is also an rd-labeling of G and hence $\eta(\omega_n) \geq rn(\mathrm{G}) \Rightarrow 1 + (n-1)k \geq rn(\mathrm{G})$. So $k \geq \frac{rn(\mathrm{G})-1}{n-1} \Rightarrow k \geq \left\lceil \frac{rn(\mathrm{G})-1}{n-1} \right\rceil$ (since $k \in Z^+$). □

Lemma 3.2 *For every* $\mathrm{G} \in \mathbb{F}$,

$$\mathrm{R}_a(\mathrm{G}) \geq 1 + \mathrm{diam(G)} - \mathrm{rad(G)}$$

Proof: Let $\mathrm{R}_a(\mathrm{G}) = k$ and $\eta : V_G \to \mathbb{A}(1,k,|V_G|)$ be an ar_k-labeling of G. Let ω_i and ω_{i+1} be any two consecutive vertices in the η-chain of G. Then $|\eta(\omega_{i+1}) - \eta(\omega_i)| + \mathrm{dist}(\omega_{i+1}, \omega_i) \geq \mathrm{diam(G)} + 1 \Rightarrow k + \mathrm{dist}(\omega_{i+1}, \omega_i) \geq \mathrm{diam} + 1$. Hence

$$k \geq 1 + \mathrm{diam(G)} - \mathrm{dist}(\omega_{i+1}, \omega_i) \tag{3.2}$$

The inequality (3.2) shall hold for every consecutive vertices in the η-chain, in particular for the central vertex also. Thus taking ω_i as central vertex we see that $\mathrm{R}_a(\mathrm{G}) = k \geq 1 + \mathrm{diam(G)} - \mathrm{dist}(\omega_i, \omega_{i+1}) \geq 1 + \mathrm{diam(G)} - \mathrm{ecc}(\omega_i) = 1 + \mathrm{diam(G)} - \mathrm{rad(G)}$. □

Above Lemma 3.2 together with Remark 2.1 yeilds;

Corollary 3.3 *For every* $\mathrm{G} \in \mathbb{F}$ *with* $\mathrm{rad(G)} = 1$, $\mathrm{R}_a(\mathrm{G}) = \mathrm{diam(G)}$.

From the above Corollary 3.3, we see that $\mathrm{R}_a(\mathrm{G} + K_1) = 2$ for every graph $\mathrm{G} \not\cong K_n$. In particular for the wheel graph $W_{1,n}$ for every $n \geq 3$ and $\mathrm{R}_a(K_n) = 1$ for every $n \geq 1$.

Corollary 3.4 *If* $\mathrm{G} \in \mathbb{F}$ *and* $\mathrm{R}_a(\mathrm{G}) = 1$, *then* G *is self-centric.*

The converse of Corollary 3.4 need not be true in general. In fact, $\mathrm{R}_a(C_6) > 1$.

4. Arithmetic radio number of a path P_n

Throughout this section let $v_0, v_1, \dots, v_{n-1}$ denote the vertices of P_n in order.

Theorem 4.1 *For any $n \in \mathbb{Z}^+$, $\mathrm{R}_a(P_n) = \lceil \frac{n}{2} \rceil$.*

Proof: The diameter and radius of the path P_n are respectively $n-1$ and $\lfloor \frac{n}{2} \rfloor$. Hence by Lemma 3.2, we have

$$\mathrm{R}_a(\mathrm{G}) \geq (n-1) + 1 - \left\lfloor \frac{n}{2} \right\rfloor = n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil \tag{4.1}$$

Now, to prove the reverse inequality we execute an ar_k-labeling with $k = \lceil \frac{n}{2} \rceil$. For which, consider sequence $\omega_1, \omega_2, \dots, \omega_n$ of vertices of P_n where

(i) $\omega_1 = v_m, \omega_2 = v_0, \omega_3 = v_{m+1}, \omega_4 = v_1, \omega_5 = v_{m+2}, \dots, \omega_{n-2} = v_{m-2}, \omega_{n-1} = v_{2m-1}, \omega_n = v_{m-1}$, if $n = 2m$.

(ii) $\omega_1 = v_m, \omega_2 = v_0, \omega_3 = v_{m+1}, \omega_4 = v_1, \omega_5 = v_{m+2}, \dots, \omega_{n-2} = v_{2m-1}, \omega_{n-1} = v_{m-1}, \omega_n = v_{2m}$, if $n = 2m+1$.

Define a function $\eta : V_{P_n} \to \mathbb{A}(1, k, n)$ by, $\eta(\omega_1) = 1$; $\eta(\omega_{i+1}) = \eta(\omega_i) + \lceil \frac{n}{2} \rceil$, for $1 \leq i \leq n-1$.

Since $|\eta(\omega_i) - \eta(\omega_{i+1})| = k$, to show η is an ar_k-labeling it is sufficient to prove that η is a radio labeling.

Let $u = \omega_i$ and $v = \omega_j$ for some $1 \leq i, j \leq n, i \neq j$. If $|i-j| = 1$, then $\mathrm{dist}(\omega_i, \omega_j) \geq \lfloor \frac{n}{2} \rfloor$ and hence $|\eta(\omega_i) - \eta(\omega_j)| + \mathrm{dist}(\omega_i, \omega_j) \geq |i-j|k + \lfloor \frac{n}{2} \rfloor \geq \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n = \mathrm{diam}(P_n) + 1$. If $|i-j| \geq 2$, then $|\eta(\omega_i) - \eta(\omega_j)| \geq 2k$ and hence $|\eta(\omega_i) - \eta(\omega_j)| + \mathrm{dist}(\omega_i, \omega_j) \geq |i-j|2k + 1 \geq 2k + 1 = 2\lceil \frac{n}{2} \rceil + 1 > n = \mathrm{diam}(P_n) + 1$. Hence η is an ar_k-labeling with $k = \lceil \frac{n}{2} \rceil$. So

$$\mathrm{R}_a(P_n) \leq \left\lceil \frac{n}{2} \right\rceil. \tag{4.2}$$

Now, Inequality (4.1) and Inequality (4.2) together proves the theorem. □

5. Arithmetic radio number of a square path

Square path of a path P_n, denoted by P_n^2, is the graph on the vertices of P_n such that two vertices in P_n^2 are adjacent if and only if they are at a distance at most 2 in P_n. Throughout this section we take in order $v_0, v_1, v_2, \dots, v_{n-1}$ are the vertices of P_n.

Theorem 5.1 *For any $n \in \mathbb{Z}^+$, $\mathrm{R}_a(P_n^2) = \lceil \frac{n+1}{4} \rceil$.*

Proof: For $n \leq 3$, $P_n^2 \equiv K_n$ and hence $rn(P_n^2) = n$ and $\mathrm{R}_a(P_n^2) = 1$. Let $n \geq 4$. The diameter d and radius r of the graph P_n^2 are respectively $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n-1}{4} \rceil$. Therefore, by Lemma 3.2, we have;

$$\mathrm{R}_a(P_n^2) \geq 1 + d - r = 1 + \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n-1}{4} \right\rceil = \left\lceil \frac{n+1}{4} \right\rceil. \tag{5.1}$$

Now to prove the reverse inequality, Consider sequence

(i) $\omega_1 = v_d, \omega_2 = v_0, \omega_3 = v_{d+1}, \omega_4 = v_1, \omega_5 = v_{d+2}, \omega_6 = v_2, \dots, \omega_{n-1} = v_{2m-1}, \omega_n = v_{d-1}$, if $n = 2m$.

(ii) $\omega_1 = v_0, \omega_2 = v_{d+1}, \omega_3 = v_1, \omega_4 = v_{d+2}, \dots, \omega_{n-2} = v_{d-1}, \omega_{n-1} = v_{2m}, \omega_n = v_d$, if $n = 2m+1$.

Define a function $\eta : V \to \mathbb{A}(1, \lceil (n+1)/4 \rceil, n)$ by $\eta(\omega_1) = 1$; $\eta(\omega_{i+1}) = \eta(\omega_i) + \lceil \frac{n+1}{4} \rceil$, for $1 \leq i \leq n-1$. Then for any $u = \omega_i$ and $v = \omega_j$ with $i \neq j$, $|\eta(\omega_i) - \eta(\omega_j)| + d_{\mathrm{G}}(\omega_i, \omega_j) \geq |i-j|k + d_{\mathrm{G}}(\omega_i, \omega_j) \geq 2k + 1 \geq 2\lceil \frac{n+1}{4} \rceil + 1 \geq \lfloor \frac{n}{2} \rfloor + 1 = d + 1$ whenever $|i-j| \geq 2$. Further, when $|i-j| = 1$, $d_{\mathrm{G}}(\omega_i, \omega_j) \geq d_{\mathrm{G}}(v_0, v_d) = \lceil \frac{n-1}{4} \rceil$ and hence, $|\eta(\omega_i) - \eta(\omega_j)| + d_{\mathrm{G}}(\omega_i, \omega_j) \geq k + \lceil \frac{n-1}{4} \rceil = \lceil \frac{n+1}{4} \rceil + \lceil \frac{n-1}{4} \rceil = \lfloor \frac{n}{2} \rfloor + 1 = d + 1$. Thus η is a $(n+1)/4$-arithmetic radio labeling.Hence, $\mathrm{R}_a(P_n^2) = \lceil \frac{n+1}{4} \rceil$. □

6. Arithmetic radio number of some self-centric graphs

We recall that $\mathrm{R}_a(\mathrm{G}) = 1$ is possible only for self-centric graphs and the converse need not be true. In this section we consider some standard self-centric graphs for which $\mathrm{R}_a(\mathrm{G}) > 1$.

6.1. For the cycles

Throughout this section, C_n denotes a cycle with $V_{C_n} = \{v_i : 0 \le i \le n-1\}$ and $E_{C_n} = \{v_i v_{i+1 \pmod n} : 0 \le i \le n-1\}$. For each integer $n \ge 3$, the diameter of the cycle C_n is $\lfloor \frac{n}{2} \rfloor$.

Theorem 6.1 *For each $n \in \mathbb{Z}^+$ with $n \ge 3$,*

$$\mathrm{R}_a(C_n) = \begin{cases} \lceil \frac{n^2+6n-8}{8(n-1)} \rceil, & if \quad n \equiv 0 \pmod 4. \\ \lceil \frac{n+3}{8} \rceil, & if \quad n \equiv 1 \pmod 4. \\ \lceil \frac{n^2+4n-4}{8(n-1)} \rceil, & if \quad n \equiv 2 \pmod 4 \ \& \ n \not\equiv 10 \pmod{16}. \\ \lceil \frac{n^2+4n-4}{8(n-1)} \rceil + 1, & if \quad n \equiv 2 \pmod 4 \ \& \ n \equiv 10 \pmod{16}. \\ \lceil \frac{n+5}{8} \rceil, & if \quad n \equiv 3 \pmod 4. \end{cases}$$

Proof: In view of Theorem 1.1 and Lemma 3.1,

$$\mathrm{R}_a(\mathrm{G}) \ge \left\lceil \frac{rn(C_n) - 1}{n-1} \right\rceil = \begin{cases} \left\lceil \frac{\frac{n^2+6n}{8} - 1}{n-1} \right\rceil = \left\lceil \frac{n^2+6n-8}{8(n-1)} \right\rceil & \text{if } n \equiv 0 \pmod 4 \\ \left\lceil \frac{\frac{n^2+2n+5}{8} - 1}{n-1} \right\rceil = \left\lceil \frac{n+3}{8} \right\rceil & \text{if } n \equiv 1 \pmod 4 \\ \left\lceil \frac{\frac{n^2+4n+4}{8} - 1}{n-1} \right\rceil = \left\lceil \frac{n^2+4n-4}{8(n-1)} \right\rceil & \text{if } n \equiv 2 \pmod 4 \\ \left\lceil \frac{\frac{n^2+4n+3}{8} - 1}{n-1} \right\rceil = \left\lceil \frac{n+5}{8} \right\rceil & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

However, $\mathrm{R}_a(\mathrm{G}) \ge \left\lceil \frac{n^2+4n-4}{8(n-1)} \right\rceil + 1$ if $n \equiv 10 \pmod{16}$. In fact, for any ar_k-lableling η of C_n with η-chain $\omega_1, \omega_2, \ldots, \omega_n$, to label any two vertices with difference k, the minimum distance between them must be $l = (1+d) - k$. Also, if we use $d(\omega_i, \omega_{i+1}) > l$, for any $1 \le i \le n-1$ then we cannot label ω_3. So, $d(\omega_i, \omega_{i+1}) = l$ for any $1 \le i \le n-1$. But, with this difference $k = 1 + d - d(\omega_i, \omega_{i+1})$, the funciton η label the vertices either only in clockwise direction or only in anticlockwise direction and in each case it label only till vertex $\omega_{\frac{n}{2}}$ and cannot label $\omega_{\frac{n}{2}+1}$ whenever $k = \left\lceil \frac{n^2+4n-4}{8(n-1)} \right\rceil$. Hence, $k > \left\lceil \frac{n^2+4n-4}{8(n-1)} \right\rceil$.

Sufficiency: We now show that the lower limits established above are actually achievable in each of the cases by executing an arithmetic radio labeling. Let k be the lower bound obtained above in each of four cases. Let $l = (1+d) - k$, where d is diameter of cycle C_n. Then $l = 1 + \lfloor \frac{n}{2} \rfloor - k$. Define a function $\eta : V \to \mathbb{A}(1, k, n)$ such that $\eta(v_0) = 1$ and for each $i = 1, 2, 3, \ldots,$ after labeling the vertex v_{i-1}, label the vertex v_j as $\eta(v_j) = \eta(v_i) + k$, where j is the least positive integer greater than or equal to $i + l \pmod n$ such that the vertex v_j is non-labeled.

Then in the η-chain $\omega_1, \omega_2, \ldots, \omega_n$ of C_n we observe that $l \le \mathrm{dist}(\omega_i, \omega_{i+1}) \le 2l$ and $|\eta(\omega_{i+1}) - \eta(\omega_i)| = k$. So, to show η is a rd-labeling it is sufficient to check for the vertices $\omega_{i+2 \pmod n}, \omega_{i+3 \pmod n}$ and $\omega_{i+\tau \pmod n}$, where $\tau \ge 4$ with ω_i. Let ω_i and $w_{j \pmod n} \in V_{C_n}$ and $j \ge i$ and $i \in \mathbb{Z}_n$ be arbitrary.

Case 1: $n \equiv 0 \pmod 4$.

In this case, we have three subcases as follows.

Subcase 1: $j = i + 2 \pmod n$.

In this case,

$$\begin{aligned} |\eta(\omega_j)-\eta(\omega_i)|+\text{dist}(\omega_j,\omega_i) &\geq 2k+n-2l-1 = 2k+n-2(1+d-k)-1 \\ &= 4k-3 \text{ (since } d=\tfrac{n}{2}) \\ &= 4\left\lceil \frac{n^2+6n-8}{8(n-1)} \right\rceil -3 \\ &= \begin{cases} \frac{n}{2}+1 \geq d+1 & \text{if } n\equiv 0 \pmod 8 \\ \frac{n}{2}+3 \geq d+1 & \text{if } n\equiv 4 \pmod 8 \end{cases} \end{aligned}$$

Subcase 2: $j = i+3 \pmod n$.
In this case,

$$\begin{aligned} |\eta(\omega_j)-\eta(\omega_i)|+\text{dist}(\omega_j,\omega_i) &\geq 3k+3l-n \\ &= 3(1+d)-n \text{ (since } l=1+d-k) \\ &= 3+3d-2d \geq d+1. \end{aligned}$$

Subcase 3: $j = i+\tau \pmod n$ for any $\tau \geq 4$.
In this case,

$$\begin{aligned} |\eta(\omega_j)-\eta(\omega_i)|+\text{dist}(\omega_j,\omega_i) &\geq 4k+1 \\ &> 4k-3 \geq d+1 \text{ (by Subcase 1).} \end{aligned}$$

Case 2: $n \equiv 1 \pmod 4$.
Again we have three subcases as above.

Subcase 1: $j = i+2 \pmod n$.
We first prove the case $n \neq 9$. In this case,

$$\begin{aligned} |\eta(\omega_j)-\eta(\omega_i)|+\text{dist}(\omega_j,\omega_i) &= 2k+n-2l \text{ (since } n \text{ and } l \text{ are relative primes)} \\ &= 2k+n-2(1+d-k) = 4k-1 \ (\because d=\tfrac{n-1}{2}) \\ &= 4\left\lceil \frac{n+3}{8} \right\rceil -1 \\ &= \begin{cases} \frac{n+5}{2} > \lfloor \frac{n}{2} \rfloor +1 \geq d+1 & \text{if } n\equiv 1 \pmod 8 \\ \frac{n+1}{2} = \lfloor \frac{n}{2} \rfloor +1 \geq d+1 & \text{if } n\equiv 5 \pmod 8 \end{cases} \end{aligned}$$

Now, if $n=9$, then $k=2$, $l=3$, so $\gcd(n,l)\neq 1$ and in this case,

$$|\eta(\omega_j)-\eta(\omega_i)|+\text{dist}(\omega_j,\omega_i) \geq 2k+n-2l-1 = 4k-2 = 4(2)-2 = 6 \geq 5 = 4+1 = d+1.$$

Subcase 2: $j = i+3 \pmod n$.
In this case,

$$\begin{aligned} |\eta(\omega_j)-\eta(\omega_i)|+\text{dist}(\omega_j,\omega_i) &\geq 3k+3l-n \\ &= 3(1+d)-n = 3(1+d)-(2d+1) > d+1. \end{aligned}$$

Subcase 3: $j = i+\tau \pmod n$ for any $\tau \geq 4$.
In this case,

$$|\eta(\omega_j)-\eta(\omega_i)|+\text{dist}(\omega_j,\omega_i) \geq 4k+1 > 4k-1 > d+1. \text{ (by Subcase 1)}$$

Case 3: $n \equiv 2 \pmod 4$.
We now consider the three subcases as follows.

Subcase 1: $j = i + 2 \pmod n$.

Two subcases of this case are;

Subcase 1a: $n \not\equiv 10 \pmod{16}$.

We first prove for $n \neq 6$.

$$\begin{aligned} |\eta(\omega_j) - \eta(\omega_i)| + \text{dist}(\omega_j, \omega_i) &= 2k + n - 2l \text{ (since } n \text{ and } l \text{ are relatively primes)} \\ &= 2k + n - 2(1 + d - k) = 4k - 2 \text{ (since } n = 2d) \\ &= 4\left\lceil \frac{n^2 + 4n - 4}{8(n-1)} \right\rceil - 2 \\ &= \begin{cases} \frac{n}{2} + 1 \geq d + 1 & \text{if } n \equiv 2 \pmod 8 \\ \frac{n}{2} + 3 \geq d + 1 & \text{if } n \equiv 6 \pmod 8 \end{cases} \end{aligned}$$

Now, when $n = 6$, we get $k = 2$, $l = 2$, $d = 3$ and hence,

$$|\eta(\omega_j) - \eta(\omega_i)| + \text{dist}(\omega_j, \omega_i) \geq 2k + n - 2l - 1 = 4 + 6 - 4 - 1 = 5 > 3 + 1 = d + 1.$$

Subcase 1b: $n \equiv 10 \pmod{16}$.

In this case,

$$\begin{aligned} |\eta(\omega_j) - \eta(\omega_i)| + \text{dist}(\omega_j, \omega_i) &= 2k + n - 2l - 1 \\ &= 2k + n - 2(1 + d - k) - 1 \\ &= 4k - 3 \text{ (since } n = 2d) \\ &= 4(\left\lceil \frac{n^2 + 4n - 4}{8(n-1)} \right\rceil + 1) - 3 \\ &= \frac{n}{2} + 4 > d + 1. \end{aligned}$$

Subcase 2: $j = i + 3 \pmod n$.

In this case,

$$\begin{aligned} |\eta(\omega_j) - \eta(\omega_i)| + \text{dist}(\omega_j, \omega_i) &\geq 3k + 3l - n \\ &= 3(1 + d) - 2d \text{ (since } n = 2d) \\ &= d + 3 > d + 1. \end{aligned}$$

Subcase 3: $j = i + \tau$ for any $\tau \geq 4$.

In this case,

$$|\eta(\omega_j) - \eta(\omega_i)| + \text{dist}(\omega_j, \omega_i) \geq 4k + 1 > 4k - 2 > d + 1. \text{ (by Subcase 1).}$$

Case 4: $n \equiv 3 \pmod 4$.

The three subcases are;

Subcase 1: $i = j + 2 \pmod n$.

In this case,

$$\begin{aligned} |\eta(\omega_j) - \eta(\omega_i)| + \text{dist}(\omega_j, \omega_i) &\geq 2k + n - 2l - 1 \\ &= 2k + n - 2(1 + d - k) - 1 \\ &= 4k - 2 \text{ (since } n = 2d + 1) \\ &= 4\left\lceil \frac{n+5}{8} \right\rceil - 2 \\ &= \begin{cases} \frac{n+1}{2} \geq d + 1 & \text{if } n \equiv 3 \pmod 8 \\ \frac{n+5}{2} \geq d + 1 & \text{if } n \equiv 7 \pmod 8 \end{cases} \end{aligned}$$

Subcase 2: $j = i + 3 \pmod n$.
In this case,

$$|\eta(\omega_j) - \eta(\omega_i)| + \mathrm{dist}(\omega_j, \omega_i) \geq 3k + 3l - n = 3(1 + d) - (2d + 1) > d + 1.$$

Subcase 3: $j = i + \tau \pmod n$ for any $\tau \geq 4$.

In this case, $|\eta(\omega_j) - \eta(\omega_i)| + \mathrm{dist}(\omega_j, \omega_i) = 4k + 1 > 4k - 2 > d + 1$. (by Subcase 1)

Also, in all the above four cases, we have

$$|\eta(\omega_{i+1}) - \eta(\omega_i)| + \mathrm{dist}(\omega_{i+1}, \omega_i) \geq k + l = k + (1 + d) - k = d + 1.$$

Thus, η is a rd-labeling. Hence the theorem. □

6.2. For join of Paths

The join of $G_1, G_2 \in \mathbb{F}$, denoted by $G_1 + G_2$, is an element of $\mathbb{F}$ such that $V_{G_1+G_2} = V_{G_1} \bigcup V_{G_2}$ and $E_{G_1+G_2} = E_{G_1} \cup E_{G_2} \cup \{uv : u \in V_{G_1} \text{ and } v \in V_{G_2}\}$. From the definition it is clear that joint of K_n and k_m is K_{m+n}. Hence we consider join of two graphs of which at least one of them is non-complete. For such graphs, the diameter of $G_1 + G_2$ is 2 and hence for any η-chain $\omega_1, \omega_2, \ldots, \omega_{|V_{G_1}|+|V_{G_2}|}$ of a rd-labeling η of $G_1 + G_2$, by Equation (3.1) we have;

$$\begin{aligned} \eta(\omega_{|V_{G_1}|+|V_{G_2}|}) &\geq 1 + (|V_{G_1}| + |V_{G_2}| - 1)(2 + 1) - Max\ S \\ &= 3(|V_{G_1}| + |V_{G_2}|) - 2 - Max\ S \end{aligned} \tag{6.1}$$

where $S = \sum_{i=1}^{2} \xi_i k_i$, with $\xi_2 + \xi_1 = |V_{G_1}| + |V_{G_2}| - 1$, $\xi_i \leq \min_i\{|E_{G_i}|, |V_{G_i}| - 1\}$, $k_i = i$, $1 \leq i \leq 2$, and $\xi_1, \xi_2 \in \mathbb{Z}^+ \cup \{0\}$.

Theorem 6.2 *For any $n, m \in \mathbb{Z}^+$ with $n \geq m > 1$,*

$$rn(P_n + P_m) = \begin{cases} n + m, & if \quad n = 2. \\ n + m + 3, & if \quad n = 3. \\ n + m + 2, & if \quad n \geq 4 \ \&\ m \in \{2, 3\}. \\ n + m + 1, & if \quad n, m \geq 4. \end{cases}$$

Proof: Let $G = P_n + P_m$ and $G_i = D_G(\{i\})$. If $m = n = 2$, then $G \equiv K_{m+n}$ which is radio graceful and hence $rn(G) = |V_G)| = n + m$. We now consider the other cases. Let η be any rd-labeling of G and $\omega_1, \omega_2, \ldots, \omega_{m+n}$ be the η-chain of G.

Case 1: $m = 2$.
If $n = 3$, then $G_2 \cong P_2$ and hence $\xi_2 \leq |E(P_2)| = 1$. Choosing $\xi_2 = 1$, the maximum possible value, we see that $\xi_1 = 3$ (since $\xi_2 + \xi_1 = m + n - 1 = 4$). So, $Max\ S = \sum_{i=1}^{2} \xi_i k_i = 1k_2 + 3k_1 = 1(2) + 3(1) = 5$. Hence from Equation (6.1), we get $rn(P_n + P_m) = min\{\eta(\omega_{m+n})\} \geq 3(m + n) - 2 - Max\ S = 3(2 + 3) - 2 - 5 = 8$.

If $n \geq 4$, then $G_2 \cong \overline{P_n}$ and hence $\xi_2 \leq |V(\overline{P_n}) - 1| = n - 1$. Choosing $\xi_2 = n - 1$, the maximum possible value, we see that $\xi_1 = 2$ (since $\xi_2 + \xi_1 = 2 + n - 1 = n + 1$). So, $Max\ S = \sum_{i=1}^{2} \xi_i k_i = (n - 1)1k_2 + 2k_1 = (n - 1)(2) + 2(1) = 2n$. Hence from Equation (6.1), we get $rn(P_n + P_m) = \min_\eta\{\eta(\omega_{m+n})\} \geq 3(m + n) - Max\ S - 2 = 3(2 + n) - 2n - 2 = n + 4$.

Case 2: $m = 3$.
If $n = 3$, then $G_2 \cong 2P_2$ and hence $\xi_2 \leq |E(2P_2)| = 2$. Choosing $\xi_2 = 2$, the maximum possible value, we see that $\xi_1 = 3$ (since $\xi_2 + \xi_1 = m + n - 1 = 5$). So, $Max\ S = \sum_{i=1}^{2} \xi_i k_i = 2k_2 + 3k_1 = 2(2) + 3(1) = 7$. Hence from Equation (6.1), we get $rn(P_n + P_m) = \min_\eta\{\eta(\omega_{m+n})\} \geq 3(m + n) -$

$Max\ S - 2 = 3(6) - 7 - 2 = 9$.
If $n \geq 4$, then $G_2 \cong P_2 \cup \overline{P}_n$ and hence $\xi_2 \leq |E_{P_2}| + [|V_{\overline{P}_n}| - 1] = 1 + n - 1 = n$. Choosing $\xi_2 = n$, the maximum possible value, we see that $\xi_1 = 2$ (since $\xi_2 + \xi_1 = m + n - 1 = 3 + n - 1 = n + 2$). So, $Max\ S = \sum_{i=1}^{2} \xi_i k_i = nk_2 + 2k_1 = n(2) + 2(1) = 2n + 2$. Hence from Equation (6.1), we get $rn(P_n + P_m) = \min_{\eta}\{\eta(\omega_{m+n})\} \geq 3(m + n) - Max - 2\ S = 3(3 + n) - (2n + 2) - 2 = n + 5$.

Case 3: $m \geq 4$.
In this case, $G_2 \equiv \overline{P}_m \cup \overline{P}_n$ and hence $\xi_2 \leq [|V_{\overline{P}_n}| - 1] + [|V_{\overline{P}_m}| - 1] = [n - 1] + [m - 1] = n + m - 2$. Choosing $\xi_2 = n + m - 2$, the maximum possible value, we see that $\xi_1 = 1$ (since $\xi_2 + \xi_1 = n + m - 1$). So, $Max\ S = \sum_{i=1}^{2} \xi_i k_i = (n + m - 2)k_2 + 1k_1 = (n + m - 2)(2) + 1(1) = 2(m + n) - 3$. Hence from Equation (6.1), we get $rn(P_n + P_m) = \min_{\eta}\{\eta(\omega_{m+n})\} \geq 3(m + n) - 2 - Max\ S = 3(m + n) - 2 - [2(m + n) - 3] = m + n + 1$.

Sufficiency: The results follows for the cases $m \in \{2, 4\}$ and $n \in \{3, 4\}$ from the labeling of the graph shown in the Figure 1. For $m \geq 2$ and $n \geq 5$, let element of V_{P_n} and V_{P_m} be respectively

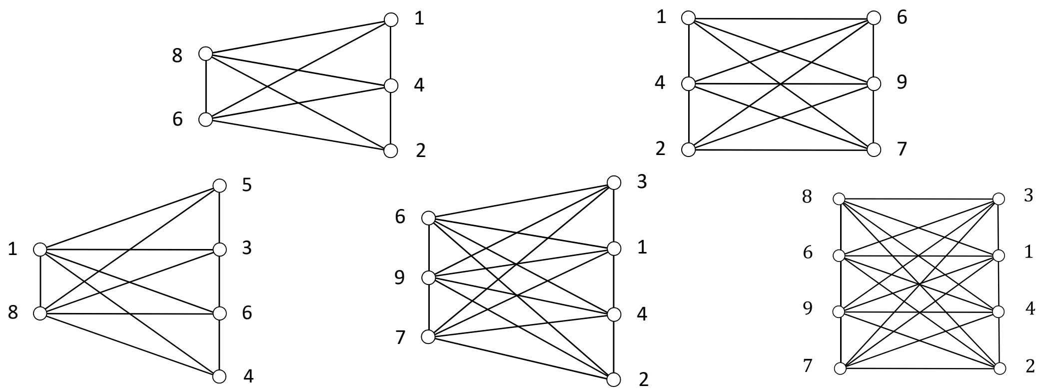

Figure 1: Radio labeling of $P_m + P_n$ for $2 \leq m \leq 4$ and $3 \leq n \leq 4$.

$v_0, v_1, v_2, \ldots, v_{n-1}$ and $u_o, u_1, u_2, \ldots, u_{m-1}$. Define a sequence of lenght $n + m$ of vertices in $P_n + P_m$ by $\omega_1 = v_0, \omega_2 = v_2, \omega_3 = v_4, \ldots, \omega_{\lfloor \frac{n-1}{2} \rfloor} = v_{2(\lfloor \frac{n-1}{2} \rfloor - 1)}, \omega_{\lfloor \frac{n-1}{2} \rfloor + 1} = v_{2\lfloor \frac{n-1}{2} \rfloor}, \omega_{\lfloor \frac{n-1}{2} \rfloor + 2} = v_1, \omega_{\lfloor \frac{n-1}{2} \rfloor + 3} = v_3$, $\ldots$, $\omega_{n-1} = v_{2\lfloor \frac{n}{2} \rfloor - 3}$, $\omega_n = v_{2\lfloor \frac{n}{2} \rfloor - 1}$, $\omega_{n+1} = u_0$, $\omega_{n+2} = u_2, \omega_{n+3} = u_4, \ldots, \omega_{n+\lfloor \frac{m-1}{2} \rfloor} = u_{2(\lfloor \frac{m-1}{2} \rfloor - 1)}$, $\omega_{n+\lfloor \frac{m-1}{2} \rfloor + 1} = u_{2\lfloor \frac{m-1}{2} \rfloor}, \omega_{n+\lfloor \frac{m-1}{2} \rfloor + 2} = u_1, \omega_{n+\lfloor \frac{m-1}{2} \rfloor + 3} = u_3, \ldots, \omega_{n+m-1} = u_{2\lfloor \frac{m}{2} \rfloor - 3}, \omega_{n+m} = u_{2\lfloor \frac{m}{2} \rfloor - 1}$.

Now, for this sequence we define a function $\eta : V(P_n + P_m) \to \mathbb{Z}_{n+m+1}/_{\{0\}}$ by $\eta(\omega_1) = 1$ and $\eta(\omega_i) = \eta(\omega_{i-1}) + 3 - \text{dist}(\omega_i, \omega_{i+1})$. The function for $P_3 + P_7$ and $P_6 + P_8$ are illustrated in Figure 2 and Figure 3 respectively.

This function is clearly a rd-labeling, because for all $\tau > 1$, $\eta(\omega_{i+\tau}) - \eta(\omega_i) + \text{dist}(\omega_i, \omega_{i+\tau}) \geq [\eta(\omega_{i+\tau}) - \eta(\omega_{i+\tau-1})] + [\eta(\omega_{i+\tau-1}) - \eta(\omega_{i+\tau-2})] + \text{dist}(\omega_i, \omega_{i+\tau}) \geq [1] + [1] + 1 = 3 = 1 + d$. Also for $\tau = 1$, $[\eta(\omega_{i+\tau}) - \eta(\omega_i)] + \text{dist}(\omega_i, \omega_{i+\tau}) \geq [1] + 2 = 1 + d$ if $i \neq n$, and $[\eta(\omega_{i+\tau}) - \eta(\omega_i)] + \text{dist}(\omega_i, \omega_{i+\tau}) \geq [2] + 1 = d + 1$ if $i = n$. Therefore, $rn(G) \leq \text{Span}\,\eta = \eta(\omega_{n+m}) = 1 + 3(n + m - 1) - \sum_{i=1}^{n+m-1} d_G(\omega_i, \omega_{i+1}) = 1 + 3(m + n - 1) - (2 \times (m + n - 2) + 1) = m + n + 1$.

Hence $rn(G) = n + m + 1$ for all $m, n \geq 4$. □

Further, the graph $P_m + P_n \in \mathbb{R}_g \iff m, n \in \{1, 2\}$. Hence $R_a(P_m + P_n) = 1 \iff m, n \in \{1, 2\}$. Hence, we record this in the form of the following corollary.

Corollary 6.3 *For any $n, m \in \mathbb{Z}^+$ with $n \geq m$,*

$$R_a(P_n + P_m) = \begin{cases} 1, & \text{if } n \leq 2. \\ 2, & \text{if } n > 2. \end{cases}$$

Theorem 6.4 *For $G, \Gamma \in \mathbb{F}$, if $\overline{G}$ and $\overline{\Gamma}$ are semi-hamiltonian, then $rn(G + \Gamma) = |V_G| + |V_\Gamma| + 1$.*

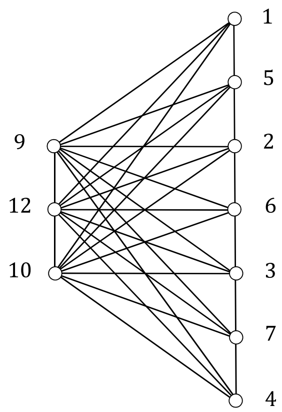

Figure 2: A radio labeling of $P_3 + P_7$.

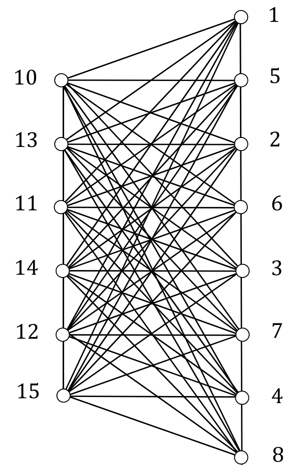

Figure 3: A radio labeling of $P_6 + P_8$.

Proof: Let $|V_{\mathrm{G}}| = m$ and $|V_\Gamma| = n$. Let $\omega_1, \omega_2, \ldots, \omega_m$ and $y_1, y_2, \ldots, y_n$ be the Hamiltonian paths of $\overline{\mathrm{G}}$ and $\overline{\Gamma}$ respectively. Then, in the graph $\mathrm{G}+\Gamma$, $\mathrm{dist}(\omega_i, \omega_{i+1}) = \mathrm{dist}(y_j, y_{j+1}) = 2$ and $\mathrm{dist}(\omega_i, y_j) = 1$, for $1 \le j < n$, $1 \le i < m$. Now, consider the sequence $\{w_i\}_{i=1,2,\ldots,m+n}$, where $w_{m+\xi} = y_\xi$ for $1 \le \ \xi \le n$. Define a function $\eta : V_{\mathrm{G}+\Gamma} \to \mathbb{Z}_{m+n+1}/_{\{0\}}$ by $\eta(\omega_1) = 1$; $\eta(\omega_i) = \eta(\omega_{i-1}) + 3 - \mathrm{dist}(\omega_i, \omega_{i+1})$, we can see as in the proof of Theorem 6.2 that η is a rd-labeling with span $m+n+1$. Hence $rn(\mathrm{G}+\Gamma) \le m+n+1$.

On the other hand, $\mathrm{G}_2 = D_{\mathrm{G}+\Gamma}(\{2\}) \equiv \overline{\mathrm{G}} \cup \overline{\Gamma}$, a disconnected graph, it follows that G_2 is not semi-hamiltonian and hence $\mathrm{G}+\Gamma$ is not radio graceful. So, $\mathrm{R}_a(\mathrm{G}) = 2$ (since $\mathrm{diam}(\mathrm{G}+\Gamma) = 2$ and by Remark 2.1) and $rn(\mathrm{G}+\Gamma) \ge |V(\mathrm{G}+\Gamma)| + 1 = m+n+1$. Thus, $rn(\mathrm{G}+\Gamma) = m+n+1$. □

Corollary 6.5 *For* $\mathrm{G}, \Gamma \in \mathbb{F}$*, if* $\overline{\mathrm{G}}$ *and* $\overline{\Gamma}$ *are semi-Hamiltonian, then* $\mathrm{R}_a(\mathrm{G}+\Gamma) = 2$.

6.3. For the Tietze's graph

Theorem 6.6 *For the Tietze's graph* G *of Figure 4,* $rn(\mathrm{G}) = 13$ *and* $\mathrm{R}_a(\mathrm{G}) = 2$.

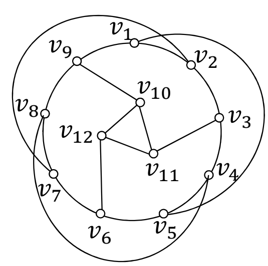

Figure 4: Tietze's graph

Proof: The graph $\mathrm{G}_3 = D_{\mathrm{G}}(\{3\})$ contains a component isomorphic to C_3 induced by $\{v_3, v_6, v_9\}$ and hence it is not semi-hamiltonian (being disconnected). Therefore, by Theorem 1.2, G is not radio graceful. So, $rn(\mathrm{G}) \ge |V_G| + 1 = 13$. Now, it is easy to verify that the labeling η defined by $\eta(v_1) = 4$, $\eta(v_2) = 1$, $\eta(v_3) = 10$, $\eta(v_4) = 3$, $\eta(v_5) = 13$, $\eta(v_6) = 8$, $\eta(v_7) = 11$, $\eta(v_8) = 6$, $\eta(v_9) = 9$, $\eta(v_{10}) = 2$, $\eta(v_{11}) = 12$, and $\eta(v_{12}) = 5$ is a rd-labeling with $rn(\eta) = 13$ and hence $rn(\mathrm{G}) \le 13$. Thus, $rn(\mathrm{G}) = 13$.

Further, as G is not radio graceful, we have $\mathrm{R}_a(\mathrm{G}) \ge 2$ (by Remark 2.2). To prove the reverse inequality, consider the labeling η of G defined by $\eta(v_1) = 11$, $\eta(v_2) = 3$, $\eta(v_3) = 21$, $\eta(v_4) = 5$, $\eta(v_5) = 15$, $\eta(v_6) = 7$, $\eta(v_7) = 17$, $\eta(v_8) = 9$, $\eta(v_9) = 19$, $\eta(v_{10}) = 1$, $\eta(v_{11}) = 13$, and $\eta(v_{12}) = 23$. This function η is an arithmetic radio 2-labeling and hence $\mathrm{R}_a(\mathrm{G}) \leq 2$. Thus, η is an arithmetic radio labeling and $\mathrm{R}_a(\mathrm{G}) = 2$. □

7. End remark

This paper considers several graphs with diameter two or three, for which the arithmetic radio graceful number is 1. The work is in progress for graphs with higher diameters.

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Ramya Hebbar,
Nitte Meenakshi Institute of Technology, NITTE (Deemed to be University), Bengaluru, India.
E-mail address: ramya.hebbar@nmit.ac.in

and

Sooryanarayana B.,
Nitte Meenakshi Institute of Technology, NITTE (Deemed to be University), Bengaluru, India.
E-mail address: sooryanaryana@nmit.ac.in

and

Vishukumar M.,
Department of Mathematics, School of Applied Sciences, REVA University, India.
E-mail address: vishukumarm@reva.edu.in

and

Sneha G. Kulkarni,
Dr. Ambekar Institute of Technology, (Affiliated to Visveshvaraya Technological University, Belagavi), Bengaluru, India.
E-mail address: snehagkulkarni1703@gmail.com