Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 4** : 1–8.
ISSN-0037-8712
doi:10.5269/bspm.80135

Energy of Signed Unit Graphs

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ABSTRACT: In this paper, the authors determine the energy of signed unit graphs $G_\Sigma(R)$ associated with finite commutative ring R. We establish sufficient conditions under which the energy of $G_\Sigma(R)$ equals the number of vertices of the graph. Moreover, it has been shown that for all local rings, the energies of the signed and underlying unit graphs are same. Furthermore, it has been shown that when a ring is isomorphic to a finite product of copies of $\mathbb{Z}_2$, the energy of its signed unit graph equals the order of the ring.

Keywords: Energy, signed graphs, commutative rings, unit graph.

Contents

1. Introduction

The study of the interplay between algebraic structures and graph theory has attracted considerable attention over the past decades. Associating graphs with rings and fields often reveals structural information about these algebraic systems, while simultaneously providing new families of graphs whose spectral and combinatorial properties are of independent interest.

One of the earliest such constructions is the *unit graph* of a commutative ring R with unity, introduced by Grimaldi [5]. The vertices of this graph are the elements of R, and two distinct vertices x and y are adjacent if and only if $x+y \in U(R)$, where $U(R)$ denotes the group of units of R. Later, Ashrafi *et al.* [2] extended this definition to arbitrary associative rings R and studied several algebraic and combinatorial properties of these graphs.

The notion of a *signed graph*, due to Harary [8], assigns to each edge of a graph a sign $+$ or $-$. Signed graphs naturally model phenomena involving antagonistic and cooperative interactions. Formally, a signed graph $\Sigma = (G, \sigma)$ consists of an underlying graph $G = (V, E)$ along with a sign function $\sigma : E \to \{+, -\}$ that assigns to each edge a positive or negative sign. The sign-reversal of Σ, denoted by $-\Sigma$, is formed by negating the sign of every edge. An edge with sign $+$ is called positive, and one with sign $-$ is called negative. The signed graph is said to be *all-positive* (resp. *all-negative*) if all edges have positive (resp. negative) signs. It is called *homogeneous* if it is either all-positive or all-negative, and *heterogeneous* otherwise; these are denoted by G^+ and G^-, respectively.

For any cycle C in G, its sign is defined by

$$\mathrm{Sign}(C) = \prod_{e \in C} \sigma(e).$$

A cycle is called *positive* if $\mathrm{Sign}(C) = +1$, and *negative* if $\mathrm{Sign}(C) = -1$. A signed graph is *balanced* if all cycles in it are positive; otherwise, it is *unbalanced*. Spectral properties of signed graphs were explored by several authors, including Acharya [1], who established a spectral criterion for balance in signed networks, and Nayak [10], who studied the energy of certain families of signed graphs.

The concept of the *energy* of a graph was introduced by Gutman [6] in connection with the total π-electron energy of conjugated hydrocarbons in Hückel molecular orbital theory. Let $\Sigma = (G, \sigma)$ be a signed

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2020 *Mathematics Subject Classification:* 05C50, 05C22, 13A99, 05C25.

Submitted November 11, 2025. Published February 26, 2026

graph, where $G = (V, E)$ is a underlying graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and $\sigma : E \to \{+1, -1\}$ is the sign function that assigns a sign to each edge. The *adjacency matrix* of Σ is the $n \times n$ matrix

$$A(\Sigma) = [a_{ij}]_{n\times n}, \quad \text{where } a_{ij} = \begin{cases} \sigma(v_iv_j), & \text{if } v_iv_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The *characteristic polynomial* of a signed graph Σ is defined as

$$\phi(\Sigma, \lambda) = \det(\lambda I_n - A(\Sigma)),$$

where I_n is the identity matrix of order n. The roots of $\phi(\Sigma, \lambda)$ are called the *eigenvalues* of Σ.

The set of all eigenvalues of $A(\Sigma)$, together with their algebraic multiplicities, is called the *spectrum* of Σ and is denoted by

$$\mathrm{Spec}(\Sigma) = \{\lambda_1(\Sigma), \lambda_2(\Sigma), \ldots, \lambda_n(\Sigma)\}.$$

Two signed graphs are cospectral if they have the same spectrum. The spectral criterion for balance in signed graph was given by B. D. Acharya as follows:

Theorem 1.1 [1] *"A signed graph is balanced if and only if it is cospectral with the underlying graph, i.e.,*

$$Spec(G_\Sigma(R)) = Spec(G(R))".$$

For a simple graph G with adjacency matrix $A(G)$ and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, the energy of G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Two signed graphs are said to be equienergetic if they have the same energy. In this paper, we combine these two frameworks and study the concept of the *energy of signed unit graphs*. Our aim is to investigate how the algebraic properties of the ring and the signing pattern influence the spectral characteristics and energy of $G_\Sigma(R)$.

Given a commutative ring R with $1 \neq 0$, the concept of *signed unit graph* $G_\Sigma(R)$ is recently introduced by Pranjali et al. [12] as a natural extension of unit graphs to the signed setting and characterized the commutative rings for which $G_\Sigma(R)$ and its negation are balanced.

Definition 1.1 [12] "A signed unit graph is an ordered pair $G_\Sigma(R) := (G(R),\ \sigma)$, where $G(R)$ is the unit graph of a commutative ring R and for an edge (a, b) of $G_\Sigma(R)$, σ is defined as

$$\sigma(a, b) = \begin{cases} + \ , & \text{if } \ a \in U(R) \text{ or } b \in U(R); \text{,,} \\ - \ , & \text{otherwise.} \end{cases}$$

We now define the energy of the signed unit graph of a finite commutative ring as follows.

Definition 1.2 Let R be a finite commutative ring with identity, and let $G_\Sigma(R)$ be its signed unit graph. The *energy* of $G_\Sigma(R)$ is defined by

$$E\big(G_\Sigma(R)\big) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix $A(G_\Sigma(R))$.

To illustrate the concept, we present an example computing the energy of the signed unit graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 1.1 Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ with unit set $U(R) = \{(1, 1)\}$. The unit graph $G(R)$ has vertex set R with edges $(x, y))$ if and only if $x + y \in U(R)$. Thus, $E(G) = \{\{(0, 0), (1, 1)\}$ $\{(1, 0), (0, 1)\}\}$, and so $G(R) \cong 2K_2$. Since in signed unit graph $G_\Sigma(R)$, an edge is positive if at least one vertex is a unit, hence $\{(0, 0), (1, 1)\}$ is positive, $\{(1, 0), (0, 1)\}$ is negative.

Thus, the adjacency matrix $A(G_\Sigma(R))$ of signed unit graph is given by

$$A(G_\Sigma(R)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

whose eigenvalues are $\{1, -1, 1, -1\}$. Therefore, the energy is

$$E(G_\Sigma(R)) = |1| + |-1| + |1| + |-1| = 4.$$

The following established result will play a crucial role to obtain the forthcoming results.

Theorem 1.2 [11, Theorem 3.5] *"The energy of the unit graph $G(\mathbb{Z}_n)$ is never an integer, where $n = p^k$, $k > 1$ and p is an odd prime."*

The paper is structured as follows. Section 1 consist of the preliminaries and notations related to the signed unit graph necessary for deriving the new results, along with its fundamental properties. Section 2 discusses the energy of the signed unit graph for specific classes of rings. Throughout the paper, we denote $\mathbb{Z}_2^t \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{t\text{-times}}$.

2. Energy of Signed Unit Graphs

In this section, we investigate the energy of signed unit graphs $G_\Sigma(R)$ associated with finite commutative rings R. After recalling when $G_\Sigma(R)$ is balanced and hence cospectral with its unit graph we obtain exact formulas for fields and local rings. We shall identify the integers n with $E(G_\Sigma(\mathbb{Z}_n)) = n$ and further derive some general bounds.

Theorem 2.1 *Let $G_\Sigma(\mathbb{Z}_n)$ denote the signed unit graph of $\mathbb{Z}_n$. Then $E\big(G_\Sigma(\mathbb{Z}_n)\big) = n$ if and only if $n = 2^k$ for some $k \in \mathbb{N}$.*

Proof: *Necessity.* Assume that $E\big(G_\Sigma(\mathbb{Z}_n)\big) = n$. We claim that n must be some power of two. Suppose, for contradiction, that n is not of the form 2^k. Then n has an odd prime divisor; let $p \mid n$ with p odd. Consider the minimal such case $n = 2p$. From [11], we obtain $E\big(G(\mathbb{Z}_{2p})\big) = 2^p\varphi(p)$ for any odd prime $p \geq 3$. However, the energy of a graph is always an even integer [3], and therefore the above expression cannot equal $2p$. This contradiction implies that no odd prime divides n, and hence n must be a power of two.

Sufficiency. Conversely, let $n = 2^k$. It is well known that $\mathbb{Z}_{2^k}$ is a local ring. For any local ring, the associated signed unit graph is balanced (all edges are positive). It can be easily seen that balanced signed graphs are cospectral with their underlying graphs [1]; hence $\operatorname{Spec}\big(G_\Sigma(\mathbb{Z}_{2^k})\big) = \operatorname{Spec}\big(G(\mathbb{Z}_{2^k})\big)$. Moreover, the unit graph $G(\mathbb{Z}_{2^k})$ is the complete bipartite graph $K_{2^{k-1},2^{k-1}}$, whose spectrum is $\operatorname{Spec}\big(G(\mathbb{Z}_{2^k})\big) = \{\, 2^{k-1}, -2^{k-1}, 0^{(2^k-2)} \,\}$. Therefore, $E\big(G(\mathbb{Z}_{2^k})\big) = |2^{k-1}| + |-2^{k-1}| = 2^k$. Since $G_\Sigma(\mathbb{Z}_{2^k})$ and $G(\mathbb{Z}_{2^k})$ are cospectral, it follows that $E\big(G_\Sigma(\mathbb{Z}_{2^k})\big) = 2^k$. Hence the result. □

Theorem 2.2 *Let $R \cong \mathbb{Z}_2^t$ be the direct product of t-copies of $\mathbb{Z}_2$. Then $E(G_\Sigma(R)) = E(G(R)) = |R|$.*

Proof: In $R = \mathbb{Z}_2^t$, the only unit is $(1, 1, \ldots, 1)$. Two elements x, y are adjacent if $x + y = (1, 1, \ldots, 1$, so each vertex has exactly one neighbor. Hence $G(R)$ is a isomorphic to $\frac{|R|}{2}$-copies of K_2.

Since each K_2 has eigenvalues 1 and -1, the unit graph $G(R)$ has $\frac{|R|}{2}$ eigenvalues equal to 1 and $\frac{|R|}{2}$ eigenvalues equal to -1. Therefore, the energy of $G(R)$ is given by $E(G(R)) = |R|$.

Under the signature σ in $G_\Sigma(R)$ the eigen values are still 1 and -1. So the absolute values are still remain same. It follows that $E(G_\Sigma(R)) = E(G(R)) = |R|$. □

Theorem 2.3 *For $p > 3$ an odd prime, the energy of the signed unit graph $G_\Sigma(\mathbb{Z}_n)$ is never an integer when n is either p or p^2.*

Proof: The signed unit graph of the ring $\mathbb{Z}_p$, where p is a prime or a prime square with $p > 3$, is balanced. Hence, by Theorem [1], $E(G_\Sigma(\mathbb{Z}_p)) = E(G(\mathbb{Z}_p))$.

From [11], we know $E(G(\mathbb{Z}_p)) = 2(p - 2 + a)$, where $0 < a < \frac{1}{2}$. Since a is not an integer, $E(G(\mathbb{Z}_p))$ is not an integer.

Similarly, for the case $n = p^2$, the energy is given by

$$E(G(\mathbb{Z}_{p^2})) = |(p-1)(p-1+a)| + |-(p+a)| + |-1|(p-1)^2.$$

Consequently, by [11, Lemma 3.3], $E(G(\mathbb{Z}_{p^2}))$ is also non-integer. Thus the result follows. □

Theorem 2.4 *For an odd prime p, the energy of the signed unit graph $G_\Sigma(\mathbb{Z}_{2p})$ is*

$$2^2\varphi(p).$$

where $\varphi(p)$ is the Euler totient function of p.

Proof: The unit graph $G(\mathbb{Z}_{2p})$ is bipartite and its associated signed graph $G_\Sigma(\mathbb{Z}_{2p})$ is balanced. Consequently, its energy equals that of the underlying unit graph $G(\mathbb{Z}_{2p})$. Thus, we have $E(G_\Sigma(\mathbb{Z}_{2p})) = E(G(\mathbb{Z}_{2p})) = 2^2\varphi(p)$, where $\varphi(p)$ is Euler's totient function. This result follows from [11, Theorem 3.4]. □

We will now determine the energy of the signed unit graph for specific classes of rings and compare it with the energy of the corresponding underlying unit graphs.

Theorem 2.5 *If R is a local ring and $G_\Sigma(R)$ be its signed unit graph, then $E(G_\Sigma(R)) = E(G(R))$.*

Proof: Let R be local with maximal ideal $M = R \setminus U(R)$. If $a, b \in M$, then $a + b \in M$ (since M is an ideal), so there is no edge between two zero-divisor in the unit graph $G(R)$. Hence every edge of $G_\Sigma(R)$ has at least one vertex from units, therefore under the σ, each edge receive a positive sign. Thus $G_\Sigma(R)$ is an all-positive signed graph. Thus $E(G_\Sigma(R)) = E(G(R))$. □

Theorem 2.6 *If R is a field of characteristic 2 and $G_\Sigma(R)$ is the signed unit graph of R, then*

$$E(G_\Sigma(R)) = 2(|R| - 1).$$

Proof: Let $R \cong \mathbb{F}_q$. In a field of characteristic 2, each nonzero element is unit and $x + y = 0$ iff $y = x$. Hence $x + y \in U(R)$ for all $x \neq y$, and the underlying unit graph $G(R)$ is K_q.

By the sign rule, every edge receive a +ive sign. Thus $G_\Sigma(R)$ is an all-positive complete graph, which is balanced. Now from Theorem 1.1, $\mathrm{Spec}(G(R)) = \mathrm{Spec}(G_\Sigma(R))$. Therefore

$$E\big(G_\Sigma(R)\big) = |(q-1)| + (q-1)|{-1}| = 2(q-1) = 2(|R| - 1).$$

□

Theorem 2.7 *Let R be a local ring with maximal ideal $\mathfrak{m}$ such that $R/\mathfrak{m} \cong \mathbb{Z}_2$. Then*

$$E(G_\Sigma(R)) = E(G(R)) = |R|.$$

Proof: In view of Theorem 2.5, we know that if R be a local ring, then its $G_\Sigma(R))$ have $E(G_\Sigma(R)) = E(G(R))$. Now we have find the common value. Note that if R be a local ring having $\mathbb{Z}_2$ as a quotient, then $G_\Sigma(R)$ ia an all-positive complete bipartite graph having each part of cardinality $\frac{|R|}{2}$. Thus the eigen values are $\frac{|R|}{2}$ and 0. Therefore the energy is $E(G_\Sigma(R)) = 2 \times \frac{|R|}{2} = |R|$. □

Theorem 2.8 *Let R be a finite commutative ring with unity such that $|R/\mathfrak{m}| = 2$, where $\mathfrak{m}$ is a maximal ideal of R. Then*

$$E(G_\Sigma(R)) = E(G(R)).$$

Proof: From [12, Theorem 2.4], it is known that the signed unit graph $G_\Sigma(R)$ is balanced if and only if R is either a local ring or $|R/\mathfrak{m}| = 2$, where $\mathfrak{m}$ is a maximal ideal.

By [1], a signed graph is balanced if and only if it is cospectral with its underlying group. Hence,

$$\mathrm{Spec}(G(R)) = \mathrm{Spec}(G_\Sigma(R)).$$

Therefore,

$$E(G(R)) = E(G_\Sigma(R)).$$

This proves the result. □

Remark 2.1 If R is a local ring with residue field $R/\mathfrak{m}$ such that $|R/\mathfrak{m}| \neq 2$, then

$$E(G_\Sigma(R)) = E(G(R)) \neq |R|,$$

as the energy of the graph is an even integer if it is rational.

Remark 2.2 If the energy of both the signed unit graph and underlying unit graphs equals the order of the ring R, then the order of R must be even, although the converse does not necessarily hold. As for instance; Consider the ring $\mathbb{Z}_6$. In light of Theorem 2.4 and Theorem 2.8, it is clear that $E(G_\Sigma(\mathbb{Z}_6)) = E(G(\mathbb{Z}_6)) \neq 6$. This shows that when the order of the ring is even, the energy equality does not imply that the energy is equals to the order.

Theorem 2.9 *Let R be a finite commutative ring with $1 \neq 0$ and $|R| < 9$. Then for each R,*

$$E(G_\Sigma(R)) = E(G(R)).$$

Proof: We know from [12] that if R is a finite commutative ring with $1 \neq 0$ and $|R| < 9$, then for each R, the signed unit graph $G_\Sigma(R)$ is balanced. Thus for all such rings

$$\mathrm{Spec}(G(R)) = \mathrm{Spec}(G_\Sigma(R)).$$

Therefore,

$$E(G_\Sigma(R)) = E(G(R)).$$

□

Theorem 2.10 *Let R be a finite local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 3$. Then*

$$E(G_\Sigma(R)) = E(G(R)) = |R| - 3 + \sqrt{|R|^2 - \tfrac{2}{3}|R| + 1}.$$

Proof: Let $(R, \mathfrak{m})$ be a finite local ring with $|R/\mathfrak{m}| = 3$ and put $s = |\mathfrak{m}|$, so $|R| = 3s$. In a local ring, the zero-divisor form the ideal $\mathfrak{m}$. Hence if $x, y \in \mathfrak{m}$, then $x + y \in \mathfrak{m}$ is a nonunit, so no two zero-divisor are adjacent in the unit graph $G(R)$. In the signed unit graph $G_\Sigma(R)$ an edge receives a negative sign only if both vertices are zero-divisor, which never occurs; thus all edges are positive and $E(G_\Sigma(R)) = E(G(R))$.

Since $|R/\mathfrak{m}| = 3$, the additive cosets of $\mathfrak{m}$ are $A = \mathfrak{m}$, $B = 1 + \mathfrak{m}$, and $C = 2 + \mathfrak{m}$, each of size s. Vertices x, y are adjacent in $G(R)$ exactly when their images in $R/\mathfrak{m} \cong \mathbb{F}_3$ sum to a nonzero element. This yields: A is independent; B and C each induce a clique K_s; every vertex of A is adjacent to every vertex of B and C; and there are no edges between B and C. With respect to the equitable partition $\{A, B, C\}$, the quotient matrix is

$$Q = \begin{pmatrix} 0 & s & s \\ s & s-1 & 0 \\ s & 0 & s-1 \end{pmatrix}.$$

Its eigenvalues are $s-1$ and $\frac{(s-1)\pm\sqrt{9s^2-2s+1}}{2}$.

The remaining eigenvalues arise from the internal structure of the blocks: B and C each contribute $s-1$ eigenvalues equal to -1, and A contributes $s-1$ eigenvalues equal to 0. Thus the spectrum of $G(R)$ consists of

$$s-1, \quad \frac{(s-1)\pm\sqrt{9s^2-2s+1}}{2}, \quad -1^{(2s-2)}, \quad 0^{(s-1)}.$$

Since the two nontrivial quotient eigenvalues have opposite signs, their absolute values sum to $\sqrt{9s^2-2s+1}$. Therefore

$$E(G(R)) = (s-1) + \sqrt{9s^2-2s+1} + 2(s-1) = 3(s-1) + \sqrt{9s^2-2s+1}.$$

Substituting $|R| = 3s$ gives

$$E(G_\Sigma(R)) = E(G(R)) = |R| - 3 + \sqrt{|R|^2 - \tfrac{2}{3}|R| + 1},$$

as claimed. □

Using the preceding results, we derive the energy of the signed unit graph $G_\Sigma(\mathbb{Z}_{p^k})$ in the following theorem:

Theorem 2.11 *Let $G_\Sigma(\mathbb{Z}_{p^k})$ be signed unit graph of $\mathbb{Z}_{p^k}$. Then its energy is given by*

$$E\big(G_\Sigma(\mathbb{Z}_{p^k})\big) = E\big(G(\mathbb{Z}_{p^k})\big) = 2p^k - p - 3p^{k-1} + \sqrt{p^{2k} - 2p^k + 4p^{k-1} + 1}\,.$$

Proof: The ring $\mathbb{Z}_{p^k}$ is a finite local ring with maximal ideal $\mathfrak{m} = p\mathbb{Z}_{p^k}$ and residue field

$$|\mathbb{Z}_{p^k}/\mathfrak{m}| = |\mathbb{Z}_p| = p.$$

Thus $|\mathfrak{m}| = p^{k-1}$ and $|\mathbb{Z}_{p^k}| = p^k$.

In any local ring $(R, \mathfrak{m})$, the zero-divisor are exactly $\mathfrak{m}$, which is closed under addition. Hence no two zero-divisors are adjacent in the unit graph $G(R)$. By the definition of the signed unit graph $G_\Sigma(R)$, an edge is negative only when both vertices are zero-divisors, which never happens here. Therefore all edges of $G_\Sigma(\mathbb{Z}_{p^k})$ are positive, so

$$E\big(G_\Sigma(\mathbb{Z}_{p^k})\big) = E\big(G(\mathbb{Z}_{p^k})\big)\,.$$

On the other hand, viewing $R = \mathbb{Z}_{p^k}$ as a local ring with residue field of order $q = p$ and $|\mathfrak{m}| = s = p^{k-1}$, the spectral analysis of the unit graph of a finite local ring with $|R/\mathfrak{m}| = q \neq 2$ yields

$$E(G(R)) = 2qs - q - 3s + \sqrt{q^2s^2 - 2qs + 4s + 1}.$$

Substituting $q = p$ and $s = p^{k-1}$ gives

$$E\big(G(\mathbb{Z}_{p^k})\big) = 2p^k - p - 3p^{k-1} + \sqrt{p^{2k} - 2p^k + 4p^{k-1} + 1}\,,$$

and the same formula holds for $E(G_\Sigma(\mathbb{Z}_{p^k}))$, as required. □

Problem 1 Does there exist a ring such that $G_\Sigma(R)$ is unbalanced and still

$$E(G_\Sigma(R)) = E(G(R)).$$

To address Problem 1, we present an explicit example of a ring R for which the signed unit graph $G_\Sigma(R)$ is unbalanced, yet its energy is exactly same with that of its underlying unit graph $G(R)$.

Example 2.1 Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $|R| = 9$. The characteristic polynomial of the adjacency matrix of $G(R)$ is

$$\lambda(\lambda+2)(\lambda^2-2)^2\big(\lambda^3-2\lambda^2-8\lambda+8\big),$$

so the spectrum is

$$\{0,\ -2,\ \pm\sqrt{2},\ \pm\sqrt{2},\ \lambda_1,\lambda_2,\lambda_3\},$$

where $\lambda_1, \lambda_2, \lambda_3$ are the real roots of $\lambda^3 - 2\lambda^2 - 8\lambda + 8 = 0$. Hence

$$E(G(R)) = 2 + 4\sqrt{2} + |\lambda_1| + |\lambda_2| + |\lambda_3| \approx 14.6448.$$

For the signed unit graph $G_\Sigma(R)$ the characteristic polynomial is

$$\lambda(\lambda-2)(\lambda^2-2)^2\big(\lambda^3+2\lambda^2-8\lambda-8\big),$$

so the spectrum is

$$\{0,\ -2,\ \pm\sqrt{2},\ \pm\sqrt{2},\ -\lambda_1,-\lambda_2,-\lambda_3\},$$

where $-\lambda_1, -\lambda_2, -\lambda_3$ are roots of $\lambda^3 + 2\lambda^2 - 8\lambda - 8 = 0$. Thus the absolute values of all eigenvalues are unchanged, and

$$E(G_\Sigma(R)) = E(G(R)) \approx 14.6448.$$

Therefore

$$E(G(\mathbb{Z}_3 \times \mathbb{Z}_3)) = E(G_\Sigma(\mathbb{Z}_3 \times \mathbb{Z}_3)) \approx 14.6448.$$

Problem 2 In general the following problem is open: Let $(R, \mathfrak{m})$ be a finite local ring with $1 \neq 0$ such that $|R/\mathfrak{m}| = q$ $(q > 3)$, where $\mathfrak{m}$ is the maximal ideal of R. Then find the $E(G_\Sigma(R))$ in term of $|R|$.

Remark 2.3 It is worth noting that there exist two non-isomorphic rings whose unit graphs are also non-isomorphic but share the property of being non-cospectral yet equienergetic. For example, consider the rings $R_1 \cong \mathbb{Z}_4$ and $R_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The unit graph of R_1 is the cycle C_4, while the unit graph of R_2 is the disjoint union $2K_2$. The signed unit graph $G_\Sigma(R_1)$ is isomorphic to C_4 with all edges positive, whereas $G_\Sigma(R_2)$ is isomorphic to the union of two K_2 graphs, where one copy has all positive edges and the other all negative edges. These signed unit graphs are non-cospectral but have the same energy, i.e., they are equienergetic.

Conclusion

In this paper we have established the characterization $E\big(G_\Sigma(\mathbb{Z}_n)\big) = n$ if and only if $n = 2^k$. It further shows that for all local rings, the energies of the signed and unsigned unit graphs coincide, $E(G_\Sigma(R)) = E(G(R))$. The study reveals that for local rings and fields of characteristic two, the signed unit graph is balanced and cospectral with its unit graph, leading to equal energies. When a ring is isomorphic to a finite product of $\mathbb{Z}_2$, the energy equals its order. For rings over odd primes $p > 3$, the energies of $G_\Sigma(\mathbb{Z}_p)$ and $G_\Sigma(\mathbb{Z}_{p^2})$ are non-integral. An explicit example with $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ illustrates an unbalanced signed unit graph whose energy equals that of its underlying unit graph.

Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions, which helped improve the quality and clarity of this paper.

Conflicts of interest

Both authors Pranjali and Manish Kumar Saini do not have any conflict of interest.

Funding

The second author sincerely thanks to Council of Scientific & Industrial Research and UGC (CSIR-UGC), Government of India, for the financial support received through the CSIR NET-JRF fellowship, File No: 231620181209.

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