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Sombor Symmetric Division Degree Index and Co-index of Graph Operations

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ABSTRACT: In the present study, we introduce two new graph invariants called Sombor symmetric division index and coindex, and discuss their properties with reference to certain graphs. Also, we compute their values for some of the graph operations such as the Cartesian product, composition, direct product and strong product of two graphs. Further, we establish bounds on these invariants in terms of other graph parameters and work on the relation between them. Through a correlative analysis with the π-electron energy of a few chosen hetero molecules and PAHs, the chemical application of these parameters is examined.

Key Words: Sombor symmetric division index, Sombor symmetric division coindex, graph operations, correlative analysis.

Contents

1. Introduction

Topological indices or molecular descriptors are numerical invariants which can be obtained for any graph and play an important role in mathematical chemistry. They are extensively used in the development of Quantitative Structure Activity Relationships (QSAR) and Quantitative Structure Property Relationships (QSPR) in the field of chemical graph theory. A topological index that depends on the degrees of the vertices of the graph G, is known as a vertex-degree-based (VDB) index. An edge-degree-based (EDB) is also introduced in a similar manner. In literature, various types of VDB and EDB indices are introduced and studied.

A new molecular descriptor, called the symmetric division degree index of a graph G, denoted by $SDD(G)$, is defined in [6] and is one among the 148 "discrete Adriatic indices" that play a vital role in QSAR/QSPR analysis of chemical compounds. Among all successful molecular descriptors, Zagreb indices called first Zagreb index and the second Zagreb index defined in [5] are more useful descriptors.

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Also, another topological index called coindex of a graph that measures the non-adjacency properties of vertices in a graph. It is often defined as the sum of the products of non-adjacent vertices in the graph. Coindices are used in various fields including Chemistry, where they are used to model the properties of molecules. When computing the weighted Wiener polynomials of certain composite graphs, non-adjacent pairs of vertices have been considered in [9]. Also, the first Zagreb coindex and the second Zagreb coindex are defined in [9].

In this paper, we consider simple, finite, connected and undirected graph $G = (V, E)$ with n vertices and m edges. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The degree of a vertex denoted by $d(u)$ is defined as the number of edges that are incident with the vertex u in G. Also, Δ and δ are respectively called the maximum degree and minimum degree of G. A graph $\overline{G}$ is the complement graph of G. Two vertices u and v in $\overline{G}$ are adjacent if and only if they are not adjacent in G. Hence, $(u, v) \in E(\overline{G})$ if and only if $(u, v) \notin E(G)$ and the number of edges in $\overline{G}$ denoted by $\overline{m}$ is defined as $\overline{m} = \binom{n}{2} - m$.

For any unexplained notations, see [2].

2. Preliminaries

We start this paper with some definitions.

Definition 2.1 [1] The Cartesian product of any two graphs G_1 and G_2, denoted by $G_1 \Box G_2$ is a graph with vertex set $V(G_1 \Box G_2) = V(G_1) \Box V(G_2)$, where two vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if $u_i = u_k$ and $v_j \sim v_l$ in G_2 or $v_j = v_l$ and $u_i \sim u_k$ in G_1. We notice that $|E(G_1 \Box G_2)| = n_1 m_2 + m_1 n_2$, and $d_{G_1 \Box G_2}(u_i, v_j) = d_{G_1}(u_i) + d_{G_2}(v_j)$.

Definition 2.2 [1] The composition of any two graphs G_1 and G_2, denoted by $G_1[G_2]$ is a graph with vertex set $V(G_1[G_2])$ in which two vertices (u_i, v_j) and (u_k, v_l) are adjacent whenever $u_i \sim u_k$ in G_1 or $u_i = u_k$ and $v_j \sim v_l$ in G_2. In the graph $G_1[G_2]$, $|V(G_1[G_2])| = n_1 n_2$, $|E(G_1[G_2])| = n_1 m_2 + m_1 n_2^2$ and $d_{G_1[G_2]}(u_i, v_j) = |V(G_2)| d_{G_1}(u_i) + d_{G_2}(v_j)$.

Definition 2.3 [7] The direct product of any two graphs G_1 and G_2, denoted by $G_1 \otimes G_2$ is the graph with vertex set $V(G_1 \otimes G_2)$ in which $(u_i, v_j) \sim (u_k, v_l)$ if $u_i \sim u_k$ in G_1 and $v_j \sim v_l$ in G_2. In this graph $|V(G_1 \otimes G_2)| = n_1 n_2$, $|E(G_1 \otimes G_2)| = 2m_1 m_2$ and $d_{G_1 \otimes G_2}(u_i, v_j) = d_{G_1}(u_i) d_{G_2}(v_j)$.

Definition 2.4 [9] The strong product of any two graphs G_1 and G_2, denoted by $G_1 \boxtimes G_2$ is a graph with vertex set $V(G_1 \boxtimes G_2)$, where $(u_i, v_j) \sim (u_k, v_l)$ if either $u_i = u_k$ and $v_j \sim v_l$ in G_2 or $v_j = v_l$ and $u_i \sim u_k$ in G_1 or $u_i \sim u_k$ in G_1 and $v_i \sim v_l$ in G_2. In $G_1 \boxtimes G_2$, we have $|V(G_1 \boxtimes G_2)| = n_1 n_2$, $|E(G_1 \boxtimes G_2)| = n_1 m_2 + m_1 n_2 + 2m_1 m_2$ and $d_{G_1 \boxtimes G_2}(u_i, v_j) = d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i) d_{G_2}(v_j)$.

Definition 2.5 [6] The SDD index of a graph G is defined as:

$$SDD(G) = \sum_{uv \in E(G)} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u} \right)$$

Where $E(G)$ is the edge set of a graph G and d_u, d_v denote the degrees of the vertices $u, v \in V(G)$ respectively.

Definition 2.6 [5] First Zagreb index and second Zagreb index of a graph G are defined as:

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

Definition 2.7 [9] The first Zagreb coindex and second Zagreb coindex of a graph G are defined as:

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)]$$

and

$$\overline{M_2}(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v)$$

Definition 2.8 [8] The Sombor coindex of a graph G is defined as:

$$\overline{SO}(G) = \sum_{uv \notin E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$$

3. Bounds on the Sombor symmetric division index of Graphs

Inspired by the work on symmetric division degree index and coindex of derived graphs defined in [3], we define a unique index called Sombor symmetric division index of a graph as follows.
The Sombor symmetric division index of a graph G, denoted by $SO_{SD}(G)$, is defined as:

$$SO_{SD}(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)^2}{d(v)^2} + \frac{d(v)^2}{d(u)^2}}$$

where $E(G)$ is the edge set of G and $d(u), d(v)$ denote the degrees of the vertices $u, v \in V(G)$ respectively.

3.0.1. Bounds on the Sombor symmetric division index.

Theorem 3.1 *Let G_1 and G_2 be two graphs having vertices n_1, n_2 and edges m_1, m_2 respectively. Then*

$$\sqrt{2}\left(\frac{\delta_1 + \delta_2}{\Delta_1 + \Delta_2}\right)(n_1 m_2 + m_1 n_2) \leq SO_{SD}(G_1 \Box G_2) \leq \sqrt{2}\left(\frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2}\right)(n_1 m_2 + m_1 n_2).$$

Further, equality holds if the given graphs G_1 and G_2 are regular.

Proof: Let $V(G_1) = \{u_i | i = 1, 2, \ldots, n_1\}$ and $V(G_2) = \{v_j | j = 1, 2, \ldots, n_2\}$ be the vertex sets of G_1 and G_2 respectively. Let δ_i be the minimum degree and Δ_i be the maximum degree of the vertex of G_i, where $i = 1, 2$. In $G_1 \Box G_2$, $|V(G_1 \Box G_2)| = n_1 n_2$ and $|E(G_1 \Box G_2)| = n_1 m_2 + m_1 n_2$.
By the definition of $SO_{SD}(G)$, we have

$$SO_{SD}(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)^2}{d(v)^2} + \frac{d(v)^2}{d(u)^2}} \quad \text{so that}$$

$$\begin{aligned}
SO_{SD}(G_1 \Box G_2) &= \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1 \Box G_2)} \sqrt{\frac{d_{G_1 \Box G_2}(u_i, v_j)^2}{d_{G_1 \Box G_2}(u_k, v_l)^2} + \frac{d_{G_1 \Box G_2}(u_k, v_l)^2}{d_{G_1 \Box G_2}(u_i, v_j)^2}} \\
&= \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1 \Box G_2)} \sqrt{\frac{(d_{G_1}(u_i) + d_{G_2}(v_j))^2}{(d_{G_1}(u_k) + d_{G_2}(v_l))^2} + \frac{(d_{G_1}(u_k) + d_{G_2}(v_l))^2}{(d_{G_2}(u_i) + d_{G_2}(v_j))^2}} \\
&\leq \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1 \Box G_2)} \sqrt{\frac{(\Delta_1 + \Delta_2)^2}{(\delta_1 + \delta_2)^2} + \frac{(\Delta_1 + \Delta_2)^2}{(\delta_1 + \delta_2)^2}} \\
&\leq \sum_{(u_i, v_j), (u_k, v_l) \in E(G_1 \Box G_2)} \sqrt{2}\left(\frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2}\right) \\
&\leq \sqrt{2}\left(\frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2}\right)(n_1 m_2 + m_1 n_2)
\end{aligned}$$

Since G_1 and G_2 are regular, the degree of each vertex is same. Therefore, for $i = 1, 2$, $\Delta_i = \delta_i$ and $SO_{SD}(G_1 \Box G_2) = \sqrt{2}(n_1 m_2 + m_1 n_2)$. Hence, equality holds if the graphs G_1 and G_2 are regular.
Similarly, the lower bound can be established.Thus,
$\sqrt{2}\left(\frac{\delta_1+\delta_2}{\Delta_1+\Delta_2}\right)(n_1m_2 + m_1n_2) \leq SO_{SD}(G_1 \Box G_2) \leq \sqrt{2}\left(\frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}\right)(n_1m_2 + m_1n_2)$ □

Theorem 3.2 *Let G_1 and G_2 be two graphs with $|V(G_1)| = n_1, |E(G_1)| = m_1$ and $|V(G_2)| = n_2$, $|E(G_2)| = m_2$ respectively. Then*

$$\sqrt{2}\left(\frac{n_2\delta_1 + \delta_2}{n_2\Delta_1 + \Delta_2}\right)(n_1m_2 + m_1n_2^2) \leq SO_{SD}(G_1[G_2]) \leq \sqrt{2}\left(\frac{n_2\Delta_1 + \Delta_2}{n_2\delta_1 + \delta_2}\right)(n_1m_2 + m_1n_2^2).$$

Further, equality holds if G_1 and G_2 are regular.

Proof: Let $V(G_1) = \{u_i | i = 1, 2, \dots, n_1\}$ and $V(G_2) = \{v_j | j = 1, 2, \dots, n_2\}$ be the vertex sets of G_1 and G_2 respectively. Let δ_i and Δ_i be the minimum degree and maximum degree of the vertex of G_i, where $i = 1, 2$.
In $G_1[G_2], |V(G_1[G_2])| = n_1n_2$ and $|E(G_1[G_2])| = n_1m_2 + m_1n_2^2$
By the definition of $SO_{SD}(G)$, we have

$$\begin{aligned}
SO_{SD}(G_1[G_2]) &= \sum_{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])} \sqrt{\frac{d_{G_1[G_2]}(u_i,v_j)^2}{d_{G_1[G_2]}(u_k,v_l)^2} + \frac{d_{G_1[G_2]}(u_k,v_l)^2}{d_{G_1[G_2]}(u_i,v_j)^2}} \\
&= \sum_{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])} \sqrt{\frac{(n_2d_{G_1}(u_i) + d_{G_2}(v_j))^2}{(n_2d_{G_1}(u_k) + d_{G_2}(v_l))^2} + \frac{(n_2d_{G_1}(u_k) + d_{G_2}(v_l))^2}{(n_2d_{G_1}(u_i) + d_{G_2}(v_j))^2}} \\
&\leq \sum_{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])} \sqrt{\frac{(n_2\Delta_1 + \Delta_2)^2}{(n_2\delta_1 + \delta_2)^2} + \frac{(n_2\Delta_1 + \Delta_2)^2}{(n_2\delta_1 + \delta_2)^2}} \\
&\leq \sum_{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2])} \sqrt{2}\left(\frac{n_2\Delta_1 + \Delta_2}{n_2\delta_1 + \delta_2}\right) \\
&\leq \sqrt{2}\left(\frac{n_2\Delta_1 + \Delta_2}{n_2\delta_1 + \delta_2}\right)(n_1m_2 + m_1n_2^2)
\end{aligned}$$

when G_1 and G_2 are regular, the degree of each vertex is same. Therefore, equality holds.
Similarly, the lower bound holds.
Thus, $\sqrt{2}\left(\frac{n_2\delta_1+\delta_2}{n_2\Delta_1+\Delta_2}\right) \leq \{SO_{SD}\}(G_1[G_2]) \leq \sqrt{2}\left(\frac{n_2\Delta_1+\Delta_2}{n_2\delta_1+\delta_2}\right)$. □

Theorem 3.3 *Let G_1 and G_2 be two graphs with $|V(G_1)| = n_1$, $|E(G_1)| = m_1$ and $|V(G_2)| = n_2$, $|E(G_2)| = m_2$ respectively. Then*

$$2\sqrt{2}\left(\frac{\delta_1\delta_2}{\Delta_1\Delta_2}\right)m_1m_2 \leq SO_{SD}(G_1 \otimes G_2) \leq 2\sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right)m_1m_2.$$

Further, equality holds if G_1 and G_2 are regular.

Proof: Let $V(G_1) = \{u_1, u_2, \dots, u_{n1}\}$ and $V(G_2) = \{v_1, v_2, \dots, v_{n2}\}$ be the disjoint vertex sets of G_1 and G_2 respectively. Let $mindeg(G_i) = \delta_i$ and $maxdeg(G_i) = \Delta_i$ for $i = 1, 2$.
In $G_1 \otimes G_2, |V(G_1 \otimes G_2)| = n_1n_2$ and $|E(G_1 \otimes G_2)| = 2m_1m_2$

By the definition of $SO_{SD}(G)$, we have

$$SO_{SD}(G) = \sum_{uv \notin E(G)} \sqrt{\frac{d(u)^2}{d(v)^2} + \frac{d(v)^2}{d(u)^2}}$$

$$SO_{SD}(G_1 \otimes G_2) = \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \otimes G_2)} \sqrt{\frac{d_{G_1 \otimes G_2}(u_i,v_j)^2}{d_{G_1 \otimes G_2}(u_k,v_l)^2} + \frac{d_{G_1 \otimes G_2}(u_k,v_l)^2}{d_{G_1 \otimes G_2}(u_i,v_j)^2}}$$

$$= \sum_{u_{ik},v_{jl} \in E(G_1 \otimes G_2)} \sqrt{\frac{(d_{G_1}(u_i)d_{G_2}(v_j))^2}{(d_{G_1}(u_k)d_{G_2}(v_l))^2} + \frac{(d_{G_1}(u_k)d_{G_2}(v_l))^2}{(d_{G_1}(u_i)d_{G_2}(v_j))^2}}$$

$$\leq \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \otimes G_2)} \sqrt{\frac{(\Delta_1\Delta_2)^2}{(\delta_1\delta_2)^2} + \frac{(\Delta_1\Delta_2)^2}{(\delta_1\delta_2)^2}}$$

$$\leq \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \otimes G_2)} \sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right)$$

$$\leq 2\sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right) m_1 m_2$$

When the graphs G_1 and G_2 are regular, we notice that, degree of each vertex is same. Terefore, equality holds .
Similarly, the lower bound can be proved.
Thus, $2\sqrt{2}\left(\frac{\delta_1\delta_2}{\Delta_1\Delta_2}\right) m_1 m_2 \leq SO_{SD}(G_1 \otimes G_2) \leq 2\sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right) m_1 m_2$. □

Theorem 3.4 *Let G_1 and G_2 be two graphs with vertices n_1, n_2 edges m_1, m_2 respectively. Then* $\sqrt{2}\left(\frac{\delta_1+\delta_2+\delta_1\delta_2}{\Delta_1+\Delta_2+\Delta_1\Delta_2}\right)(n_1m_2 + m_1n_2 + 2m_1m_2) \leq SO_{SD}(G_1 \boxtimes G_2) \leq \sqrt{2}\left(\frac{\Delta_1+\Delta_2+\Delta_1\Delta_2}{\delta_1+\delta_2+\delta_1\delta_2}\right)(n_1m_2 + m_1n_2 + 2m_1m_2)$.
Equality holds if G_1 and G_2 are regular.

Proof: Let $V(G_1) = \{u_1, u_2, \ldots, u_{n1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n2}\}$ be the disjoint vertex sets of G_1 and G_2 respectively. Let $mindeg(G_i) = \delta_i$ and $maxdeg(G_i) = \Delta_i$ for $i = 1, 2$.
In $G_1 \boxtimes G_2, |V(G_1 \boxtimes G_2)| = n_1n_2$ and $|E(G_1 \boxtimes G_2)| = n_1m_2 + m_1n_2 + 2m_1m_2$
By the definition of $SO_{SD}(G)$, we have

$$SO_{SD}(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u)^2}{d(v)^2} + \frac{d(v)^2}{d(u)^2}}$$

$$SO_{SD}(G_1 \boxtimes G_2) = \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \boxtimes G_2)} \sqrt{\frac{d_{G_1 \boxtimes G_2}(u_i,v_j)^2}{d_{G_1 \boxtimes G_2}(u_k,v_l)^2} + \frac{d_{G_1 \boxtimes G_2}(u_k,v_l)^2}{d_{G_1 \boxtimes G_2}(u_i,v_j)^2}}$$

$$= \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \boxtimes G_2)} \sqrt{\frac{(d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i)d_{G_2}(v_j))^2}{(d_{G_1}(u_k) + d_{G_2}(v_l) + d_{G_1}(u_k)d_{G_2}(v_l))^2} + \frac{(d_{G_1}(u_k) + d_{G_2}(v_l) + d_{G_1}(u_k)d_{G_2}(v_l))^2}{(d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i)d_{G_2}(v_j))^2}}$$

$$\leq \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \boxtimes G_2)} \sqrt{\frac{(\Delta_1 + \Delta_2 + \Delta_1\Delta_2)^2}{(\delta_1 + \delta_2 + \delta_1\delta_2)^2} + \frac{(\Delta_1 + \Delta_2 + \Delta_1\Delta_2)^2}{(\delta_1 + \delta_2 + \delta_1\delta_2)^2}}$$

Therefore,

$$SO_{SD}(G) \leq \sum_{(u_i,v_j),(u_k,v_l) \in E(G_1 \boxtimes G_2)} \sqrt{2}\left(\frac{\Delta_1 + \Delta_2 + \Delta_1\Delta_2}{\delta_1 + \delta_2 + \delta_1\delta_2}\right)$$

$$\leq \sqrt{2}\left(\frac{\Delta_1 + \Delta_2 + \Delta_1\Delta_2}{\delta_1 + \delta_2 + \delta_1\delta_2}\right)(n_1m_2 + m_1n_2 + 2m_1m_2)$$

when the graphs G_1 and G_2 are regular, the degree of each vertex is same. Therefore, equality holds. Similarly, we follow the lower bound.Thus,
$\sqrt{2}\left(\frac{\delta_1+\delta_2+\delta_1\delta_2}{\Delta_1+\Delta_2+\Delta_1\Delta_2}\right)(n_1m_2+m_1n_2+2m_1m_2) \leq SO_{SD}(G_1 \boxtimes G_2) \leq \sqrt{2}\left(\frac{\Delta_1+\Delta_2+\Delta_1\Delta_2}{\delta_1+\delta_2+\delta_1\delta_2}\right)(n_1m_2+m_1n_2+2m_1m_2)$. □

4. Sombor symmetric division coindex of a Graph

Motivated by the work on Sombor coindex of graphs defined in [8] and Zagreb coindices of composite graphs defined in [8], we defined a new index called Sombor symmetric division coindex of a graph. The Sombor symmetric division coindex of a graph G, denoted by $\overline{SO_{SD}}(G)$ is defined as:

$$\overline{SO_{SD}}(G) = \sum_{uv \notin E(G)} \sqrt{\frac{d(u)^2}{d(v)^2} + \frac{d(v)^2}{d(u)^2}}$$

Where $E(G)$ is the edge set of G and $d(u), d(v)$ denote the degrees of the vertices $u, v \in V(G)$ respectively. In the following propositions, an edge $e = (u, v)$ of a complement graph $\overline{G}$ is referred to as (a, b)-edge, where $a = d_G(u)$ and $b = d_G(v)$.

Proposition 4.1 *For $n \geq 3$, the complement of C_n has $\frac{n(n-3)}{2}$ number of $(2, 2)$-edges. Therefore, by the definition, we have*

$$\overline{SO_{SD}}(C_n) = \frac{1}{\sqrt{2}}n(n-3).$$

Proposition 4.2 *Let K_{n_1,n_2}, where $n_1n_2 \geq 2$ be a complete bipartite graph with (n_1+n_2) vertices and n_1, n_2 edges. Then the complement of K_{n_1,n_2} has $\frac{n_1(n_1-1)}{2}$ number of (n_2, n_2)-edges and $\frac{n_2(n_2-1)}{2}$ number of (n_1, n_1)-edges. Hence,*

$$\overline{SO_{SD}}(K_{n_1,n_2}) = \frac{1}{\sqrt{2}}\left[n_1(n_1-1) + n_2(n_2-1)\right].$$

Proposition 4.3 *For $n \geq 5$ the complement of W_n has $\frac{(n-1)(n-4)}{2}$ number of $(3, 3)$-edges. Hence,*

$$\overline{SO_{SD}}(W_n) = \frac{1}{\sqrt{2}}(n-1)(n-4).$$

Proposition 4.4 *For $n \geq 3$, the complement of a star graph $K_{1,n-1}$ has $\frac{(n-1)(n-2)}{2}$ number of $(1, 1)$-edges. Hence,*

$$\overline{SO_{SD}}(K_{1,n-1}) = \frac{1}{\sqrt{2}}(n-1)(n-2).$$

Proposition 4.5 *In $\overline{P_n}$, we obtain only one $(1, 1)$-edge, $2(n-3)$ number of $(1, 2)$-edges and $\frac{(n-3)(n-4)}{2}$ number of $(2, 2)$-edges. Therefore,*

$$\overline{SO_{SD}}(P_n) = (n-3)\sqrt{17} + \frac{1}{\sqrt{2}}(n-3)(n-4).$$

4.0.1. Properties of Sombor symmetric division coindex of Graphs. In the present section, we determine Sombor symmetric division coindex of some graph operations. In the following graph operations, the cardinality of the edges of a complement graph $\overline{G}$ that are connected by the vertices (u_i, v_j) and (u_k, v_l) is denoted by $|E_{(a,b)}|$, where $a = d(u_i, v_j)$ and $b = d(u_k, v_l)$

Theorem 4.6 *For any two path graphs P_{n_1} and P_{n_2}, where $n_1, n_2 \geq 3$,*

$$\overline{SO_{SD}}(P_{n_1}\Box P_{n_2}) = \frac{1}{\sqrt{2}}(n_1n_2)^2 - n_1n_2(4n_1+4n_2-19) + \sqrt{2}\left[n_1(4n_1-21)+n_2(4n_2-21)+48\right] + (n_1+n_2-5)\frac{4}{5}\sqrt{97} + \left[n_1n_2 - 2(n_1+n_2)+4\right]2\sqrt{17} + (n_1+n_2-4)\left[n_1n_2-2(n_1+n_2)+3\right]\frac{1}{6}\sqrt{337}$$

Proof: For $V(P_{n_1}) = \{u_1, u_2, u_3, \cdots, u_{n_1}\}$ and $V(P_{n_2}) = \{v_1, v_2, v_3, \cdots, v_{n_2}\}$ the cardinality of the edges respect to the degree of vertices (u_i, v_j) and (u_k, v_l) in the Cartesian product $P_{n_1}\Box P_{n_2}$ are given in the following table.

$\lvert E_{(a,b)}\rvert$	Cardinality
$\lvert E_{(2,2)}\rvert$	6
$\lvert E_{(2,3)}\rvert$	$8(n_1+n_2-5)$
$\lvert E_{(2,4)}\rvert$	$4(n_1-2)(n_2-2)$
$\lvert E_{(3,3)}\rvert$	$4(n_1-2)(n_2-2)+(n_1-3)(2n_1-5)+(n_2-3)(2n_2-5)+2$
$\lvert E_{(3,4)}\rvert$	$2(n_1+n_2-4)[(n_1-3)(n_2-2)+(n_2-3)]$
$\lvert E_{(4,4)}\rvert$	$\frac{1}{2}(n_1-2)(n_2-3)(n_2-4)+\frac{1}{2}(n_1-3)(n_1-4)(n_2-2)^2+(n_1-3)(n_2-2)(n_2-3)$

By the definition, we have

$$\overline{SO_{SD}}(P_{n_1}\Box P_{n_2}) = \sum_{(u_i,v_j),(u_k,v_l)\notin E(P_{n_1}\Box P_{n_2})} \sqrt{\frac{(d(u_i)+d(v_j))^2}{(d(u_k)+d(v_l))^2} + \frac{(d(u_k)+d(v_l))^2}{(d(u_i)+d(v_j))^2}}$$

Using the above table, we get
$\overline{SO_{SD}}(P_{n_1}\Box P_{n_2})$

$$=6\sqrt{2} + 8(n_1+n_2-5)\sqrt{\frac{4}{9}+\frac{9}{4}} + 4(n_1-2)(n_2-2)\sqrt{\frac{4}{16}+\frac{16}{4}} + [4(n_1-2)(n_2-2)+ (n_1-3)(2n_1-5)+(n_2-3)(2n_2-5)+2]\sqrt{2} + 2(n_1+n_2-4)[(n_1-3)(n_2-2)+(n_2-3)] \sqrt{\frac{9}{16}+\frac{16}{9}} + \left[\frac{1}{2}(n_1-2)(n_2-3)(n_2-4)+\frac{1}{2}(n_1-3)(n_1-4)(n_2-2)^2+(n_1-3)(n_2-2)(n_2-3)\right]\sqrt{2}$$

$$=\frac{1}{\sqrt{2}}\left[(n_1n_2)^2 - n_1n_2(4n_1+4n_2-19)\right] + \sqrt{2}\left[n_1(4n_1-21)+n_2(4n_2-21)+48\right] + (n_1+n_2-5)\frac{4}{5}\sqrt{97} + \left[n_1n_2-2(n_1+n_2)+4\right]2\sqrt{17} + (n_1+n_2-4)\left[n_1n_2-2(n_1+n_2)+3\right]\frac{1}{6}\sqrt{337}$$

□

Theorem 4.7 *For the path graph P_{n_1} and the cycle graph C_{n_2}, where $n_1 \geq 3$ and $n_2 \geq 4$,*
$\overline{SO_{SD}}(P_{n_1}\Box C_{n_2}) = \frac{1}{\sqrt{2}}[(n_1n_2)^2 - n_1n_2(4n_2+5)+2n_2(4n_2+3)] + n_2(n_1n_2-2n_2-1)\frac{1}{6}\sqrt{337}$

Proof: For $V(P_{n_1}) = \{u_1, u_2, u_3, \cdots, u_{n_1}\}$ and $V(C_{n_2}) = \{v_1, v_2, v_3, \cdots, v_{n_2}\}$ the cardinality of the edges respect to the degree of vertices (u_i, v_j) and (u_k, v_l) are given in the following table.

$\lvert E_{(a,b)}\rvert$	Cardinality
$\lvert E_{(3,3)}\rvert$	$n_2(n_2-3)+n_2^2$
$\lvert E_{(3,4)}\rvert$	$2n_2(n_1n_2-2n_2-1)$
$\lvert E_{(4,4)}\rvert$	$\frac{1}{2}(n_1n_2-2n_2-2)(n_1n_2-2n_2-3)+(n_2-3)$

The proof is similar to the above theorem. □

Theorem 4.8 *If $P_{n_1}[P_{n_2}]$ is a composition of P_{n_1} and P_{n_2}, then*
(i) $\overline{SO_{SD}}(P_{n_1}[P_{n_2}]) = (n_2^2-7n_2+14)\sqrt{2}+4(n_2-3)\sqrt{\frac{(n_2+1)^2}{(n_2+2)^2}+\frac{(n_2+2)^2}{(n_2+1)^2}}\, for\, n_1=2\, and\, n_2 \geq 3$
(ii) $\overline{SO_{SD}}(P_{n_1}[P_{n_2}]) = \frac{1}{\sqrt{2}}[(n_1n_2)^2 - n_1n_2(4n_1+6n_2-21)+2n_1(3n_1-14)+14n_2(n_2-4)+88]$
$+4(2n_2-5)\sqrt{\frac{(n_2+1)^2}{(n_2+2)^2}+\frac{(n_2+2)^2}{(n_2+1)^2}}+[n_1n_2(n_1-5)-2n_1(n_1-4)+4(2n_2-3)]\sqrt{\frac{(2n_2+1)^2}{(2n_2+2)^2}+\frac{(2n_2+2)^2}{(2n_2+1)^2}}$
$for\, n_1, n_2 \geq 3$

Proof: Consider a composition $P_{n_1}[P_{n_2}]$, where, $V(P_{n_1}) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(P_{n_2}) = \{v_1, v_2, \cdots, v_{n_2}\}$ be the vertex sets of P_{n_1} and P_{n_2} respectively. The cardinality of the edges respect to the degree of vertices (u_i, v_j) and (u_k, v_l) in the graph $P_{n_1}[Pn_2]$ are given in the following table.

$\lvert E_{(a,b)}\rvert$	Cardinality
$for\, n_1=2, n_2 \geq 3$	
$\lvert E_{(n_2+1,n_2+2)}\rvert$	$4(n_2-3)$
$\lvert E_{(n_2+1,n_2+1)}\rvert$	2
$\lvert E_{(n_2+2,n_2+2)}\rvert$	$(n_2-3)(n_2-4)$
$for\, n_1, n_2 \geq 3$	
$\lvert E_{(n_2+1,n_2+2)}\rvert$	$4(2n_2-5)$
$\lvert E_{(n_2+1,n_2+1)}\rvert$	6
$\lvert E_{(n_2+2,n_2+2)}\rvert$	$(n_2-3)(2n_2-5)+1$
$\lvert E_{(2n_2+1,2n_2+2)}\rvert$	$2(n_1-2)(n_2-3)+(n_1-3)(n_1-4)(n_2-2)$
$\lvert E_{(2n_2+1,2n_2+1)}\rvert$	$(n_1-2)+(n_1-3)(n_1-4)$
$\lvert E_{(2n_2+2,2n_2+2)}\rvert$	$\frac{1}{2}[(n_1-2)(n_2-3)(n_2-4)+(n_2-2)^2(n_1-3)(n_1-4)$

By the definition, we have

$$\overline{SO_{SD}}(P_{n_1}[P_{n_2}]) = \sum_{(u_i,v_j),(u_k,v_l)\notin E(P_{n_1}[P_{n_2}])} \sqrt{\frac{d_{P_{n_1}[P_{n_2}]}(u_i,v_j)^2}{d_{P_{n_1}[P_{n_2}]}(u_k,v_l)^2}+\frac{d_{P_{n_1}[P_{n_2}]}(u_k,v_l)^2}{d_{P_{n_1}[P_{n_2}]}(u_i,v_j)^2}}$$

Here, clearly we notice that, in a composition graph $P_{n_1}[P_{n_2}]$ an edge $e = ((u_i, v_j), (u_k, v_l))$ and
$d_{P_{n_1}[P_{n_2}]}(u_i, v_j) = n_2 d_{P_{n_1}}(u_i) + d_{P_{n_2}}(v_j)$
$d_{P_{n_1}[P_{n_2}]}(u_k, v_l) = n_2 d_{P_{n_1}}(u_k) + d_{P_{n_2}}(v_l)$
substituting these in the above formula, we get

$$\overline{SO_{SD}}(P_{n_1}[P_{n_2}]) = \sum_{(u_i,v_j),(u_k,v_l)\notin E(P_{n_1}[P_{n_2}])} \sqrt{\frac{(n_2 d_{P_{n_1}}(u_i) + d_{P_{n_2}}(v_j))^2}{(n_2 d_{P_{n_1}}(u_k) + d_{P_{n_2}}(v_l))^2} + \frac{(n_2 d_{P_{n_1}}(u_k) + d_{P_{n_2}}(v_l))^2}{(n_2 d_{P_{n_1}}(u_i) + d_{P_{n_2}}(v_j))^2}}$$

Using the above table, we get
(i)

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1}[P_{n_2}]) =& 4(n_2-3)\sqrt{\frac{(n_2+1)^2}{(n_2+2)^2} + \frac{(n_2+2)^2}{(n_2+1)^2}} + 2\sqrt{\frac{(n_2+1)^2}{(n_2+1)^2} + \frac{(n_2+1)^2}{(n_2+1)^2}} \\ &+ (n_2-3)(n_2-4)\sqrt{\frac{(n_2+2)^2}{(n_2+2)^2} + \frac{(n_2+2)^2}{(n_2+2)^2}} \\ =& (n_2^2 - 7n_2 + 14)\sqrt{2} + 4(n_2-3)\sqrt{\frac{(n_2+1)^2}{(n_2+2)^2} + \frac{(n_2+2)^2}{(n_2+1)^2}}\end{aligned}$$

ii)

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1}[P_{n_2}]) =& 4(2n_2-5)\sqrt{\frac{(n_2+1)^2}{(n_2+2)^2} + \frac{(n_2+2)^2}{(n_2+1)^2}} + 6\sqrt{\frac{(n_2+1)^2}{(n_2+1)^2} + \frac{(n_2+1)^2}{(n_2+1)^2}} + \\ & [(n_2-3)(2n_2-5)+1]\sqrt{\frac{(n_2+2)^2}{(n_2+2)^2} + \frac{(n_2+2)^2}{(n_2+2)^2}} + \\ & [2(n_1-2)(n_2-3) + (n_1-3)(n_1-4)(n_2-2)]\sqrt{\frac{(2n_2+1)^2}{(2n_2+2)^2} + \frac{(2n_2+2)^2}{(2n_2+1)^2}} + \\ & [(n_1-2) + (n_1-3)(n_1-4)]\sqrt{\frac{(2n_2+1)^2}{(2n_2+1)^2} + \frac{(2n+1)^2}{(2n+1)^2}} + \\ & \frac{1}{2}[(n_1-2)(n_2-3)(n_2-4) + (n_2-2)^2(n_1-3)(n_1-4)]\sqrt{\frac{(2n_2+2)^2}{(2n_2+2)^2} + \frac{(2n_2+2)^2}{(2n_2+2)^2}} \\ =& \frac{1}{\sqrt{2}}[(n_1n_2)^2 - n_1n_2(4n_1+6n_2-21) + 2n_1(3n_1-14) + 14n_2(n_2-4) + 88] + \\ & 4(2n_2-5)\sqrt{\frac{(n_2+1)^2}{(n_2+2)^2} + \frac{(n_2+2)^2}{(n_2+1)^2}} + [n_1n_2(n_1-5) - 2n_1(n_1-4) + 4(2n_2-3)] \\ & \sqrt{\frac{(2n_2+1)^2}{(2n_2+2)^2} + \frac{(2n_2+2)^2}{(2n_2+1)^2}}\end{aligned}$$

□

Theorem 4.9 *For P_{n_1} and C_{n_2} with $n_1, n_2 \geq 3$,*
$\overline{SO_{SD}}(P_{n_1}[C_{n_2}]) = \frac{n_2}{\sqrt{2}}[n_1n_2(n_1-6) - 3n_1 + 14n_2] + 2n_2^2(n_1-3)\sqrt{\frac{(n_2+2)^2}{(2n_2+2)^2} + \frac{(2n_2+2)^2}{(n_2+2)^2}}$

Proof: For $V(P_{n_1}) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(C_{n_2}) = \{v_1, v_2, \cdots, v_{n_2}\}$ be the vertex sets of P_{n_1} and C_{n_2} respectively. The following table shows the cardinality of the edges respect to the degree of vertices (u_i, v_j) and (u_k, v_l) in the composition graph $P_{n_1}[C_{n_2}]$.

$\lvert E_{a,b}\rvert$	Cardinality
$\lvert E_{(n_2+2,n_2+2)}\rvert$	$n_2(2n_2-3)$
$\lvert E_{(n_2+2,2n_2+2)}\rvert$	$2(n_1-3)n_2^2$
$\lvert E_{(2n_2+2,2n_2+2)}\rvert$	$\frac{n_2}{2}[(n_1-2)(n_2-3)+n_2(n_1-3)(n_1-4)]$

In a composition graph $P_{n_1}[C_{n_2}]$,
$d_{P_{n_1}[C_{n_2}]}(u_i,v_j)=n_2 d_{P_{n_1}}(u_i)+d_{C_{n_2}}(v_j)$
$d_{P_{n_1}[C_{n_2}]}(u_k,v_l)=n_2 d_{P_{n_1}}(u_k)+d_{C_{n_2}}(v_)$. Therefore,

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1}[C_{n_2}]) =& n_2(2n_2-3)\sqrt{2}+2(n_1-3)n_2^2\sqrt{\frac{(n_2+2)^2}{(2n_2+2)^2}+\frac{(2n_2+2)^2}{(n_2+2)^2}}\\ &+\frac{n_2}{2}[(n_1-2)(n_2-3)+n_2(n_1-3)(n_1-4)]\sqrt{2}\\ =&\frac{n_2}{\sqrt{2}}\left[n_1n_2(n_1-6)-3n_1+14n_2\right]+2n_2^2(n_1-3)\sqrt{\frac{(n_2+2)^2}{(2n_2+2)^2}+\frac{(2n_2+2)^2}{(n_2+2)^2}}\end{aligned}$$

□

Theorem 4.10 *If P_{n_1} and P_{n_2} are any two path graphs of the direct product $P_{n_1}\otimes P_{n_2}$, then*
(i)

$$\overline{SO_{SD}}(P_{n_1}\otimes P_{n_2}) =(2n_2^2-11n_2+22)\sqrt{2}+4(2n_2-5)2\sqrt{17}\, for\, n_1=2\, and\, n_2>2$$

(ii)

$$\overline{SO_{SD}}(P_{n_1}\otimes P_{n_2}) =(2n_1^2-11n_1+22)\sqrt{2}+4(2n_1-5)2\sqrt{17}\, for\, n_1>2\, and\, n_2=2$$

(iii)

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1}\otimes P_{n_2}) =& \frac{1}{\sqrt{2}}\left[(n_1n_2)^2-n_1n_2(4n_1+4n_2-19)\right]+4\left[n_1(2n_1-9)+n_2(2n_2-9)+13\right]\\ &+\left[(n_1n_2)(n_1+n_2-8)-2n_1(n_1-7)-2n_2(n_2-7)-20\right]\sqrt{17}+(n_1-3)\\ &(n_2^2-5n_2+6)\sqrt{257}\, for\, n_1\,,n_2\geq 3\end{aligned}$$

Proof: Let $V(P_{n_1})=\{u_1,u_2,\cdots,u_{n_1}\}$ and $V(P_{n_2})=\{v_1,v_2,\cdots,v_{n_2}\}$ be the vertex sets of P_{n_1} and P_{n_2} respectively. In the direct product $P_{n_1}\otimes P_{n_2}$ we obtain the cardinality of the edges respect to the degree of vertices (u_i,v_j) and (u_k,v_l) and are shown in the following table.

$\lvert E_{(u_{ik},v_{jl})}\rvert$	Cardinality
$for\, n_1 = 2, n_2 > 2$	
$\lvert E_{(1,1)}\rvert$	6
$\lvert E_{(1,2)}\rvert$	$4(2n_2 - 5)$
$\lvert E_{(2,2)}\rvert$	$(n_2 - 3)(2n_2 - 5) + 1$
$for\, n_1 > 2, n_2 = 2$	
$\lvert E_{(1,1)}\rvert$	6
$\lvert E_{(1,2)}\rvert$	$4(2n_1 - 5)$
$\lvert E_{(2,2)}\rvert$	$(n_1 - 3)(2n_1 - 5) + 1$
$for\, n_1, n_2 \geq 3$	
$\lvert E_{(1,1)}\rvert$	6
$\lvert E_{(1,2)}\rvert$	$8(n_1 + n_2 - 4)$
$\lvert E_{(2,2)}\rvert$	$(n_1 + 1)(n_1 - 3) + (n_1 - 2)(n_1 + 3n_2 - 11) + (n_2 - 1)(n_2 - 3) + (n_2 - 2)(n_1 + n_2 - 5)$
$\lvert E_{(1,4)}\rvert$	$4(n_1 - 3)(n_2 - 2)(n_2 - 3)$
$\lvert E_{(2,4)}\rvert$	$(n_1 - 3)(n_2 - 2)(2n_2 - 4) + (n_2 - 3)(n_1 - 2)(2n_1 - 4) + 2[(n_1 - 4)^2 + (n_2 - 4)^2] + 4(n_1 + n_2 - 6)$
$\lvert E_{(4,4)}\rvert$	$\frac{1}{2}(n_1 - 2)(n_2 - 2)(n_2 - 3) + \frac{1}{2}(n_1 - 3)(n_1 - 4)(n_2 - 2)^2 + (n_1 - 3)[2(n_2 - 3) + (n_2 - 4)^2]$

□

By the definition, we have
Here, we notify that, in $P_{n_1} \otimes P_{n_2}$
$d_{P_{n_1} \otimes P_{n_2}}(u_i, v_j) = d_{P_{n_1}}(u_i) d_{P_{n_2}}(v_j)$
$d_{P_{n_1} \otimes P_{n_2}}(u_k, v_l) = d_{P_{n_1}}(u_k) d_{P_{n_2}}(v_j)$

(i)

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1} \otimes P_{n_2}) &= 6\sqrt{2} + (2n_2 - 5)2\sqrt{17} + [(n_2 - 3)(2n_2 - 5) + 1]\sqrt{2} \\ &= (2n_2^2 - 11n_2 + 22)\sqrt{2} + 4(2n_2 - 5)2\sqrt{17}\end{aligned}$$

(ii)

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1} \otimes P_{n_2}) &= 6\sqrt{2} + (2n_1 - 5)2\sqrt{17} + [(n_1 - 3)(2n_1 - 5) + 1]\sqrt{2} \\ &= (2n_1^2 - 11n_1 + 22)\sqrt{2} + 4(2n_1 - 5)2\sqrt{17}\end{aligned}$$

(iii)

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1}\otimes P_{n_2}) = {} & 6\sqrt{2}+(n_1+n_2-4)4\sqrt{17}+\{[(n_1+1)(n_1-3)+(n_1-2)(n_1+3n_2-11)]+\\ & [(n_2-1)(n_2-3)+(n_2-2)(n_1+n_2-5)]\}\sqrt{2}+[(n_1-3)(n_2-2)+(n_2-3)]\\ & \sqrt{257}+\{[(n_1-3)(n_2-2)(2n_2-4)+(n_2-3)(n_1-2)(2n_1-4)]+2[(n_1-4)^2\\ & +(n_2-4)^2]+4(n_1+n_2-6)\}\frac{\sqrt{17}}{2}+\{\frac{1}{2}(n_1-2)(n_2-2)(n_2-3)+\frac{1}{2}(n_1-3)\\ & (n_1-4)(n_2-2)^2+(n_1-3)[2(n_2-3)+(n_2-4)^2]\}\sqrt{2}\end{aligned}$$

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1}\otimes P_{n_2}) = {} & \frac{1}{\sqrt{2}}\left[(n_1n_2)^2-n_1n_2(4n_1+4n_2-19)\right]+4\left[n_1(2n_1-9)+n_2(2n_2-9)+13\right]+\\ & [(n_1n_2)(n_1+n_2-8)-2n_1(n_1-7)-2n_2(n_2-7)-20]\sqrt{17}+\\ & (n_1-3)(n_2^2-5n_2+6)\sqrt{257}\end{aligned}$$

Theorem 4.11 *For the path graph P_{n_1} and the cycle graph C_{n_2} with $n_1, n_2 \geq 3$,*

$$\overline{SO_{SD}}(P_{n_1}\otimes C_{n_2}) = \frac{1}{\sqrt{2}}[(n_1n_2)^2-n_1n_2(4n_2+5)+4n_2(2n_2+3)+n_2(n_1n_2-2n_2-2)]\sqrt{17}$$

Proof: Let $V(P_{n_1}) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(C_{n_2}) = \{v_1, v_2, \cdots, v_{n_2}\}$ be the vertex sets of P_{n_1} and C_{n_2} respectively. In the direct graph $P_{n_1}\otimes C_{n_2}$, the cardinality of the edges respect to the degree of vertices (u_i, v_j) and u_k, v_l are given in the following table.

$\lvert E_{(a,b)}\rvert$	Cardinality
$\lvert E_{(2,2)}\rvert$	$n_2(2n_2-1)$
$\lvert E_{(2,4)}\rvert$	$2n_2[n_1n_2-2(n_2+1)]$
$\lvert E_{(4,4)}\rvert$	$\frac{1}{2}[(n_1n_2-3n_2)(n_1n_2-n_2-5)+n_2(n_2-1)]$

Using the above table, we get

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1}\otimes C_{n_2}) = {} & n_2(2n_2-1)\sqrt{2}+n_2[n_1n_2-2(n_2+1)]\sqrt{17}+\frac{1}{2}[(n_1n_2-3n_2)(n_1n_2-n_2-5)+\\ & n_2(n_2-1)]\sqrt{2}\\ = {} & \frac{1}{\sqrt{2}}[(n_1n_2)^2-n_1n_2(4n_2+5)+4n_2(2n_2+3)+n_2(n_1n_2-2n_2-2)]\sqrt{17}\end{aligned}$$

□

Theorem 4.12 *If $P_{n_1}\boxtimes P_{n_2}$ is the strong product of P_{n_1} and P_{n_2}, then*

(i)

$$\overline{SO_{SD}}(P_{n_1}\boxtimes P_{n_2}) = (n_2^2-7n_2+14)2\sqrt{2}+(n_2-3)\frac{8}{15}\sqrt{706}\, for\, n_1 = 2\, and\, n_2 \geq 3$$

(ii)

$$\overline{SO_{SD}}(P_{n_1} \boxtimes P_{n_2}) = (n_1^2 - 7n_1 + 14)2\sqrt{2} + (n_1 - 3)\frac{8}{15}\sqrt{706}\, for\, n_1 \geq 3\, and\, n_2 = 2$$

(iii)

$$\begin{aligned}\overline{SO_{SD}}(P_{n_1} \boxtimes P_{n_2}) = & \frac{1}{\sqrt{2}}[(n_1n_2)^2 - n_1n_2(4n_1 + 4n_2 - 15) + 2n_1(4n_1 + 15) + 2n_2(4n_2 + 15) + 52] + \\ & (n_1 + n_2 - 5)\frac{8}{15}\sqrt{706} + [n_1n_2 - 2(n_1 + n_2 + 3)]\,\frac{1}{6}\sqrt{4177} + \\ & [n_1n_2(n_1 + n_2 - 8) - n_1(2n_1 - 9) - n_2(2n_2 - 9)]\,\frac{1}{20}\sqrt{4721}\, for\, n_1, n_2 \geq 3\end{aligned}$$

Proof: In $P_{n_1} \boxtimes P_{n_2}$ the cardinality of the edges respect to the degree of vertices (u_i, v_j) and (u_k, v_l) are given in the following table.

$\|E_{(a,b)}\|$	Cardinality
$for\, n_1 = 2, n_2 \geq 3$	
$\|E_{(3,3)}\|$	4
$\|E_{(3,5)}\|$	$8(n_2 - 3)$
$\|E_{(5,5)}\|$	$2(n_2 - 3)(n_2 - 4)$
$for\, n_1 \geq 3, n_2 = 2$	
$\|E_{(3,3)}\|$	4
$\|E_{(3,5)}\|$	$8(n_1 - 3)$
$\|E_{(5,5)}\|$	$2(n_1 - 3)(n_1 - 4)$
$for\, n_1, n_2 \geq 3$	
$\|E_{(3,3)}\|$	6
$\|E_{(3,5)}\|$	$8(n_1 + n_2 - 5)$
$\|E_{(5,5)}\|$	$(n_1 - 3)(2n_1 - 3) + (2n_2 - 5)(2n_1 + n_2 - 7)$
$\|E_{(3,8)}\|$	$4[n_1n_2 - 2(n_1 + n_2) + 3]$
$\|E_{(5,8)}\|$	$2(n_1 - 3)[(n_1 - 4) + (n_2 - 2)^2] + 2(n_2 - 3)[(n_2 - 4) + (n_1 - 2)^2]$
$\|E_{(8,8)}\|$	$\frac{1}{2}(n_1 - 2)(n_2 - 3)(n_2 - 4) + \frac{1}{2}(n_1 - 3)(n_1 - 4)(n_2 - 2)^2$ $+(n_1 - 3)(n_2 - 3)(n_2 - 4)$

By the definition, we have

$$\overline{SO_{SD}}(P_{n_1} \boxtimes P_{n_2}) = \sum_{(u_i,v_j),(u_k,v_l) \notin E(P_{n_1} \boxtimes P_{n_2})} \sqrt{\frac{d_{P_{n_1} \boxtimes P_{n_2}}(u_i, v_j)^2}{d_{P_{n_1} \boxtimes P_{n_2}}(u_k, v_l)^2} + \frac{d_{P_{n_1} \boxtimes P_{n_2}}(u_k, v_l)^2}{d_{P_{n_1} \boxtimes P_{n_2}}(u_i, v_j)^2}}$$

Clearly,

$$\begin{aligned} d_{P_{n_1}\boxtimes P_{n_2}}(u_i,v_j) =& d_{P_{n_1}}(u_i)+d_{P_{n_2}}(v_j)+d_{P_{n_1}}(u_i)d_{P_{n_2}}(v_j)\\ d_{P_{n_1}\boxtimes P_{n_2}}(u_k,v_l) =& d_{P_{n_1}}(u_k)+d_{P_{n_2}}(v_l)+d_{P_{n_1}}(u_k)d_{P_{n_2}}(v_l) \end{aligned}$$

Using the above table, we get
(i)

$$\begin{aligned} \overline{SO_{SD}}(P_{n_1}\boxtimes P_{n_2}) =& 4\sqrt{\frac{3^2}{3^2}+\frac{3^2}{3^2}}+8(n_2-3)\sqrt{\frac{3^2}{5^2}+\frac{5^2}{3^2}}+2(n_2-3)(n_2-4)\sqrt{\frac{5^2}{5^2}+\frac{5^2}{5^2}}\\ =& 4\sqrt{2}+8(n_2-3)\sqrt{\frac{9}{25}+\frac{25}{9}}+(n_2-3)(n_2-4)2\sqrt{2}\\ =& (n_2^2-7n_2+14)2\sqrt{2}+(n_2-3)\frac{8}{15}\sqrt{706}\, for\, n_1=2\, and\, n_2\geq 3 \end{aligned}$$

(ii)

$$\begin{aligned} \overline{SO_{SD}}(P_{n_1}\boxtimes P_{n_2}) =& 4\sqrt{2}+8(n_1-3)\sqrt{\frac{9}{25}+\frac{25}{9}}+(n_1-3)(n_1-4)2\sqrt{2}\\ =& (n_1^2-7n_1+14)2\sqrt{2}+(n_1-3)\frac{8}{15}\sqrt{706}\, for\, n_1\geq 3\, and\, n_2=2 \end{aligned}$$

(iii)

$$\begin{aligned} \overline{SO_{SD}}(P_{n_1}\boxtimes P_{n_2}) =& 6\sqrt{\frac{3^2}{3^2}+\frac{3^2}{3^2}}+8(n_1+n_2-5)\sqrt{\frac{3^2}{5^2}+\frac{5^2}{3^2}}+[(n_1-3)(2n_1-3)+(2n_2-5)\\ & (2n_1+n_2-7)]\sqrt{\frac{5^2}{5^2}+\frac{5^2}{5^2}}+4[n_1n_2-2(n_1+n_2)+3]\sqrt{\frac{3^2}{8^2}+\frac{8^2}{3^2}}+\{2(n_1-3)[(n_1-4)+\\ & (n_2-2)^2]+2(n_2-3)[(n_2-4)+(n_1-2)^2]\}\sqrt{\frac{5^2}{8^2}+\frac{8^2}{5^2}}+[\frac{1}{2}(n_1-2)(n_2-3)(n_2-4)+\\ & \frac{1}{2}(n_1-3)(n_1-4)(n_2-2)^2+(n_1-3)(n_2-3)(n_2-4)]\sqrt{\frac{8^2}{8^2}+\frac{8^2}{8^2}}\\ =& 6\sqrt{2}+8(n_1+n_2-5)\sqrt{\frac{9}{25}+\frac{25}{9}}+[(n_1-3)(2n_1-3)+(2n_2-5)(2n_1+n_2-7)]\sqrt{2}\\ & +4[n_1n_2-2(n_1+n_2)+3]\sqrt{\frac{9}{64}+\frac{64}{9}}+\{2(n_1-3)[(n_1-4)+(n_2-2)^2]+2(n_2-3)\\ & [(n_2-4)+(n_1-2)^2]\}\sqrt{\frac{25}{64}+\frac{64}{25}}+[\frac{1}{2}(n_1-2)(n_2-3)(n_2-4)+\frac{1}{2}(n_1-3)\\ & (n_1-4)(n_2-2)^2+(n_1-3)(n_2-3)(n_2-4)]\sqrt{2}\\ =& \frac{1}{\sqrt{2}}[(n_1n_2)^2-n_1n_2(4n_1+4n_2-15)+2n_1(4n_1+15)+2n_2(4n_2+15)+52]+\\ & (n_1+n_2-5)\frac{8}{15}\sqrt{706}+[n_1n_2-2(n_1+n_2+3)]\,\frac{1}{6}\sqrt{4177}+\\ & [n_1n_2(n_1+n_2-8)-n_1(2n_1-9)-n_2(2n_2-9)]\,\frac{1}{20}\sqrt{4721}\, for\, n_1,n_2\geq 3 \end{aligned}$$

□

Theorem 4.13 *If P_{n_1} is a path graph and C_{n_2} is a the cycle graph, $n_1,n_2\geq 4$. Then*

$$\overline{SO_{SD}}(P_{n_1}\boxtimes C_{n_2}) = \frac{1}{\sqrt{2}}[(n_1n_2)^2-4n_1n_2(n_2+9)+2n_2(4n_2+9)+n_2(n_1n_2-2n_2-3)]\frac{1}{20}\sqrt{4721}$$

Proof: The following table contains the cardinality of the edges respect to the degree of vertices (u_i,v_j) and (u_k,v_l), where $V(P_{n_1})=\{u_1,u_2,u_3,\cdots,u_{n_1}\}$ and $V(C_{n_2})=\{v_1,v_2,v_3,\cdots,v_{n_2}\}$.

$\lvert E_{(a,b)}\rvert$	Cardinality
$\lvert E_{(5,5)}\rvert$	$n_2(2n_2-3)$
$\lvert E_{(5,8)}\rvert$	$2n_2(n_1n_2-2n_2-3)$
$\lvert E_{(8,8)}\rvert$	$\frac{1}{2}[n_2^2(n_1^2-4n_1+4)+3n_2(8-3n_1)]$

The proof is similar to the above theorem. □

5. Bounds on the Sombor symmetric division coindex

Theorem 5.1 *Let G_1 and G_2 be two graphs having vertices n_1, n_2 and edges m_1, m_2 respectively. Then* $\overline{m}\sqrt{2}\left(\frac{\delta_1+\delta_2}{\Delta_1+\Delta_2}\right) \leq \overline{SO_{SD}}(G_1\Box G_2) \leq \overline{m}\sqrt{2}\left(\frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}\right)$.
Equality holds if the given graphs G_1 and G_2 are regular.

Proof: Let $V(G_1)=\{u_i|i=1,2,\ldots,n_1\}$ and $V(G_2)=\{v_j|j=1,2,\ldots,n_2\}$ be the vertex sets of G_1 and G_2 respectively. Let δ_i be the minimum degree and Δ_i be the maximum degree of the vertex of G_i, where $i=1,2$.
In $G_1\Box G_2$, $|V(G_1\Box G_2)|=n_1n_2$ and $|E(G_1\Box G_2)|=n_1m_2+m_1n_2$ and the number of non-adjacent edges in $G_1\Box G_2$ is $\overline{m}=\left(\begin{array}{c} n_1n_2 \\ 2 \end{array}\right)-(n_1m_2+m_1n_2)$.
By the definition of $\overline{SO_{SD}}(G)$, we have

$$\begin{aligned}
\overline{SO_{SD}}(G_1\Box G_2) &= \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\Box G_2)} \sqrt{\frac{d_{G_1\Box G_2}(u_i,v_j)^2}{d_{G_1\Box G_2}(u_k,v_l)^2}+\frac{d_{G_1\Box G_2}(u_k,v_l)^2}{d_{G_1\Box G_2}(u_i,v_j)^2}} \\
&= \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\Box G_2)} \sqrt{\frac{(d_{G_1}(u_i)+d_{G_2}(v_j))^2}{(d_{G_1}(u_k)+d_{G_2}(v_l))^2}+\frac{(d_{G_1}(u_k)+d_{G_2}(v_l))^2}{(d_{G_2}(u_i)+d_{G_2}(v_j))^2}} \\
&\leq \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\Box G_2)} \sqrt{\frac{(\Delta_1+\Delta_2)^2}{(\delta_1+\delta_2)^2}+\frac{(\Delta_1+\Delta_2)^2}{(\delta_1+\delta_2)^2}} \\
&\leq \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\Box G_2)} \sqrt{2}\left(\frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}\right) \\
&\leq \overline{m}\sqrt{2}\left(\frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}\right)
\end{aligned}$$

Since G_1 and G_2 are regular, the degree of each vertex is same. Therefore, for $i=1,2$, $\Delta_i=\delta_i$ and $\overline{SO_{SD}}(G_1\Box G_2)=\overline{m}\sqrt{2}$. Hence, equality holds if the graphs G_1 and G_2 are regular.
Similarly, we follow the lower bound.
Thus, $\overline{m}\sqrt{2}\left(\frac{\delta_1+\delta_2}{\Delta_1+\Delta_2}\right) \leq \overline{SO_{SD}}(G_1\Box G_2) \leq \overline{m}\sqrt{2}\left(\frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}\right)$

□

Theorem 5.2 *Let G_1 and G_2 be two graphs with $|V(G_1)|=n_1, |E(G_1)|=m_1$ and $|V(G_2)|=n_2$, $|E(G_2)|=m_2$ respectively. Then*

$\overline{m}\sqrt{2}\left(\frac{n_2\delta_1+\delta_2}{n_2\Delta_1+\Delta_2}\right) \leq \overline{SO_{SD}}(G_1[G_2]) \leq \overline{m}\sqrt{2}\left(\frac{n_2\Delta_1+\Delta_2}{n_2\delta_1+\delta_2}\right)$.
Equality holds if G_1 and G_2 are regular.

Proof: Let $V(G_1)=\{u_i|i=1,2,\ldots,n_1\}$ and $V(G_2)=\{v_j|j=1,2,\ldots,n_2\}$ be the vertex sets of G_1 and G_2 respectively. Let δ_i and Δ_i be the minimum degree and maximum degree of the vertex of G_i, where $i=1,2$.
In $G_1[G_2], |V(G_1[G_2])| = n_1n_2$ and $|E(G_1[G_2])| = n_1m_2+m_1n_2^2$ and the number of non-adjacent edges in $G_1[G_2]$ is $\overline{m} = \begin{pmatrix} n_1n_2 \\ 2 \end{pmatrix} - \left(n_1m_2+m_1n_2^2\right)$.
By the definition of $\overline{SO_{SD}}(G)$, we have

$$\begin{aligned}
\overline{SO_{SD}}(G_1[G_2]) &= \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1[G_2])} \sqrt{\frac{d_{G_1[G_2]}(u_i,v_j)^2}{d_{G_1[G_2]}(u_k,v_l)^2}+\frac{d_{G_1[G_2]}(u_k,v_l)^2}{d_{G_1[G_2]}(u_i,v_j)^2}} \\
&= \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1[G_2])} \sqrt{\frac{(n_2d_{G_1}(u_i)+d_{G_2}(v_j))^2}{(n_2d_{G_1}(u_k)+d_{G_2}(v_l))^2}+\frac{(n_2d_{G_1}(u_k)+d_{G_2}(v_l))^2}{(n_2d_{G_1}(u_i)+d_{G_2}(v_j))^2}} \\
&\leq \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1[G_2])} \sqrt{\frac{(n_2\Delta_1+\Delta_2)^2}{(n_2\delta_1+\delta_2)^2}+\frac{(n_2\Delta_1+\Delta_2)^2}{(n_2\delta_1+\delta_2)^2}} \\
&\leq \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1[G_2])} \sqrt{2}\left(\frac{n_2\Delta_1+\Delta_2}{n_2\delta_1+\delta_2}\right) \\
&\leq \overline{m}\sqrt{2}\left(\frac{n_2\Delta_1+\Delta_2}{n_2\delta_1+\delta_2}\right)
\end{aligned}$$

when G_1 and G_2 are regular, the degree of each vertex is same. Therefore, equality holds.
Similarly, we follow the lower bound.
Thus, $\overline{m}\sqrt{2}\left(\frac{n_2\delta_1+\delta_2}{n_2\Delta_1+\Delta_2}\right) \leq \overline{SO_{SD}}(G_1[G_2]) \leq \overline{m}\sqrt{2}\left(\frac{n_2\Delta_1+\Delta_2}{n_2\delta_1+\delta_2}\right)$ □

Theorem 5.3 *Let G_1 and G_2 be two graphs with $|V(G_1)|=n_1, |E(G_1)|=m_1$ and $|V(G_2)|=n_2$, $|E(G_2)|=m_2$ respectively. Then*
$\overline{m}\sqrt{2}\left(\frac{\delta_1\delta_2}{\Delta_1\Delta_2}\right) \leq \overline{SO_{SD}}(G_1\otimes G_2) \leq \overline{m}\sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right)$.
Equality holds if G_1 and G_2 are regular.

Proof: Let $V(G_1)=\{u_1,u_2,\ldots,u_{n1}\}$ and $V(G_2)=\{v_1,v_2,\ldots,v_{n2}\}$ be the disjoint vertex sets of G_1 and G_2 respectively. Let $mindeg(G_i)=\delta_i$ and $maxdeg(G_i)=\Delta_i$ for $i=1,2$.
In $G_1\otimes G_2, |V(G_1\otimes G_2)|=n_1n_2$ and $|E(G_1\otimes G_2)|=2m_1m_2$ and the number of non-adjacent edges in $G_1\otimes G_2$ is $\overline{m} = \begin{pmatrix} n_1n_2 \\ 2 \end{pmatrix} - 2m_1m_2$.

By the definition, we have

$$\begin{aligned}
\overline{SO_{SD}}(G_1 \otimes G_2) &= \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\otimes G_2)} \sqrt{\frac{d_{G_1\otimes G_2}(u_i,v_j)^2}{d_{G_1\otimes G_2}(u_k,v_l)^2} + \frac{d_{G_1\otimes G_2}(u_k,v_l)^2}{d_{G_1\otimes G_2}(u_i,v_j)^2}} \\
&= \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\otimes G_2)} \sqrt{\frac{((d_{G_1}(u_i)d_{G_2}(v_j))^2}{(d_{G_1}(u_k)d_{G_2}(v_l))^2} + \frac{(d_{G_1}(u_k)d_{G_2}(v_l))^2}{(d_{G_1}(u_i)d_{G_2}(v_j))^2}} \\
&\leq \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\otimes G_2)} \sqrt{\frac{(\Delta_1\Delta_2)^2}{(\delta_1\delta_2)^2} + \frac{(\Delta_1\Delta_2)^2}{(\delta_1\delta_2)^2}} \\
&\leq \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\otimes G_2)} \sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right) \\
&\leq \overline{m}\sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right)
\end{aligned}$$

When the graphs G_1 and G_2 are regular, we notice that, degree of each vertex is same. Therefore, equality holds.
Similarly, we follow the lower bound.
Thus, $\overline{m}\sqrt{2}\left(\frac{\delta_1\delta_2}{\Delta_1\Delta_2}\right) \leq \overline{SO_{SD}}(G_1 \otimes G_2) \leq \overline{m}\sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right)$. □

Theorem 5.4 *Let G_1 and G_2 be two graphs with vertices n_1, n_2 edges m_1, m_2 respectively. Then* $\overline{m}\sqrt{2}\left(\frac{\delta_1+\delta_2+\delta_1\delta_2}{\Delta_1+\Delta_2+\Delta_1\Delta_2}\right) \leq \overline{SO_{SD}}(G_1 \boxtimes G_2) \leq \overline{m}\sqrt{2}\left(\frac{\Delta_1+\Delta_2+\Delta_1\Delta_2}{\delta_1+\delta_2+\delta_1\delta_2}\right)$.
Equality holds if G_1 and G_2 are regular.

Proof: Let $V(G_1) = \{u_1, u_2, \ldots, u_{n1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n2}\}$ be the disjoint vertex sets of G_1 and G_2 respectively. Let $mindeg(G_i) = \delta_i$ and $maxdeg(G_i) = \Delta_i$ for $i = 1, 2$.
In $G_1 \boxtimes G_2, |V(G_1 \boxtimes G_2)| = n_1n_2$ and $|E(G_1 \boxtimes G_2)| = n_1m_2 + m_1n_2 + 2m_1m_2$ and the number of non-adjacency edges in $G_1 \boxtimes G_2$ is $\overline{m} = \begin{pmatrix} n_1n_2 \\ 2 \end{pmatrix} - (n_1m_2 + m_1n_2 + 2m_1m_2)$.
By the definition of $\overline{SO_{SD}}(G)$, we have

$$\overline{SO_{SD}}(G_1 \boxtimes G_2) = \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\boxtimes G_2)} \sqrt{\frac{d_{G_1\boxtimes G_2}(u_i,v_j)^2}{d_{G_1\otimes G_2}(u_k,v_l)^2} + \frac{d_{G_1\boxtimes G_2}(u_k,v_l)^2}{d_{G_1\boxtimes G_2}(u_i,v_j)^2}}$$

Here,

$$\begin{aligned}
d_{G_1\boxtimes G_2}(u_i,v_j) &= d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i)d_{G_2}(v_j) \\
&\leq \Delta_1 + \Delta_2 + \Delta_1\Delta_2
\end{aligned}$$

$$\begin{aligned}
d_{G_1\boxtimes G_2}(u_k,v_l) &= d_{G_1}(u_k) + d_{G_2}(v_l) + d_{G_1}(u_k)d_{G_2}(v_l) \\
&\leq \delta_1 + \delta_2 + \delta_1\delta_2
\end{aligned}$$

Hence,

$$\begin{aligned}
\overline{SO_{SD}}(G_1 \boxtimes G_2) &\leq \sum_{(u_i,v_j),(u_k,v_l)\notin E(G_1\boxtimes G_2)} \sqrt{2}\left(\frac{\Delta_1+\Delta_2+\Delta_1\Delta_2}{\delta_1+\delta_2+\delta_1\delta_2}\right) \\
&\leq \overline{m}\sqrt{2}\left(\frac{\Delta_1+\Delta_2+\Delta_1\Delta_2}{\delta_1+\delta_2+\delta_1\delta_2}\right)
\end{aligned}$$

when the graphs G_1 and G_2 are regular, the degree of each vertex is same. Therefore, equality holds. Similarly, we follow the lower bound. Thus
$\overline{m}\sqrt{2}\left(\frac{\delta_1+\delta_2+\delta_1\delta_2}{\Delta_1+\Delta_2+\Delta_1\Delta_2}\right) \leq \overline{SO_{SD}}(G_1 \boxtimes G_2) \leq \overline{m}\sqrt{2}\left(\frac{\Delta_1+\Delta_2+\Delta_1\Delta_2}{\delta_1+\delta_2+\delta_1\delta_2}\right)$. □

6. Relations between Sombor symmetric division index and coindex

Theorem 6.1 *For the graph G_i, where $i \in \{1,2\}$ with n_1, n_2 vertices and m_1, m_2 edges,*

$$\overline{SO_{SD}}(G_1 \Box G_2) \leq \sqrt{2}\left(\frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}\right)\begin{pmatrix} n_1n_2 \\ 2 \end{pmatrix} - SO_{SD}(G_1 \Box G_2)$$

Equality holds if G_i is regular.

Proof: From Theorem 5.1 we have

$$\overline{SO_{SD}}(G_1 \Box G_2) \leq \overline{m}\sqrt{2}\left(\frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}\right) \, and \, \overline{m} = \begin{pmatrix} n_1n_2 \\ 2 \end{pmatrix} - (n_1m_2 + m_1n_2)$$

From Theorem 3.1

$$SO_{SD}(G_1 \Box G_2) \leq \sqrt{2}\left(\frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}\right)(n_1m_2 + m_1n_2)$$

Clearly, we notice that

$$\overline{SO_{SD}}(G_1 \Box G_2) \leq SO_{SD}(G_1 \Box G_2)$$

From the above three inequalities, we obtain the result as

$$\overline{SO_{SD}}(G_1 \Box G_2) \leq \sqrt{2}\left(\frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}\right)\begin{pmatrix} n_1n_2 \\ 2 \end{pmatrix} - SO_{SD}(G_1 \Box G_2)$$

□

Theorem 6.2 *For the graph G_i, where $i \in \{1,2\}$ with n_1, n_2 vertices and m_1, m_2 edges,*

$$\overline{SO_{SD}}(G_1[G_2]) \leq \sqrt{2}\left(\frac{n_2\Delta_1+\Delta_2}{n_2\delta_1+\delta_2}\right)\begin{pmatrix} n_1n_2 \\ 2 \end{pmatrix} - SO_{SD}(G_1[G_2])$$

Equality holds if G_i is regular.

Proof: From Theorem 5.2 we have

$$\overline{SO_{SD}}(G_1[G_2]) \leq \overline{m}\sqrt{2}\left(\frac{n_2\Delta_1+\Delta_2}{n_2\delta_1+\delta_2}\right) \, and \, \overline{m} = \begin{pmatrix} n_1n_2 \\ 2 \end{pmatrix} - \left(n_1m_2 + m_1n_2^2\right)$$

From Theorem 3.2

$$SO_{SD}(G_1[G_2]) \leq \sqrt{2}\left(\frac{n_2\Delta_1+\Delta_2}{n_2\delta_1+\delta_2}\right)(n_1m_2 + m_1n_2^2)$$

Also, we clarify that

$$\overline{SO_{SD}}(G_1[G_2]) \leq SO_{SD}(G_1[G_2])$$

Using above three inequalities, we obtain the result as

$$\overline{SO_{SD}}(G_1[G_2]) \leq \sqrt{2}\left(\frac{n_2\Delta_1+\Delta_2}{n_2\delta_1+\delta_2}\right)\begin{pmatrix} n_1n_2 \\ 2 \end{pmatrix} - SO_{SD}(G_1[G_2])$$

□

Theorem 6.3 *For the graph G_i, where $i \in \{1,2\}$ with n_1, n_2 vertices and m_1, m_2 edges,*

$$\overline{SO_{SD}}(G_1 \otimes G_2) \leq \sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right)\left(\begin{array}{c} n_1n_2 \\ 2 \end{array}\right) - SO_{SD}(G_1 \otimes G_2)$$

Equality holds if G_i is regular.

Proof: From Theorem 5.3 we have

$$\overline{SO_{SD}}(G_1 \otimes G_2) \leq \overline{m}\sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right) \; and \, \overline{m} = \left(\begin{array}{c} n_1n_2 \\ 2 \end{array}\right) - 2m_1m_2$$

From Theorem 3.3

$$SO_{SD}(G_1 \otimes G_2) \leq 2\sqrt{2}\left(\frac{\Delta_1\Delta_2}{\delta_1\delta_2}\right) m_1m_2$$

Clearly, we notice that

$$\overline{SO_{SD}}(G_1 \otimes G_2) \leq SO_{SD}(G_1 \otimes G_2)$$

From the above three inequalities, we obtain the result as

$$\overline{SO_{SD}}(G_1 \square G_2) \leq \sqrt{2}\left(\frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2}\right)\left(\begin{array}{c} n_1n_2 \\ 2 \end{array}\right) - SO_{SD}(G_1 \square G_2)$$

□

Theorem 6.4 *For the graph G_i, where $i \in \{1,2\}$ with n_1, n_2 vertices and m_1, m_2 edges,*

$$\overline{SO_{SD}}(G_1 \boxtimes G_2) \leq \sqrt{2}\left(\frac{\Delta_1 + \Delta_2 + \Delta_1\Delta_2}{\delta_1 + \delta_2 + \delta_1\delta_2}\right)\left(\begin{array}{c} n_1n_2 \\ 2 \end{array}\right) - SO_{SD}(G_1 \boxtimes G_2)$$

Equality holds if G_i is regular.

Proof: From Theorem 5.4 we have

$$\overline{SO_{SD}}(G_1 \boxtimes G_2) \leq \overline{m}\sqrt{2}\left(\frac{\Delta_1 + \Delta_2 + \Delta_1\Delta_2}{\delta_1 + \delta_2 + \delta_1\delta_2}\right) \; and \, \overline{m} = \left(\begin{array}{c} n_1n_2 \\ 2 \end{array}\right) - (n_1m_2 + m_1n_2 + 2m_1m_2)$$

From Theorem 3.4

$$SO_{SD}(G_1 \boxtimes G_2) \leq \sqrt{2}\left(\frac{\Delta_1 + \Delta_2 + \Delta_1\Delta_2}{\delta_1 + \delta_2 + \delta_1\delta_2}\right)(n_1m_2 + m_1n_2 + 2m_1m_2)$$

Clearly, we notice that

$$\overline{SO_{SD}}(G_1 \boxtimes G_2) \leq SO_{SD}(G_1 \boxtimes G_2)$$

From the above three inequalities, we obtain the result as

$$\overline{SO_{SD}}(G_1 \boxtimes G_2) \leq \sqrt{2}\left(\frac{\Delta_1 + \Delta_2 + \Delta_1\Delta_2}{\delta_1 + \delta_2 + \delta_1\delta_2}\right)\left(\begin{array}{c} n_1n_2 \\ 2 \end{array}\right) - SO_{SD}(G_1 \boxtimes G_2)$$

□

7. Chemical applicability of $\overline{SO_{SD}}(G)$

As explained in [4], we explained the correlation analysis between Sombor symmetric division coindex and pi-electron energy of some selected hetero molecules and PAHs.

7.1. Correlation analysis between Sombor symmetric division coindex and pi-electron energy of some selected hetero molecules.

This section explores the chemical applicability of $\overline{SO_{SD}}(G)$ index upon performing correlation analysis with total pi-electron energy of selected Hetero molecules. For this we have calculated $\overline{SO_{SD}}(G)$ values of some selected Hetero molecules . The following Table 1 consists of $\overline{SO_{SD}}(G)$ values and pi-electron energy of selected Hetero molecules. Figure 1 illustrate the graph corresponding to Table 1.

Code	$\overline{SO_{SD}}(G)$	**Total π-electron energy**
$H2$	3.799	5.66
$H3$	3.799	5.76
$H5$	3.799	6.82
$H6$	3.536	5.23
$H7$	4.243	6.69
$H8$	4.243	9.06
$H9$	4.243	9.1
$H10$	4.243	9.07
$H11$	4.243	9.65
$H12$	5.042	8.19
$H13$	5.808	12.21
$H14$	5.832	12.22
$H15$	5.832	12.21
$H16$	5.715	11
$H17$	7.105	14.23
$H18$	7.105	14.23
$H19$	7.838	16.15
$H20$	7.826	16.12
$H21$	6.434	13.46
$H22$	6.411	13.59
$H23$	9.912	20.1
$H24$	9.912	21.02
$H25$	9.932	20.56
$H26$	9.912	21.62
$H27$	11.44	24.23
$H28$	9.232	19.39

Table 1: Sombor symmetric division co-index of Hetero molecules

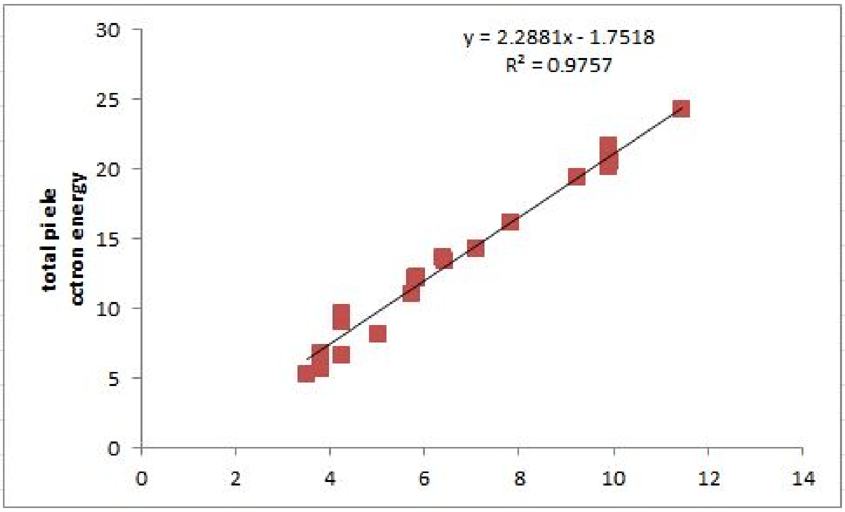

Figure 1:

7.2. Correlation analysis between Sombor symmetric division coindex and π-electron energy of some selected PAHs

This section explores the chemical applicability of $\overline{SO_{SD}}(G)$ index upon performing correlation analysis with total π-electron energy of selected PAH's. For this we have calculated $\overline{SO_{SD}}(G)$ values of some selected PAH's. The following Table 2 consists of $\overline{SO_{SD}}(G)$ values and π-electron energy of selected PAH's. Table 2 shows the Sombor symmetric division co-index of PAHs. The graph corresponding to Table 2 is shown in Figure 2.

Molecules	$\overline{SO_{SD}}(G)$	**Total π-electron energy**
Benzene	4.243	8
Naphthalene	7.188	13.68
Phenanthrene	10.09	19.44
Anthracene	9.925	19.31
Chrysene	12.78	25.19
Benzanthracene	12.75	25.1
Triphenylene	12.75	25.27
Tetracene	12.76	25.18
Benzo[a]pyrene	14.17	28.22
Benzo[e]pyrene	14.17	28.22
Perylene	14.18	28.24
Anthanthrene	15.58	31.25
Benzoperylene	15.58	31.42
Dibenzo(a,c)*anthracene*	15.58	30.94
Dibenzo(a,h)*anthracene*	15.58	30.88
Dibenzo(a,j)*anthracene*	15.58	30.94
Picene	15.59	33.95
Dibenzo(a,h)*pyrene*	17.0	30.94
Dibenzo(a,e)*pyrene*	16.99	30.94
Pyrene	11.36	22.5
Coronene	16.99	34.57
Fluoranthene	11.35	22.5
Pentacene	15.58	30.54
Acenapthalene	8.564	30.54
Azulene	7.188	30.54

Table 2: Sombor symmetric division co-index of PAHs

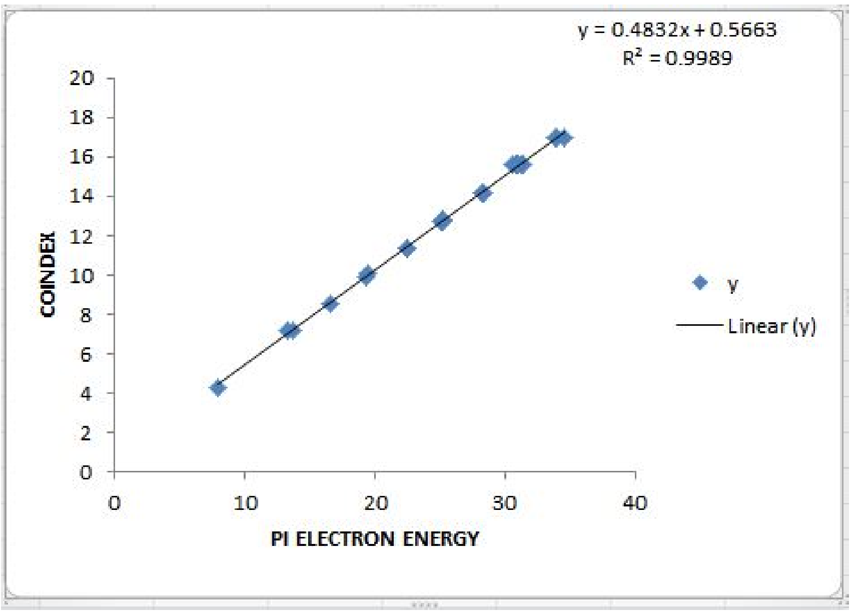

Figure 2:

8. Conclusion

In this study, we have defined the topological indices called Sombor symmetric division index and coindex of graph and give some properties there on. We have also computed some bounds on the the Sombor symmetric division degree index and coindex under some standard graph operations, such as Cartesian product, composition, direct product and strong product. Additionally, we have established a significant correlation between the Sombor symmetric division index and the π-electron energy of selected heteroatomic molecules and PAHs.

Work in this paper is entirely the result of the authors' independent work. No external contributions or assistance were involved in the preparation of this study.

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