



Eta Quotients of Level 32 and Weight 1: Classification and Applications

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ABSTRACT: We find all the eta quotients in the spaces $M_1\left(\Gamma_0(32), \left(\frac{d}{*}\right)\right)$ with $d = -4, -8$ of modular forms, and we determine their Fourier coefficients, where $\left(\frac{d}{*}\right)$ is the Legendre-Jacobi-Kronecker symbol in the group of Dirichlet characters modulo 32 with values in the rational field \mathbb{Q} .

Key Words: Eta quotients, modular forms, Fourier coefficients of cusp forms, eta function.

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1. Introduction

Let \mathbb{N} , \mathbb{N}_0 , \mathbb{Z} , and \mathbb{C} denote the sets of positive integers, nonnegative integers, integers, and complex numbers, respectively. Let $N \in \mathbb{N}$, $k \in \mathbb{Z}$, and let χ be a Dirichlet character whose modulus divides N . We define the congruence subgroup $\Gamma_0(N)$ of level N by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \equiv 0 \pmod{N} \right\}.$$

Let $M_k(\Gamma_0(N), \chi)$ denote the space of modular forms of weight k and character χ on the congruence subgroup $\Gamma_0(N)$. The subspaces of Eisenstein series and cusp forms are denoted by $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$, respectively. It is well-known that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi). \tag{1.1}$$

For instance, (see [7, p. 83]).

The Dedekind eta function $\eta(z)$ is a holomorphic function on the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\},$$

defined by

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).$$

A product of the form

$$f(z) = \prod_{1 \leq \delta | N} \eta^{r_\delta}(\delta z), \tag{1.2}$$

where $r_\delta \in \mathbb{Z}$, not all zero, is called an eta quotient.

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Let χ and ψ be Dirichlet characters. For $n \in \mathbb{N}$, we define the generalized divisor sum $\sigma_{(\chi, \psi)}(n)$ by

$$\sigma_{(\chi, \psi)}(n) = \sum_{1 \leq m|n} \chi(m)\psi(n/m). \quad (1.3)$$

If $n \notin \mathbb{N}$, we set $\sigma_{(\chi, \psi)}(n) = 0$.

Let χ_0 denote the trivial character, *i.e.*, $\chi_0(m) = 1$ for all $m \in \mathbb{Z}$. In particular, we observe that $\sigma_{(\chi_0, \chi_0)}(n)$ coincides with the classical divisor-counting function

$$\sigma_0(n) = \sum_{1 \leq m|n} 1.$$

We now define two Dirichlet characters modulo 32 by

$$\chi_1(m) = \left(\frac{-32}{m} \right), \quad \chi_2(m) = \left(\frac{-4}{m} \right), \quad (m \in \mathbb{Z}),$$

where $\left(\frac{d}{\cdot} \right)$ denotes the Legendre-Jacobi-Kronecker symbol considered as a Dirichlet character modulo 32 with values in the field of rational numbers. These are precisely nontrivial Dirichlet characters modulo 32.

The cusps of $\Gamma_0(N)$ can be represented by rational numbers $\frac{a}{c}$, where $a \in \mathbb{Z}$, $c \in \mathbb{N}$, $c \mid N$, and $\gcd(a, c) = 1$; (see [6, p. 320]).

For the group $\Gamma_0(32)$, a complete set of representatives of the cusps can be chosen as

$$\frac{1}{32}, \quad \frac{1}{16}, \quad \frac{1}{8}, \quad \frac{1}{4}, \quad \frac{1}{2}, \quad \frac{1}{1}, \quad \frac{3}{4}, \quad \frac{3}{8}. \quad (1.4)$$

Let $f(z)$ be an eta quotient as defined in (1.2). A formula for the order $v_{a/c}(f)$ of f at the cusp $\frac{a}{c}$ (see [6, p. 320]) is given by

$$v_{a/c}(f) = \frac{N}{24 \gcd(c^2, N)} \sum_{1 \leq \delta \mid N} \frac{\gcd(\delta, c)^2 r_\delta}{\delta}. \quad (1.5)$$

It follows from the dimension formulae [7, Section 6.3] that the only nontrivial modular form spaces of level 32 with trivial cuspidal subspaces are

$$M_1(\Gamma_0(32), \chi_1) \quad \text{and} \quad M_1(\Gamma_0(32), \chi_2).$$

Moreover, we have that

$$\begin{aligned} \dim(M_1(\Gamma_0(32), \chi_1)) &= \dim(E_1(\Gamma_0(32), \chi_1)) = 4, \\ \dim(M_1(\Gamma_0(32), \chi_2)) &= \dim(E_1(\Gamma_0(32), \chi_2)) = 4. \end{aligned} \quad (1.6)$$

In this paper, we find all the eta quotients in $M_1(\Gamma_0(32), \chi_1)$ and $M_1(\Gamma_0(32), \chi_2)$, and we determine their Fourier coefficients.

2. Preliminary results

Throughout the remainder of this paper we use the notation $q = q(z) = e^{2\pi iz}$ with $z \in \mathbb{H}$. We define the Eisenstein series $E_{\chi_1, \chi_0}(q)$, $E_{\chi_2, \chi_0}(q)$ and $E_{\chi_2, \chi_1}(q)$ by

$$\begin{aligned} E_{\chi_1, \chi_0}(q) &= \frac{1}{4} + \sum_{n=1}^{\infty} \sigma_{(\chi_1, \chi_0)}(n)q^n, \quad E_{\chi_2, \chi_0}(q) = \frac{1}{2} + \sum_{n=1}^{\infty} \sigma_{(\chi_2, \chi_0)}(n)q^n, \\ E_{\chi_2, \chi_1}(q) &= \sum_{n=1}^{\infty} \sigma_{(\chi_2, \chi_1)}(n)q^n. \end{aligned}$$

In view of (2.1) for $N = 32$, we define an eta quotient $f(z)$ by

$$f(z) = \eta^{r_1}(z)\eta^{r_2}(2z)\eta^{r_4}(4z)\eta^{r_8}(8z)\eta^{r_{16}}(16z)\eta^{r_{32}}(32z). \quad (2.1)$$

Theorem 2.1 *Let $f(z) \in M_1(\Gamma_0(32), \chi_1)$ be an eta quotient given by (2.1), and let $f(z) = \sum_{n=0}^{\infty} a_n q^n$ be its Fourier series expansion. Then*

$$f(z) = b_1 E_{\chi_1, \chi_0}(q) + b_2 E_{\chi_1, \chi_0}(q^2) + b_3 E_{\chi_1, \chi_0}(q^4) + b_4 E_{\chi_1, \chi_0}(q^8)$$

for unique scalars $b_1, b_2, b_3, b_4 \in \mathbb{C}$, and the Fourier coefficients a_n are given by

$$\begin{aligned} a_n &= b_1 \sigma_{\chi_1, \chi_0}(n) + b_2 \sigma_{\chi_1, \chi_0}(n/2) + b_3 \sigma_{\chi_1, \chi_0}(n/4) + b_4 \sigma_{\chi_1, \chi_0}(n/8) \text{ for } n \geq 1, \\ a_0 &= \frac{1}{4}(b_1 + b_2 + b_3 + b_4). \end{aligned}$$

Proof: It follows from (1.6) and [7, Theorem 5.9] that the set of Eisenstein series

$$\{E_{\chi_1, \chi_0}(q), E_{\chi_1, \chi_0}(q^2), E_{\chi_1, \chi_0}(q^4), E_{\chi_1, \chi_0}(q^8)\}$$

is a basis for $M_1(\Gamma_0(32), \chi_1)$. Thus,

$$f(z) = b_1 E_{\chi_1, \chi_0}(q) + b_2 E_{\chi_1, \chi_0}(q^2) + b_3 E_{\chi_1, \chi_0}(q^4) + b_4 E_{\chi_1, \chi_0}(q^8)$$

for some unique scalars $b_1, b_2, b_3, b_4 \in \mathbb{C}$, from which the assertion follows by equating the coefficients of q^n on both sides. \square

Similarly to Theorem 2.1, we prove the following theorem.

Theorem 2.2 *Let $f(z) \in M_1(\Gamma_0(32), \chi_2)$ be an eta quotient given by (2.1), and let $f(z) = \sum_{n=0}^{\infty} a_n q^n$ be its Fourier series expansion. Then*

$$f(z) = b_1 E_{\chi_2, \chi_0}(q) + b_2 E_{\chi_2, \chi_0}(q^2) + b_3 E_{\chi_2, \chi_0}(q^4) + b_4 E_{\chi_2, \chi_1}(q)$$

for unique scalars $b_1, b_2, b_3, b_4 \in \mathbb{C}$, and the Fourier coefficients a_n are given by

$$\begin{aligned} a_n &= b_1 \sigma_{\chi_2, \chi_0}(n) + b_2 \sigma_{\chi_2, \chi_0}(n/2) + b_3 \sigma_{\chi_2, \chi_0}(n/4) + b_4 \sigma_{\chi_2, \chi_1}(n) \text{ for } n \geq 1, \\ a_0 &= \frac{1}{4}(b_1 + b_2 + b_3). \end{aligned}$$

We use the following lemma to determine if certain eta quotients are modular forms. See [3, Theorem 5.7, p. 99], [4, Corollary 2.3, p. 37], [2, p. 174], and [5].

Lemma 2.3 *Let $f(z)$ be an eta quotient given by (2.1), and let $k = \frac{1}{2} \sum_{1 \leq \delta | N} r_\delta$ and $s = \prod_{1 \leq \delta | N} \delta^{r_\delta}$. Suppose that the following conditions are satisfied:*

- (i) $\sum_{1 \leq \delta | N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,
- (ii) $\sum_{1 \leq \delta | N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,
- (iii) $\sum_{1 \leq \delta | N} \frac{\gcd(d, \delta)^2 \cdot r_\delta}{\delta} \geq 0$ for each positive divisor d of N ,
- (iv) k is an integer.

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, where the character χ is given by

$$\chi(m) = \left(\frac{(-1)^k s}{m} \right).$$

We take $N = 32$ and $k = 1$ in Lemma 2.3 to obtain the following theorem.

Theorem 2.4 *Let $f(z)$ be an eta quotient given by (2.1), which satisfies the conditions (i)-(iv) in Lemma 2.3 with*

$$r_1 + r_2 + r_4 + r_8 + r_{16} + r_{32} = 2. \quad (2.2)$$

Then $f(z) \in M_1(\Gamma_0(32), \chi)$, where the character χ is determined by

$$f(z) \in \begin{cases} M_1(\Gamma_0(32), \chi_1), & \text{if } r_4 + r_{16} + r_{32} \equiv 1 \pmod{2}; \\ M_1(\Gamma_0(32), \chi_2), & \text{if } r_4 + r_{16} + r_{32} \equiv 0 \pmod{2}. \end{cases} \quad (2.3)$$

Proof: For $N = 32$, we have

$$s = \prod_{1 \leq \delta | 32} \delta^{r_\delta} = 1^{r_1} 2^{r_2} 3^{r_4} 8^{r_8} 16^{r_{16}} 32^{r_{32}} = 2^{r_2 + 2r_4 + 3r_8 + 4r_{16} + 5r_{32}}.$$

The conditions (i) and (ii) in Lemma 2.3 becomes

$$r_1 + 2r_2 + 4r_4 + 8r_8 + 16r_{16} + 32r_{32} \equiv 0 \pmod{24},$$

$$32r_1 + 16r_2 + 8r_4 + 4r_8 + 2r_{16} + r_{32} \equiv 0 \pmod{24}.$$

Let

$$\nu_2(s) = \sum_{\delta | 32} r_\delta \cdot \nu_2(\delta) = r_2 + 2r_4 + 3r_8 + 4r_{16} + 5r_{32}. \quad (2.4)$$

Reducing (2.4) modulo 2, we obtain

$$\nu_2(s) \equiv r_2 + r_8 + r_{32} \pmod{2}. \quad (2.5)$$

Let us denote:

$$\begin{aligned} r_1 + r_2 + r_4 + r_8 + r_{16} + r_{32} &= 2, \\ r_1 + 2r_2 + 4r_4 + 8r_8 + 16r_{16} + 32r_{32} &\equiv 0 \pmod{24}, \\ 32r_1 + 16r_2 + 8r_4 + 4r_8 + 2r_{16} + r_{32} &\equiv 0 \pmod{24}. \end{aligned} \quad (2.6)$$

Reducing (2.6) modulo 2, we find:

$$r_1 \equiv 0 \pmod{2}, \quad r_{32} \equiv 0 \pmod{2}. \quad (2.7)$$

Substituting (2.7) into (2.6), we get:

$$r_2 + r_4 + r_8 + r_{16} \equiv 0 \pmod{2}. \quad (2.8)$$

From (2.8), we solve for r_2 :

$$r_2 \equiv r_4 + r_8 + r_{16} \pmod{2}. \quad (2.9)$$

Substituting (2.9) into (2.5), we have:

$$\nu_2(s) \equiv (r_4 + r_8 + r_{16}) + r_8 + r_{32} \pmod{2}, \quad (2.10)$$

$$\nu_2(s) \equiv r_4 + 2r_8 + r_{16} + r_{32} \pmod{2}. \quad (2.11)$$

Hence:

$$\nu_2(s) \equiv r_4 + r_{16} + r_{32} \pmod{2}. \quad (2.12)$$

The character associated to $f(z)$ is given by

$$\chi(m) = \left(\frac{(-1)^k s}{m} \right) = \left(\frac{-s}{m} \right).$$

Since s is a power of 2, say $s = 2^{\nu_2(s)}$, we have $\chi(m) = \left(\frac{-2^{\nu_2(s)}}{m}\right)$, which depends only on the parity of $\nu_2(s)$. If $\nu_2(s) \equiv 1 \pmod{2}$, then $\chi(m) = \left(\frac{-8}{m}\right) = \chi_1(m)$. If $\nu_2(s) \equiv 0 \pmod{2}$, then $\chi(m)(s) = \left(\frac{-4}{m}\right) = \chi_2(m)$. To confirm this, we compute:

$$\begin{aligned}\chi_1(5) &= \left(\frac{-8}{5}\right) = -1, \\ \chi_1(31) &= \left(\frac{-8}{31}\right) = -1, \\ \chi_2(5) &= \left(\frac{-4}{5}\right) = 1, \\ \chi_2(31) &= \left(\frac{-4}{31}\right) = -1.\end{aligned}$$

These match the given definitions of χ_1 and χ_2 , and the result follows. \square

3. Main results

Theorem 3.1 *Let $f(z)$ be an eta quotient given by (2.1). Then we have $f(z) \in M_1(\Gamma_0(32), \chi_1)$ if and only if*

$$\begin{aligned}r_1 + 2r_2 + 4r_4 + 8r_4 + 8r_8 + 16r_{16} + 32r_{32} &\equiv 0 \pmod{24}, \\ 32r_1 + 16r_2 + 8r_4 + 4r_8 + 2r_{16} + r_{32} &\equiv 0 \pmod{24}, \\ 0 \leq v_{1/c}(f) < 4 \text{ for } c &= \frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{1}{1}, \\ r_1 + r_2 + r_4 + r_8 + r_{16} + r_{32} &= 2, \\ r_4 + r_{16} + r_{32} &\equiv 1 \pmod{2}.\end{aligned}$$

Proof: Let $f(z) \in M_1(\Gamma_0(32), \chi_1)$ be an eta quotient given by (2.1). By (1.6) we have $\dim(M_1(\Gamma_0(32), \chi_1)) = 4$. We define the eta quotients $f_1(z), f_2(z), f_3(z), f_4(z)$ by

$$\begin{aligned}f_1(z) &= \frac{\eta(2z)^3 \cdot \eta(4z)^3}{\eta(z)^2 \cdot \eta(8z)^2}, \\ f_2(z) &= \frac{\eta(2z)^5 \cdot \eta(16z)^5}{\eta(z)^2 \cdot \eta(4z)^2 \cdot \eta(8z)^2 \cdot \eta(32z)^2}, \\ f_3(z) &= \frac{\eta(2z)^5 \cdot \eta(32z)^2}{\eta(z)^2 \cdot \eta(4z)^2 \cdot \eta(16z)}, \\ f_4(z) &= \frac{\eta(2z)^7}{\eta(z)^2 \cdot \eta(4z)^3}.\end{aligned}$$

By Lemma 2.3, we have $f_1(z), f_2(z), f_3(z), f_4(z) \in M_1(\Gamma_0(32), \chi_1)$. One can easily see that the set $\{f_1(z), f_2(z), f_3(z), f_4(z)\}$ is linearly independent, and so it is a basis for $M_1(\Gamma_0(32), \chi_1)$. Appealing to (1.4) and (1.5), we have

$v_{1/c}(\cdot)/c$	1	1/2	1/4	1/8	1/16	1/32
f_1	0	2	1	0	0	0
f_2	0	2	0	0	2	0
f_3	0	2	0	0	0	2
f_4	1	3	0	0	0	0

Thus, for any $b_1, b_2, b_3, b_4 \in \mathbb{C}$ we have

$$v_{1/c}(b_1 f_1 + b_2 f_2 + b_3 f_3 + b_4 f_4) \in \mathbb{N}_0.$$

As $f(z)$ can be expressed as a linear combination of $f_1(z)$, $f_2(z)$, $f_3(z)$, and $f_4(z)$, we have

$$v_{1/c}(f) \in \mathbb{N}_0,$$

from which the second and first assertions follow, respectively. The third assertion follows from [4, Corollary 2.3] and the fifth assertion follows from (2.3). The converse follows from Theorem 2. \square

Theorem 3.2 *Let $f(z)$ be an eta quotient given by (2.1). Then we have $f(z) \in M_1(\Gamma_0(32), \chi_2)$ if and only if*

$$\begin{aligned} r_1 + 2r_2 + 4r_4 + 8r_8 + 16r_{16} + 32r_{32} &\equiv 0 \pmod{24}, \\ 32r_1 + 16r_2 + 8r_4 + 4r_8 + 2r_{16} + r_{32} &\equiv 0 \pmod{24}, \\ 0 \leq v_{1/c}(f) &< 4 \text{ for } c = 1, 2, 4, 8, 16, 32, \\ r_1 + r_2 + r_4 + r_8 + r_{16} + r_{32} &= 2, \\ r_4 + r_{16} + r_{32} &\equiv 0 \pmod{2}. \end{aligned}$$

Proof: Let $f(z) \in M_1(\Gamma_0(32), \chi_2)$ be an eta quotient given by (2.1). By (1.6) we have $\dim(M_1(\Gamma_0(32), \chi_2)) = 4$. We define the eta quotients $f_1(z)$, $f_2(z)$, $f_3(z)$ and $f_4(z)$ by

$$\begin{aligned} f_1(z) &= \frac{\eta(4\tau)^{10}}{\eta(2\tau)^4\eta(8\tau)^4}, \\ f_2(z) &= \frac{\eta(4\tau)^5\eta(16\tau)^5}{\eta(2\tau)^2\eta(8\tau)^4\eta(32\tau)^2}, \\ f_3(z) &= \frac{\eta(4\tau)^5\eta(32\tau)^2}{\eta(2\tau)^2\eta(8\tau)^2\eta(16\tau)}, \\ f_4(z) &= \frac{\eta(8\tau)^{10}}{\eta(4\tau)^4\eta(16\tau)^4}. \end{aligned}$$

By Lemma 2.3, we have $f_1(z), f_2(z) \in M_1(\Gamma_0(32), \chi_2)$.

One can easily see that the set $\{f_1(z), f_2(z), f_3(z), f_4(z)\}$ is a basis for $M_1(\Gamma_0(32), \chi_2)$. By (1.4) and (1.5), we have

$v_{1/c}(\cdot)/c$	1	1/2	1/4	1/8	1/16	1/32
f_1	0	0	2	0	0	0
f_2	0	0	1	0	2	0
f_3	0	0	1	0	0	2
f_4	0	0	2	0	0	0

Then, $v_{1/c}(b_1f_1 + b_2f_2 + b_3f_3 + b_4f_4) \in \mathbb{N}_0$ for any $b_1, b_2, b_3, b_4 \in \mathbb{C}$. As $f(z)$ can be expressed as a linear combination of $f_1(z)$, $f_2(z)$, $f_3(z)$ and $f_4(z)$, we have $v_{1/c}(f) \in \mathbb{N}_0$, from which the second and first assertions follow, respectively. The third assertion follows from [4, Corollary 2.3] and the fifth assertion follows from (2.3). The converse follows from Theorem 2. \square

There are 14 eta quotients in $M_1(\Gamma_0(32), \chi_1)$ and 15 eta quotients in $M_1(\Gamma_0(32), \chi_2)$. We found all the eta quotients with Sagemath using Theorems 3.1 and 3.2. We then determined their Fourier coefficients using Theorems 2.1 and 2.2. All these eta quotients and their Fourier coefficients are listed in the appendices (Table 1 and 2).

4. Applications and remarks

Theorem 4.1 *Let $f(z) \in M_1(\Gamma_0(32), \chi_1)$ with the Fourier series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n.$$

Then

1. For all $m \geq 0$, we have $a_{16m+5} \equiv 0 \pmod{\sigma_{\chi_2, \chi_1}(16m+5)}$ and $a_{16m+13} \equiv 0 \pmod{\sigma_{\chi_2, \chi_1}(16m+5)}$.
2. For all $m \geq 0$, we have $a_{16m+7} = a_{16m+15} \equiv 0 \pmod{\sigma_{\chi_2, \chi_1}(16m+7)}$.

Proof: Suppose $n \equiv 5, 7 \pmod{8}$. Then

$$\chi_1(n) = \left(\frac{-8}{n}\right) = \left(\frac{n}{8}\right) = -1.$$

Also, for all positive divisors d of n , we have

$$\chi_1(n/d) = \left(\frac{-8}{n/d}\right) = \left(\frac{-8}{nd}\right) = \left(\frac{-8}{n}\right) \left(\frac{-8}{d}\right) = -\left(\frac{-8}{d}\right) = -\chi_1(d).$$

By pairing $\chi_1(d)$ and $\chi_1(n/d)$ for all $d \mid n$, we obtain

$$\sum_{d \mid n} \chi_1(d) = 0.$$

The assertion now follows from (1.4), (1.3), and Theorem 2. □

Theorem 4.2 Let $f(z) \in M_1(\Gamma_0(32), \chi_2)$ with the Fourier series representation

$$f(z) = \sum_{n=0}^{\infty} a_n q^n.$$

Then for all $m \geq 0$, we have $a_{12m+7} = a_{12m+11} = 0$.

Proof: Suppose $n \equiv 3 \pmod{4}$. Arguing as in the proof of Theorem 4.1, we obtain

$$\sum_{d \mid n} \chi_2(d) = 0. \tag{4.1}$$

The assertion now follows from (4.1), (1.3), and Theorem 2.2. □

The following corollary follows immediately from Theorem 4.1 and 4.2.

Corollary 4.3 If an eta quotient $f(z)$ given by (2.1) is a modular form of weight 1 with the Fourier series representation $f(z) = \sum_{n=0}^{\infty} a_n q^n$, then for all $m \geq 0$ we have

$$a_n \equiv \begin{cases} 0 \pmod{\sigma_{\chi_2, \chi_1}(16m+5)}, & \text{if } n = 16m+5, r_4 + r_{16} + r_{32} \equiv 0 \pmod{2}; \\ 0 \pmod{\sigma_{\chi_2, \chi_1}(16m+5)}, & \text{if } n = 16m+7, r_4 + r_{16} + r_{32} \equiv 0 \pmod{2}; \\ 0, & \text{if } n = 12m+5, r_3 + r_{16} + r_{32} \equiv 1 \pmod{2}. \end{cases}$$

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Conflict of interest

The authors declare that there is no conflict of interest.

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5. Appendices

Table 1: Eta quotients in $M_1(\Gamma_0(32), \chi_1)$ and their Fourier coefficients

No.	r_1	r_2	r_4	r_8	r_{16}	r_{32}	(b_1, b_2, b_3, b_4)
1	-2	3	3	-2	0	0	(2, 0, 0, 0)
2	-2	5	-2	-2	5	-2	(2, -2, 2, 0)
3	-2	5	-2	0	-1	2	(0, 1, -1, 0)
4	-2	7	-3	0	0	0	(2, -4, 4, 0)
5	0	-2	3	3	-2	0	(0, 2, 0, 0)
6	0	-2	5	-3	2	0	(1, -1, 0, 0)
7	0	0	-2	3	3	-2	(0, 0, 2, 0)
8	0	0	-2	5	-3	2	(0, 1, -1, 0)
9	0	0	0	-3	7	-2	(1, -1, 0, 0)
10	0	2	-3	5	-2	0	(0, -2, 4, 0)
11	0	2	-1	-1	2	0	(1, -1, 0, 0)
12	2	-3	5	-2	0	0	(-2, 4, 0, 0)
13	2	-1	0	-2	5	-2	(-2, 2, 2, 0)
14	2	-1	0	0	-1	2	(0, 1, -1, 0)

Table 2: Eta quotients in $M_1(\Gamma_0(32), \chi_2)$ and their Fourier coefficients

No.	r_1	r_2	r_4	r_8	r_{16}	r_{32}	(b_1, b_2, b_3, b_4)
1	0	-4	10	-4	0	0	(0, 4, 0, 0)
2	0	-2	5	-4	5	-2	(0, 2, -2, 4)
3	0	-2	5	-2	-1	2	(0, 1, 1, -2)
4	0	0	-4	10	-4	0	(0, 0, 4, 0)
5	0	0	-2	4	0	0	(1, 0, -2, 1)
6	0	0	0	-4	10	-4	(0, 0, 0, 4)
7	0	0	0	-2	4	0	(0, 1, -1, 0)
8	0	0	0	0	-2	4	(0, 0, 1, -1)
9	0	0	4	-2	0	0	(0, 0, -4, 8)
10	0	2	-1	-2	5	-2	(0, -2, 2, 4)
11	0	2	-1	0	-1	2	(0, 1, -3, 2)
12	0	4	-2	0	0	0	(0, -4, 8, 0)
13	2	-1	-2	5	-2	0	(-2, 0, 8, -2)
14	2	-1	0	-1	2	0	(1, -2, 0, 1)
15	4	-2	0	0	0	0	(-4, 4, 8, -4)

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