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A Study of Singular Solutions for a p-Laplacian Equation with Combined Power-Type Nonlinearity

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ABSTRACT: This paper is devoted to the analysis of the existence and asymptotic behavior of singular positive radial solutions of the equation

$$\Delta_p v + v^q - v^{-\delta} = 0, \qquad \text{in } \mathbb{R}^N,$$

where $N > p > 2$, $q > 1$ and $\delta > 0$. We investigate radial solutions to specific initial conditions, with a particular focus on singular solutions that vanish at a finite radius. By employing scaling techniques and comparison principles, we derive constraints on the parameters q and δ ensuring the existence of such singular solutions. Furthermore, we characterize the asymptotic behavior of these solutions.

Key Words: p-Laplacian operator, Power-Law term, singular solution, asymptotic behavior.

Contents

1. Introduction

The purpose of this article is to study the following singular p-Laplacian equation

$$\left(|v'(r)|^{p-2}v'(r)\right)' + \frac{N-1}{r}|v'(r)|^{p-2}v'(r) + v^q - v^{-\delta} = 0, \quad r > 0, \tag{1.1}$$

where $N > p > 2$, $q > 1$ and $\delta > 0$.
For convenience, we introduce the next function

$$f(v) = v^q(r) - v^{-\delta}(r), \quad \text{for each} \quad r > 0. \tag{1.2}$$

When $p = 2$ and $f(v) = v^q$, the first substantial contributions to the radial solutions were made by Emden–Fowler (see [10,11,12])), who established existence results and provided a complete classification of radial solutions. For higher dimensions, it is known that this equation involves certain critical exponents that characterize the behavior of solutions. Regarding the non-radial case, important results were obtained by Lions [15], Aviles [1], Gidas and Spruck [13], and later extended by Caffarelli, Gidas, and Spruck [7]. These works describe the existence and qualitative properties of solutions under various critical regimes. For further information about this type we refer to see [9,6].

When $p > 2$ and $f(v) = v^q$, the first results were obtained by Ni and Serrin [17], who established the existence of two critical exponents, $\frac{N(p-1)}{N-p}$ and $\frac{N(p-1)+p}{N-p}$. The case where $p > 2$ and $f(v) = v^q - v$ is studied in [5], where the existence of positive solutions in an exterior domain is established. In addition, in [2], the same equation is investigated under the assumption that the nonlinearity f is odd, behaves like v^q for large v, and admits a single zero. However, it should be noted that in these previous works, the authors impose certain regularity conditions on the second term of the equation, which are not assumed in our case.

Our aim in this article is to address an interesting question, which is to analyze the existence of singular solutions to (1.1). The presence of the mixed-power nonlinearity $f(v) = v^p - v^{-\delta}$ considerably increases the analytical difficulty. First, the corresponding solutions do not belong to standard functional

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spaces. Second, the solution v of (1.1) may exhibit a singular behavior when it vanishes at a finite point, which prevents us from using several classical methods that require smoothness. More precisely, we define a singular solution as a solution satisfying $v(\xi) = 0$ for some $\xi > 0$. We then establish the asymptotic behavior of such solutions in a neighborhood of ξ, that is, we prove that v satisfies the following behavior

$$v(r) = \mathcal{L}_2\,(\xi - r)^{p/(\delta+p-1)} + o\left((\xi - r)^{p/(\delta+p-1)}\right), \quad \text{when } r \text{ is near to } \xi,$$

and $\mathcal{L}_2$ will be introduced later.

The structure of this paper is as follows. In the section 2, we study the initial problem, focusing on the case where the solution is regular at zero. We classify the solutions into two types: monotone solutions and non-monotone solutions. By establishing several key properties, in particular the existence of $r_0 > 0$ for which $v(r_0) < 1$, we show that any monotone solution must become singular at a finite point. In the section 3, we provide additional findings about the behavior of the solution near this finite point and derive an asymptotic equivalent in its vicinity. Finally, we present concluding remarks and outline possible directions for future research.

2. Existence of Singular Solution

In this section, we are concerned with the initial value problem given by

$$(Q_a)\begin{cases} \left(|v'(r)|^{p-2}v'(r)\right)' + \frac{N-1}{r}|v'(r)|^{p-2}v'(r) + v^q - v^{-\delta} = 0, \quad r > 0, \\ v(0) = a, \quad v'(0) = 0, \end{cases} \tag{2.1}$$

where $N > p > 2$, $q > 1$, $\delta > 0$ and $a > 0$. We begin by establishing the (Q_a) has a local solution. Since $a > 0$, it follows that v remains strictly positive for r sufficiently small. This approach is inspired by the previous works [3,4].

Proposition 2.1 *Let $a > 0$. Problem (Q_a) admits a unique solution u defined on a maximal interval $[0, r_{max}[$.*

Proof: Consider v as a solution of (Q_a). Hence if $r \in [0, r_{max}[$, v satisfies the following equation

$$v(r) = a - \int_0^r L(J(v)(s))\,\mathrm{d}s, \tag{2.2}$$

$$L(s) = |s|^{\frac{2-p}{p-1}}s, \;\; s \in \mathbb{R} \tag{2.3}$$

and the nonlinear function J is given by

$$J(v)(s) = s^{1-N}\int_0^s \left(\sigma^{N-1}v^q + \sigma^{N-1}v^{-\delta}\right)\,\mathrm{d}\sigma. \tag{2.4}$$

Fix $R > 0$ and two real numbers a and m for which $a > m > 0$. Let $C([0, R], \mathbb{R}^N)$ be the Banach space of continuous functions defined on $[0, R]$. We next consider the complete metric space defined below

$$V_{a,m,R} = \left\{\psi \in C^1[0, R] : \|\psi - a\|_0 \le m\right\}. \tag{2.5}$$

Define the following mapping H on $V_{a,m,R}$ by

$$T(\psi)(r) = a - \int_0^r L(J(\psi)(s))\,\mathrm{d}s. \tag{2.6}$$

First, we demonstrate that T maps E into itself for small R.
Fix $\psi \in V_{a,m,R}$ and $r \in [0, R]$, then $H(\psi)(r) \in C([0, R])$. So we arrive at the next estimate

$$|H(\psi)(s) - a| \le \int_0^r |J(\psi)(s)|^{\frac{1}{p-1}}\,ds. \tag{2.7}$$

As $\psi(r) \in [a-m, a+m]$, from the above, we obtain if $s \in (0, R)$,

$$\left(\frac{(a-m)^q}{N} + \frac{(a+m)^{-\delta}}{N}\right) s \leq J(\psi)(s) \leq \left(\frac{(m+a)^q}{N} + \frac{(a-m)^{-\delta}}{N}\right) s. \tag{2.8}$$

Which implies that there is a small R for which $H(\psi) \in V_{a,m,R}$.
Next, we claim that H is a contraction. For this, fix ψ_1, $\psi_2 \in V_{a,m,R}$ and $0 \leq r \leq R$

$$|H(\psi_1) - H(\psi_2)| \leqslant \int_0^r |L(J(\psi_1)(r)) - L(J(\psi_2)(r))| ds, \tag{2.9}$$

where $J(\psi)$ is given by (2.4). Next let $\phi(s) = \min(|J(\psi_1)(s)|, |J(\psi_2)(s)|)$. Then

$$|H(\psi_1)(r) - H(\psi_2)(r)| \leqslant \int_0^r (\phi(s))^{\frac{2-p}{p-1}} |J(\psi_1)(s) - J(\psi_2)(s)| ds. \tag{2.10}$$

Using the boundless of J in (2.8) and taking R small enough and we conclude that H is a contraction. Finally, by the Banach theorem we deduce that there is a unique function v solving the problem (Q_a) on $[0, R_{max}]$. □

Note that if $a = 1$, then by the uniqueness of the solution we have $v(r, 1) \equiv 1$. Therefore, in the remainder of this work, we restrict our attention to the case $a > 1$. Now, following some ideas from [16], we present some properties of the solutions to problem (Q_a).

Lemma 2.1 *Given that $a > 1$, then one can find $r_0 > 0$ for which $v(r_0, a) < 1$.*

Proof: Assume by contradiction that, for all $r \in (0, +\infty)$, $v(r) = v(r, a) \geq 1$. This implies necessarily that $v(r) > 1$ for each $r > 0$. Indeed, if we can find a $r_1 > 0$ for which $v(r_1) = 1$, since $v(r) \geq 1$ then $v'(r_1) = 0$. Which yields that $v \equiv 1$ and this is impossible. Hence $v(r) > 1$ when $r > 0$. We have

$$\left(r^{N-1}|v'|^{p-2}v'\right)' + r^{N-1}f(v) = 0 \tag{2.11}$$

Initially we prove that $v'(r) < 0$ given any $r > 0$. Assume otherwise that there is $r_2 > 0$ for which $v'(r_2) = 0$. We integrate (2.11) on $[0, r_2]$ we obtain

$$0 = \int_0^{r_2} r^{N-1} f(v(r))\, dr > 0,$$

which is impossible. Thus, $v'(r) < 0$ for each $r > 0$.
Let us define the following function

$$w(r) = \frac{p'(r)}{p(r)} \quad \text{given any} \quad r > 0, \tag{2.12}$$

where

$$p(r) = r^{N-1}\, |v'|^{p-2} v'(r), \quad \text{given any} \quad r > 0 \tag{2.13}$$

and

$$p'(r) + r^{N-1}\, f(v) = 0 \quad \text{given any} \quad r > 0. \tag{2.14}$$

Differentiating (2.12) we arrive at

$$w'(r) = \frac{p''(r)}{p(r)} - w^2(r). \tag{2.15}$$

Otherwise, using (2.13) we conclude

$$p''(r) = -(N-1)r^{N-2}f(v) - |v'|^{2-p}p(r)f'(v).$$

Thus, if $r > 0$

$$w'(r) = -w^2 - |v'|^{2-p} f'(v) + \frac{N-1}{r}\, w. \tag{2.16}$$

As $v'(r) < 0$, then $p(r) < 0$ when $r > 0$, also we have f is strictly increasing if $r > 0$. Therefore $w'(r) < \dfrac{N-1}{r}\, w(r) - w^2(r)$, which gives $w'(r) < (N-1)\, w(r) - w^2(r)$ when $r > 1$. Let us define $G(r) = w(r) - N + 1$. By a simple study of the function G we conclude that

$$w(r) < N - 1, \quad \text{ given any } \ r > 1. \tag{2.17}$$

Based on (2.16) and (2.17), we get

$$w'(r) < -w^2(r) - |v'|^{2-p} f'(v) + \frac{(N-1)^2}{r}. \tag{2.18}$$

Moreover, observe that since $v(r) > 1$, we can find $C > 0$ for which $f'(v(r)) \geq C$. Consequently, we can pick r_3 large enough for which

$$w'(r) \leq -C \quad \text{for all} \quad r \geq r_3,$$

where $w(r_3) > 0$. Hence, we can found necessarily $r_4 > r_3$ for which $w(r_4) = 0$, which implies that $f(v(r_4)) = 0$, that is $v(r_4) = 1$ which gives a contradiction.

□

We now present the result concerning the monotone solutions of problem (Q_a).

Theorem 2.1 *Let $a > 1$ and v be a monotone solution of problem (Q_a). Then v is a singular solution of (Q_a).*

Proof: Assume otherwise that v is not a singular solution. Since $(|v'|^{p-2}v')'(0) = a^{-\delta}(1 - a^{q+\delta}) < 0$ because $a > 1$, so $v'(r) < 0$ near the origin and thus v is monotone when $r > 0$. In addition v is positive, then it is strictly decreasing on $(0, +\infty)$, thus we deduce that $L \geq 0$ for which $\lim\limits_{r \to +\infty} v(r) = L$. Multiplying equation (1.1) by v' and integrating over $(0, r)$, then we get

$$\frac{p-1}{p} |v'(r)|^p + (N-1) \int_0^r \frac{|v'(s)|^p}{s}\, ds = -\int_a^{v(r)} f(s)\, ds.$$

Hence by tending $r \to +\infty$ we have $\displaystyle\int_L^a f(s)\, ds$ converges, that implies $\displaystyle\int_0^{+\infty} |v'(r)|^p s\, ds < \infty$, $\lim\limits_{r \to +\infty} v'(r) = 0$ and $\lim\limits_{r \to +\infty} (|v'|^{p-2}v')' = 0$. Next tending r to $+\infty$ in equation (1.1) and using the fact that v converges, then necessarily we have $\lim\limits_{r \to +\infty} v(r) = 1$, that gives a contradiction with Lemma 2.1. □

Given that $a > 1$, it can be seen that $v(r, a) > 0$ and $v'(r, a) < 0$ for small r. Then we can define

$$\xi(a) = \sup \{ r : v(s, a) > 0 \;\; v'(s, a) < 0 \;\; \text{for all } s \in (0, r) \}.$$

Lemma 2.2 *Given any $a \in (1, +\infty)$, then $0 < \xi(a) < +\infty$.*

Proof: We argue by contradiction and assume that $\xi(a) = +\infty$. Noting that in this case $v(r) = v(r, a)$ decreases strictly if $r \in (0, +\infty)$, then as a consequence of Lemma 2.1, there is $r_0 > 0$ for which $0 < v(r) \leq v(r_0) < 1$ and $v'(r) < 0$ when $r \geq r_0$. Let ρ be arbitrary large enough, then there is $\rho_1 \in (\rho, r)$ for which $v(r) - v(\rho) = v'(\rho_1)(r - \rho)$. Since $v'(\rho_1) < 0$, then by letting r to $+\infty$. we deduce that v is unbounded for large r, which gives a contradiction. □

3. Behavior near the Singularity

From the definition of $\xi(a)$, we remark that $v(r,a)$ is a decreasing function of r for $0<r<\xi(a)$. That gives $\lim_{r\to\xi(a)} v(r,a)$ always exists. In addition, by Lemma 2.1, we conclude that $\lim_{r\to\xi(a)} v(r,a)<1$. A further observation is $\lim_{r\to\xi(a)} v'(r,a)$ always exists. Indeed, we distinguish two cases. First, if $\lim_{r\to\xi(a)} v(r,a)=0$, we can choose $r_0>0$ for which $v(r_0,a)<1$. Therefore, for each $r\in[r_0,\xi(a))$, we have $f(v(r,a))<1$. Using relation (2.11) we deduce that $r^{N-1}v'(r,a)$ is increasing in $[r_0,\xi(a))$, and so $\lim_{r\to\xi(a)} v'(r,a)$ exists. Second, if $\lim_{r\to\xi(a)} v(r,a)>0$, then by regularity we conclude easily that $\lim_{r\to\xi(a)} v'(r,a)=0$.
Therefore, we define

$$\mathcal{M}=\{a\in(1,\infty): v(\xi(a),a)=0;\ v'(r,a)\le 0 \quad \text{given any} \quad r\in[0,\xi(a)]\},$$
$$\overline{\mathcal{M}}=\{a\in(1,\infty): v(\xi(a),a)>0;\ v'(\xi(a),a)=0\}.$$

We start this section by the following proposition which gives an information about the behavior of $\xi(a)$ as $a\to+\infty$.

Proposition 3.1 *Let $N>p$ and $p-1<q<\dfrac{N(p-1)+p}{N-p}$. Consider v as a solution of (Q_a). Then*

$\limsup_{a\to+\infty} a^{(q+1-p)/p}\,\xi(a)<\infty$ *and* $\lim_{a\to+\infty}\xi(a)=0$.

Proof: The proof will be done in three steps.

Step 1. We prove that $|v'|\le\left(\dfrac{p}{(p-1)(q+1)}\right)^{1/p} a^{(q+1)/p}$ given any $r\in[0,\xi(a)]$.
First we note that we have $v'(r)<0$ given any $0\le r\le\xi(a)$. Now we define, for each $r\in[0,\xi(a)]$ the following energy function

$$G(r)=\frac{p-1}{p}|v'(r)|^p+\frac{v^{q+1}(r)}{q+1}-\frac{v^{1-\delta}(r)}{1-\delta}. \tag{3.1}$$

It is evident that

$$G'(r)=-\frac{N-1}{r}\,|v'|^p, \quad \text{given any} \quad r\in[0,\xi(a)].$$

Which gives that $G(r)$ is decreasing in $[0,\xi(a)]$ and

$$\frac{p-1}{p}|v'(r)|^p\le\frac{a^{q+1}}{q+1}+\frac{v^{1-\delta}(r)}{1-\delta}-\frac{a^{1-\delta}}{1-\delta} \quad \text{given any} \quad r\in[0,\xi(a)].$$

Since the function $t\to\dfrac{t^{1-\delta}}{1-\delta}$ is increasing given any $t>0$ and v is decreasing on $[0,\xi(a)]$, we get the result.

Step 2. $\limsup_{a\to+\infty} a^{(q+1-p)/p}\,\xi(a)<\infty$ and $\lim_{a\to+\infty}\xi(a)=0$.
We set

$$\omega_a(r)=a^{-1}\,v(r\,a^{-(q+1-p)/p},a) \quad \text{given any} \quad r\in[0,a^{(q+1-p)/p}\,\xi(a)). \tag{3.2}$$

It is easy to verify that ω_a satisfies the following problem

$$\begin{aligned}&\Big(|\omega_a'|^{p-2}|\omega_a'|(r)\Big)'+\frac{N-1}{r}\,|\omega_a'|^{p-2}|\omega_a'|(r)+\omega_a^q-a^{-\delta-q}\,\omega_a^{-\delta}=0, \quad r\in[0,a^{(q+1-p)/p}\,\xi(a)),\\ &\omega_a(0)=1, \quad \omega_a'(0)=0.\end{aligned} \tag{3.3}$$

Also, $\omega_a'(r)<0$ and $\omega_a(r)>0$ given any $r\in[0,a^{(q+1-p)/p}\,\xi(a))$. Therefore

$$0\le\omega_a(r)\le 1 \quad \text{given any} \quad 0\le r\le a^{(q+1-p)/p}\,\xi(a) \tag{3.4}$$

and

$$\omega_a'(r) \le \left(\frac{p}{(p-1)(q+1)}\right)^{1/p} \quad \text{given any} \quad 0 \le r \le a^{(q+1-p)/p}\,\xi(a). \tag{3.5}$$

Hence ω_a and ω_a' are uniformly bounded. Next assume the existence of a sequence $a_k \to +\infty$ for which $a_k^{(q+1-p)/p}\,\xi(a_k) \to +\infty$ as $k \to +\infty$. Based on the Arzelà-Ascoli theorem, we can found a subsequence denoted also ω_a and a continuous function $\omega : [0,\infty) \to [0,1]$ for which $\omega_{a_k} \to \omega$ uniformly on all compact subsets $[0,m] \subset [0,\infty)$ and ω satisfies

$$\left(|\omega^{'}|^{p-2}|\omega^{'}|(r)\right)' + \frac{N-1}{r}|\omega^{'}|^{p-2}|\omega^{'}|(r) + \omega^q(r) = 0, \quad r > 0. \tag{3.6}$$

Since $p-1 < q < \dfrac{N(p-1)+p}{N-p}$, then this gives a contradiction, because such ω does not exist follow the work proved in [14]. This gives the results. □

Here, we prove that if $v(0) = a \in \mathcal{M}$ then we give the following theorem which describes the boundary behavior of solutions of (1.1).

Theorem 3.1 *Assume that $0 < \delta < 1$. Let v be a solution of (Q_a), then if $r \to \xi(a)$ we have*

$$v(r) = \begin{cases} \mathcal{K}\,(\xi(a)-r)^{p/(\delta+p-1)} + o\left((\xi(a)-r)^{p/(\delta+p-1)}\right), & as \;\; v'(\xi(a)) = 0 \\ \mathcal{K}\,(\xi(a)-r) + o\left((\xi(a)-r)\right), & as \;\; v'(\xi(a)) \ne 0, \end{cases}$$

where

$$\mathcal{K} = \begin{cases} \left[\dfrac{1}{(p-1)\,(p/(\delta+p-1)-1)}\left(\dfrac{p}{\delta+p-1}\right)^{-(p-1)}\right]^{1/(\delta+p-1)}, & as \;\; v'(\xi(a)) = 0, \\ -v'(\xi(a)), & as \;\; v'(\xi(a)) \ne 0. \end{cases}$$

Proof: Since $v(0) \in M$, then two cases present. Considering the second scenario, in which $v'(\xi) \ne 0$, then necessarily $v'(\xi) < 0$. The result is obvious. Now, If $v(r) \to 0$ and $v'(r) \to 0$ as $r \to \xi(a) = \xi$. Let us define the, given any $r \in (0,\xi)$, the following function

$$v(r) = z^{p/(\delta+p-1)}(r). \tag{3.7}$$

Then

$$v'(r) = \frac{p}{\delta+p-1}\, z'(r)\ (z(r))^{p/(\delta+p-1)-1},$$

$$|v'|^{p-2}v'(r) = \left(\frac{p}{\delta+p-1}\right)^{p-1} |z'|^{p-2}z'\,(z(r))^{(p/(\delta+p-1)-1)(p-1)}$$

and

$$\begin{aligned}\left(|v'|^{p-2}v'(r)\right)' &= \left(\frac{p}{\delta+p-1}\right)^{p-1}\left(|z'|^{p-2}z'\right)'(z(r))^{(p/(\delta+p-1)-1)(p-1)} \\ &+ (p-1)\left(\frac{p}{\delta+p-1}-1\right)\left(\frac{p}{\delta+p-1}\right)^{p-1} |z'|^p\,(z(r))^{(p/(\delta+p-1)-1)(p-1)-1}\end{aligned}$$

Hence, given any $r \in (0,\xi)$, z verifies the following equation

$$\begin{aligned}&\left(\frac{p}{\delta+p-1}\right)^{p-1}\left(|z'|^{p-2}z'\right)' z(r) + (p-1)\left(\frac{p}{\delta+p-1}-1\right)\left(\frac{p}{\delta+p-1}\right)^{p-1}|z'|^p \\ &+ \frac{N-1}{r}\left(\frac{p}{\delta+p-1}\right)^{p-1}|z'|^{p-2}z'z(r) + z^{p(q+\delta)/(\delta+p-1)} - 1 = 0.\end{aligned} \tag{3.8}$$

As

$$\lim_{r\to\xi^-} v^\delta(r)\,\left(|v'|^{p-2}v'(r)\right)' = 1, \tag{3.9}$$

then

$$\lim_{r\to\xi^-}\left(\frac{p}{\delta+p-1}\right)^{p-1}\left(|z'|^{p-2}z'\right)' z(r)$$
$$+ (p-1)\left(\frac{p}{\delta+p-1}-1\right)\left(\frac{p}{\delta+p-1}\right)^{p-1}|z'|^p = 1$$

we get, by using (3.8), that

$$\lim_{r\to\xi^-}\frac{N-1}{r}\left(\frac{p}{\delta+p-1}\right)^{p-1}|z'|^{p-2}z'z(r) = 0$$

which gives that $\lim_{r\to\xi^-}|z'|^{p-2}z'z(r) = 0$. Then we can define that

$$\chi(r) = \begin{cases} |z'|^{p-2}z'(r)z(r), & \text{for each } r\in[0,\xi) \\ \chi(\xi) = 0. \end{cases} \tag{3.10}$$

Now we show that $\limsup_{r\to\xi^-}|z'|^{p-2}z'(r) < 0$. As $|z'|^{p-2}z'(r) \le 0$ for $r < \xi$, then we conclude $\limsup_{r\to\xi^-}|z'|^{p-2}z'(r) \le 0$. Assume otherwise that $\limsup_{r\to\xi^-}|z'|^{p-2}z'(r) = 0$. In the present case, we prove that $\lim_{r\to\xi^-}|z'|^{p-2}z'(r) = 0$. In fact, suppose that $\liminf_{r\to\xi^-}|z'|^{p-2}z'(r) = k < 0$, then there is two sequences $\{\eta_i\}$ and $\{\beta_i\}$ tend to ξ as $i\to+\infty$ for which $\eta_i < \beta_i < \eta_{i+1}$ and satisfy

$$\lim_{i\to+\infty}|z'|^{p-2}z'(r)(\beta_i) = 0, \quad \left(|z'|^{p-2}z'(r)\right)'(\beta_i) = 0, \tag{3.11}$$
$$\lim_{i\to+\infty}|z'|^{p-2}z'(r)(\eta_i) = k, \quad \left(|z'|^{p-2}z'(r)\right)'(\eta_i) = 0.$$

By using (3.8) we get a contradiction. As a consequence $\lim_{r\to\xi^-} z'|^{p-2}z'(r) = 0$. Moreover, relation (3.9) gives that

$$\lim_{r\to\xi^-}\left[\left(|z'|^{p-2}z'\right)' z(r) + (p-1)\left(\frac{p}{\delta+p-1}-1\right)|z'|^p\right] = \left(\frac{p}{\delta+p-1}\right)^{-(p-1)}, \tag{3.12}$$

as follows

$$\lim_{r\to\xi^-}\left[\chi'(r) + \left(\frac{p}{\delta+p-1}(p-1)-p\right)|z'|^p\right] = \left(\frac{p}{\delta+p-1}\right)^{-(p-1)}, \tag{3.13}$$

thus

$$\lim_{r\to\xi^-}\chi'(r) = \left(\frac{p}{\delta+p-1}\right)^{-(p-1)}. \tag{3.14}$$

As χ' is a function depending on z and z' that are continue function, then χ' is continue on $[0,\xi]$ and

$$\chi'(\xi) = \lim_{r\to\xi^-}\chi'(r) = \left(\frac{p}{\delta+p-1}\right)^{-(p-1)}. \tag{3.15}$$

Let $\epsilon > 0$, then given any $R-\epsilon < r < R$, we have

$$\chi(r) = \chi'(\xi)(r-\xi) + \chi(\xi).$$

Which yields that

$$\int_r^\xi |\chi(t)|dt \geq \frac{1}{4}\left(\frac{p}{\delta+p-1}\right)^{-(p-1)} (\xi - r)^2. \tag{3.16}$$

On the other side, by assuming that $\mathcal{C} = \max_{0\leq t\leq\xi} |z'(t)|$, we have

$$\begin{aligned} \int_r^\xi |\chi(t)|dt &= \int_r^\xi |z'|^{p-1}|z|dt \\ &\leq \mathcal{C}^p \int_r^\xi (\xi - t)\,dt, \\ &\leq \frac{\mathcal{C}^p}{2}\,(\xi - r)^2. \end{aligned} \tag{3.17}$$

Hence

$$\frac{1}{4}\left(\frac{p}{\delta+p-1}\right)^{-(p-1)} \leq \frac{\mathcal{C}^p}{2} \tag{3.18}$$

Next, by tending $r \to \xi^-$, we have $\mathcal{C} \to 0$, which implies a contradiction by using (3.18). Therefore we have $\limsup_{r\to\xi^-} |z'|^{p-2}z'(r) = \mathcal{L} < 0$ and as follows

$\limsup_{r\to\xi^-} z'(r) = -\mathcal{L}^{1/(p-1)} < 0$, then given any $\xi - \varepsilon < r < \xi$, we have

$$\begin{aligned} v'(r) &= \frac{p}{\delta+p-1} z'(r)((z(r))^{p/(\delta+p-1)-1} \\ &< -\frac{p}{\delta+p-1}\frac{\mathcal{L}^{1/(p-1)}}{2}\,(v(r))^{(1-\delta)/p}. \end{aligned} \tag{3.19}$$

By integrating the last inequality on $[r, \xi]$, we get

$$v^{(\delta+p-1)/p}(\xi) - v^{(\delta+p-1)/p}(r) \leq -\frac{\mathcal{L}^{1/(p-1)}}{2}(\xi - r), \quad \text{for each } r \in [\xi - \epsilon,\, \xi],$$

that is

$$v(r) \geq \left(\frac{\mathcal{L}^{1/(p-1)}}{2}\right)^{p/(\delta+p-1} (\xi - r)^{p/(\delta+p-1)}.$$

In the same manner, assume otherwise that $\liminf_{r\to\xi^-} v'(r) = -l < -\mathcal{L}$, then by using (3.9) we get a contradiction and we obtain that $\lim_{r\to\xi^-} v'(r) = -\mathcal{L} < 0$. Next we conclude the existence of $\mathcal{L}_1 > 0$ so that

$$v(r) \leq \mathcal{L}_1\,(\xi - r)^{p/(\delta+p-1)}, \quad \text{given any } \ \xi - \epsilon < r < \xi.$$

In conclusion, we deduce the existence of $\mathcal{L}_2 > 0$ so that

$$v(r) = \mathcal{L}_2\,(\xi - r)^{p/(\delta+p-1)} + o\left((\xi - r)^{p/(\delta+p-1)}\right), \quad \text{given any } \ \xi - \epsilon < r < \xi,$$

where

$$\mathcal{L}_2 = \left[\frac{1}{(p-1)\,(p/(\delta+p-1)-1)}\,\left(\frac{p}{\delta+p-1}\right)^{-(p-1)}\right]^{1/(\delta+p-1)}.$$

□

Conclusion

Throughout the present paper, we have addressed the existence and behavior of singular solutions for equation (1.1), motivated by the challenges introduced by the mixed-power nonlinearity $f(v) = v^p - v^{-\delta}$. By first analyzing the regular behavior near the origin and then studying the asymptotic behavior near the finite singular point, we have shown that any monotone solution necessarily develops a singularity at a finite distance.

The present study has focused on the case where the initial data is strictly greater than one, the opposite case will be investigated in a future work. In addition, we will treat the case where solutions u are not necessarily monotone, that is $v \in \overline{\mathcal{M}}$. More precisely, we aim to analyze non-monotone solutions and describe their behavior. Furthermore, it would be of interest to extend these results to more general classes of nonlinearities.

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