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Some Results on Contra Harmonic Cordial Mean Graphs

Nithin J. V., Veeresha R. G.*, Natesha M. K., Shankaralingappa B. M. and Sridevi M. J.

ABSTRACT: Let f be a map from the vertex set $V(G)$ to $\{0, 1, 2\}$. For each edge uv assign the label

$$\left\lceil \frac{(f(u))^2 + (f(v))^2}{f(u) + f(v)} \right\rceil .$$

Then f is called a contra harmonic cordial mean labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j \in 0, 1, 2$ where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with $x = 0, 1, 2$. A graph with a contra harmonic cordial mean labeling is called a contra harmonic cordial mean graph. In this paper we investigate contra harmonic cardial mean labeling behavior of path, cycles, triangular snake, complete graphs and some more standard graphs.

Keywords: Graphs, cordial mean graph, labeling, contra harmonic cordial mean graph.

Contents

1. Introduction and Preliminaries

The mean defined as a midway between the value of other quantities or as an average which is important for the researchers of Mathematics and Statistics for their investigations and justifications. The arithmetic mean, geometric mean and harmonic mean [2], [11] are the three classical means among ten Greek means which are defined on the basis of proportions. These means were studied by Pythagoreans and later developed by Greek Mathematicians because of their importance in geometry and music.
Mean: A mean is defined as a function

$$M : \mathbb{R}_+^2 \to \mathbb{R}_+$$

which has the property

$$\min(x_1, x_2) \leq M(x_1, x_2) \leq \max(x_1, x_2)$$

where x_1 and x_2 are positive real numbers.

Recently, the researchers have effectively utilized Mathematical means for labeling the graphs, for processing the digital images [4]. According to Wang, Bin Yao and Ming Yao [13], graph labelings are used for incorporating redundancy in disks, designing drilling machines, creating layouts for circuit boards, and configuring resistor networks. Labeled graphs serve as useful models for a broad range of applications such as coding theory, X-ray crystallography, astronomy, circuit design, communication network addressing, database management, secret sharing schemes and models for constraint programming over finite domains.

The notion of mean labeling was introduced by Somasundaram and Ponraj [10], [11], they have also studied the concept of geometric mean labeling of a graph G with p vertices and q edges and proved paths, cycles, combs, ladders are geometric mean graphs and K_n $(n > 4)$ and $K_{1,n}$ $(n > 5)$ are not geometric mean graphs. They have also proved that $C_m \cup P_n$; $C_m \cup C_n$; nK_3; $nK_3 \cup P_n$; $nK_3 \cup C_m$; P_n^2 and crowns are geometric mean graphs. In [11], the Authors investigated geometric mean labelings in the context of duplication of graph elements in cycle C_n and path P_n. According to Beineke and Hegde [1], graph labeling serves as a frontier between number theory and structure of graphs. In [6] authors proved

* Corresponding author.
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that some disconnected graphs are harmonic mean graphs. In [8], Contra harmonic mean labeling of a graph is introduced and also investigated that for a polygonal chain, square of the path and dragon are harmonic mean graph of order at most 5.

Abundant literature exists as of today concerning the structure of graphs admitting a variety of function assigning real numbers to their elements so that given conditions are satisfied. In [12] authors have studied the vertex function $f : V(G) \to A$, where $A \subseteq \mathbb{N}$, for which the induced edge function $f^* : E(G) \to \mathbb{N}$ is defined as

$$f^*(uv) = \left\lfloor \frac{2[f(u)^2 + f(u)f(v) + f(v)^2]}{3(f(u)+f(v))} \right\rfloor$$

or

$$f^*(uv) = \left\lceil \frac{2[f(u)^2 + f(u)f(v) + f(v)^2]}{3(f(u)+f(v))} \right\rceil$$

for every $uv \in E(G)$ are all distinct, known as Centroidal mean labeling and introduced Centroidal mean labeling of some standard graphs like triangular snake, Double triangular snake, triangular ladder and so on.

Graph: A graph G is a pair (V, E), where V is a nonempty set and E is a set of unordered pairs of elements taken from the set V. A graph which does not contain loops and multiple edges is a simple graph, a finite number of vertices and edges in a graph is a finite graph and undirected with p vertices and q edges. The cardinality of vertex set V of a graph is the order and the cardinality of edge set E is called the size of the graph G. The graph $G - e$ is obtained from G by deleting an edge e.

The concept of cardial labeling was introduced by Cahit [3] in the year 1987. Let f from $V(G)$ to $\{0, 1\}$ be a function. For each edge uv assign the label $|f(u) - f(v)|$. Then f is called a cardial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for $i, j \in \{0, 1\}$ where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labelled with $x(x = 0, 1)$. A graph which admits a cardial labeling is called a cardial graph. Product cardial labeling was introduced by M. Sundaram, R. Ponraj and Somasundaram [12]. Here we introduce a new notation called Contra harmonic cardial mean labeling. The symbol $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms not defined here are used in the sense of Harary [3].

2. Contra Harmonic Cordial Mean Labeling of a Graph

In this paper, the contra harmonic cordial mean labeling of graphs containing paths, cycles such as triangular snake $T_n \odot K_1$, double triangular snake $D_n(T_n) \odot K_1$, and $TL_n \odot K_1$ are discussed using the following definition.

Definition. [7], [10] A contra harmonic cordial mean labeling of a connected graph G is a one-to-one map f from the vertex set $V(G)$ of G to set $\{1, 2\}$. For each edge uv of G, assign the label $\left\lceil \frac{(f(u))^2 + (f(v))^2}{f(u)+f(v)} \right\rceil$. Then f is called a contra harmonic cordial labeling of G if $|V_f(i) - V_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1, i, j \in \{0, 1, 2\}$ where $V_f(x)$ and $e_f(x)$ denote the number of vertices and edges labelled with $x(x = 0, 1, 2)$ respectively. A graph with a contra harmonic cordial mean labeling graph is called contra harmonic cordial mean graph.

3. Main Results

Theorem 3.1 *A path P_n is a contra harmonic cordial mean graph.*

Proof: Consider a path P_n with vertices $u_1, u_2, u_3, \ldots, u_n$.

Case (1): If the number of vertices n in the path is even, then the number of edges in P_n is odd. Define $f : V \to \{1, 2\}$ as

$$f(u_i) = \begin{cases} 1 & 1 \leq i \leq \frac{n}{2} \\ 2 & \frac{n}{2} < i \leq n \end{cases},$$

$V_f(1) = (n-2)$ and $V_f(2) = (n-2)$, then $|V_f(1) - V_f(2)| = |(n-2) - (n-2)| = 0$. Also $e_f(1) = \left(\frac{n}{2} - 1\right)$ and $e_f(2) = \frac{n}{2}$, then $|e_f(1) - e_f(2)| = \left|\frac{n}{2} - 1 - \frac{n}{2}\right| = 1$.
Hence the path P_n is a contra harmonic cordial mean graph.

Case (2): If n is odd, then the path P_n is of even length.
Define

$$f(u_i) = \begin{cases} 1 & 1 \leq i < \frac{n+1}{2} \\ 2 & \frac{n+1}{2} \leq i \leq n \end{cases}$$

$$|V_f(1) - V_f(2)| < 1$$
$$|e_f(1) - e_f(2)| \leq 1$$

therefore the path P_n is a contra harmonic cordial mean graph. □

Example: Consider the graphs $T_3 \bigodot K_1$ and $T_4 \bigodot K_1$. The contra harmonic mean cordial labeling is as shown in the figures - 1 and 2.

Figure 1: Triangular snake $T_3 \bigodot K_1$

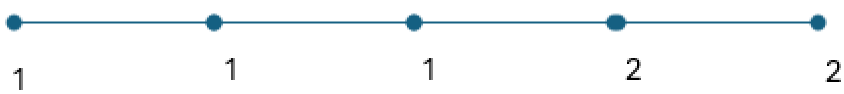

Figure 2: Triangular snake $T_4 \bigodot K_1$

Theorem 3.2 *The Comb $P_n \bigodot K_1$ is a Contra harmonic mean cordial graph.*

Proof: Let u_i for $1 \leq i \leq n$ be the vertices of path P_n. Let the vertices $V(P_n \bigodot K_1) = V(P_n) \cup V_i$, where $1 \leq i \leq n$ and $E(P_n \bigodot K_1) = E(P_n) \cup U_iV_i$ where $1 \leq i \leq (n-1)$.

Case(1): Let $n = 4t \ \forall \ t \in Z$.
Define

$$\begin{aligned} f(u_i) &= 3, \quad 1 \leq i \leq t \\ f(u_{t+i}) &= 2, \quad 1 \leq i \leq t \\ f(u_{2t+i}) &= 1, \quad 1 \leq i \leq t \\ f(u_{3t+i}) &= 0, \quad 1 \leq i \leq t \\ f(v_i) &= 3, \quad 1 \leq i \leq t \\ f(v_{t+i}) &= 2, \quad 1 \leq i \leq t \\ f(v_{2t+i}) &= 1, \quad 1 \leq i \leq t \\ f(v_{3t+i}) &= 0, \quad 1 \leq i \leq t. \end{aligned}$$

Now $V_f(0) = V_f(1) = V_f(2) = V_f(3) = 2t$ and $e_f(0) = (2t-1), e_f(1) = e_f(2) = e_f(3) = 2t$.

Case(2): Let $n = (4t+1)\ \forall\ t \in Z$. Assign labels to the vertices u_i and v_i where $1 \leq i \leq (n-1)$, as in the case (1). Then assign the label 0 and 1 to the vertices u_n and v_n.
Now $V_f(0) = V_f(1) = (2t+1),\ V_f(2) = V_f(3) = 2t$ and $e_f(0) = e_f(2) = e_f(3) = 2t,\ e_f(1) = (2t+1)$.

Case(3): Let $n = (4t+2)\ \forall\ t \in Z$. Assign labels to the vertices u_i and v_i where $1 \leq i \leq (n-2)$ as in the case (1). Then assign the label $0, 1$ and $1, 2$ to the vertices u_{n-1}, u_n and v_{n-1}, v_n respectively.
Now $V_f(0) = V_f(2) = (2t+1),\ V_f(1) = (2t+2),\ V_f(3) = 2t$ and $e_f(0) = e_f(3) = 2t,\ e_f(1) = (2t+2),\ e_f(2) = (2t+1)$.

Case(4): Let $n = (4t+3)\ \forall\ t \in Z$. Assign labels to the vertices u_i and v_i where $1 \leq i \leq (n-3)$ as in the case (1). Then assign the label $0, 1, 2$ and $1, 2, 3$ to the vertices u_{n-2}, u_{n-1}, u_n and v_{n-2}, v_{n-1}, v_n respectively.
Now $V_f(0) = V_f(3) = (2t+1),\ V_f(1) = v_f(2) = (2t+2)$ and $e_f(0) = 2t,\ e_f(1) = e_f(2) = (2t+2),\ e_f(3) = (2t+1)$.

It may be observed in all the above cases that

$$|V_f(i) - V_f(j)| \leq 1$$

and

$$|e_f(i) - e_f(j)| \leq 1\ for\ i, j \in \{0, 1, 2\},$$

where $V_f(x)$ and $e_j(x)$ denote the number of vertices and edges labeled with $x (x = 0, 1, 2)$ respectively. Therefore, the comb graph is a contra harmonic mean cordial graph. □

Example: Consider the graph $P_8 \bigodot K_1$. The contra harmonic mean cordial labeling is as shown in figure-3.

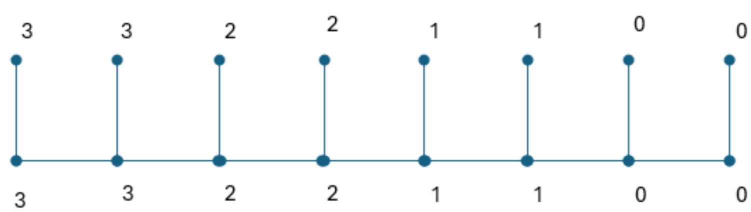

Figure 3: The graph $P_8 \bigodot K_1$

Theorem 3.3 *The quadrilateral snake Q_n is a Contra harmonic mean cordial graph.*

Proof: Let P_n be the path with n vertices u_i, where $1 \leq i \leq n$.
Let the vertex set $V(Q_n) = V(P_n) \cup V_i, W_i$ where $1 \leq i \leq (n-1)$ and $E(Q_n) = E(P_n) \cup U_iV_i, U_{i+1}W_i$ where $1 \leq i \leq (n-1)$.

Case(1): Let $n = 4t\ \forall\ t \in Z$.

Define

$$\begin{aligned}
f(u_i) &= 0, \quad 1 \le i \le t \\
f(u_{t+i}) &= 1, \quad 1 \le i \le t \\
f(u_{2t+i}) &= 2, \quad 1 \le i \le t \\
f(u_{3t+i}) &= 3, \quad 1 \le i \le t+1 \\
f(v_i) &= 0, \quad 1 \le i \le t \\
f(v_{t+i}) &= 1, \quad 1 \le i \le t \\
f(v_{2t+i}) &= 2, \quad 1 \le i \le t \\
f(v_{3t+i}) &= 3, \quad 1 \le i \le t+1 \\
f(w_i) &= 0, \quad 1 \le i \le t \\
f(w_{t+i}) &= 1, \quad 1 \le i \le t \\
f(w_{2t+i}) &= 2, \quad 1 \le i \le t \\
f(w_{3t+i}) &= 3, \quad 1 \le i \le t+1
\end{aligned}$$

Now $V_f(0) = V_f(1) = V_f(2) = (2t+2),\ V_f(3) = (2t+3)$ and $e_f(0) = (2t+2),\ e_f(1) = e_f(2) = 4t,\ e_f(3) = (4t+2)$.

Case(2): Let $n = (4t+1)\ \forall\ t \in Z$. Assign labels to the vertices $u_i,\ v_i$ and w_i where $1 \le i \le (n-1)$ as in the Case (1). Then assign the label 2 to the vertex u_n, v_n and w_n.
Now $V_f(0) = V_f(1) = (2t+2),\ V_f(2) = (4t+1),\ V_f(3) = (2t+3)$ and $e_f(0) = (2t+2),\ e_f(1) = 4t,\ e_f(2) = (4t+2),\ e_f(3) = (4t+4)$.

Case(3): Let $n = (4t+2)\ \forall\ t \in Z$. Assign labels to the vertices $u_i,\ v_i$ and w_i where $1 \le i \le (n-1)$ as in the case (1). Then assign the label 2 to the vertices $u_{n-1},\ v_{n-1},\ w_{n-1}$ and the label 1 to the vertices $u_n,\ v_n$, and w_n.
Now $V_f(0) = (2t+2),\ V_f(1) = v_f(2) = (4t+1),\ V_f(3) = (2t+3)$. and $e_f(0) = (2t+2),\ e_f(1) = (4t+2),\ e_f(2) = e_f(3) = (4t+4)$.

Case(4): Let $n = (4t+3)\ \forall\ t \in Z$. Assign labels to the vertices $u_i,\ v_i$ and w_i where $1 \le i \le (n-1)$ as in the case (1), then the label 2 to the vertices $u_{n-2}, v_{n-2}, w_{n-2}$ and the label 1 to the vertices $u_{n-1}, v_{n-1}, w_{n-1}$ and the label 0 to the vertices u_n, v_n, w_n.
Now $V_f(0) = V_f(1) = V_f(2) = (4t+1),\ V_f(3) = (2t+3)$ and $e_f(0) = 4t,\ e_f(1) = e_f(2) = e_f(3) = (4t+4)$.

It may be observed in all the above cases that

$$|V_f(i) - V_f(j)| \le 1$$

and

$$|e_f(i) - e_f(j)| \le 1,\ for\ i, j \in \{0, 1, 2\},$$

where $V_f(x)$ and $e_j(x)$ denote the number of vertices and edges labeled with $x(x = 0, 1, 2)$ respectively. Therefore the Quadrilateral Snake Q_n is a contra harmonic mean cordial graph. □

Example: Consider the graph $Q_9 \bigodot K_1$. The contra harmonic mean cordial labeling is as shown in figure-4.

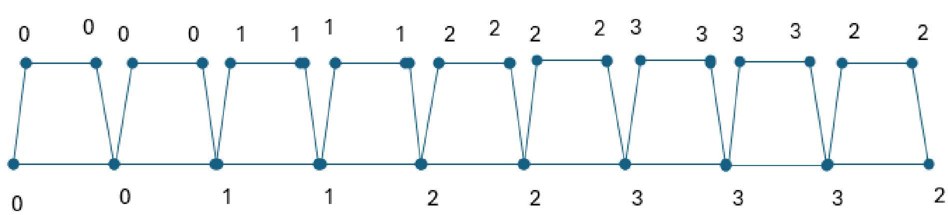

Figure 4: The graph $Q_9 \bigodot K_1$

Theorem 3.4 *A Triangular snake T_n is a contra harmonic cordial mean graph.*

Proof: Consider a path P_n with vertices $u_1, u_2, u_3, \dots, u_n$ and joining u_i and u_{i+1} to a new vertex v_i for $1 \le i \le (n-1)$. Hence the total number of vertices in T_n is $p = (2n-1)$ and the number of edges in T_n is $q = (3n-3)$. The result is proved by two cases.

Case (1): The number of vertices n in the path is even then the number vertices in T_n is odd.
Define $f : V \to \{1, 2\}$ as

$$f(u_i) = \begin{cases} 1, & 1 \le i < \left(\frac{n}{2}+1\right) \\ 2, & \left(\frac{n}{2}+1\right) \le i < (n+1) \end{cases}$$

and

$$f(v_i) = \begin{cases} 2, & \frac{n}{2} < i \le n \\ 1, & \text{otherwise.} \end{cases}$$

$V_f(1) = n$ and $V_f(2) = n$, then $|V_f(1) - V_f(2)| = |n-n| = 0$.
Also, $e_f(1) = \frac{3n-4}{2}$ and $e_f(2) = \frac{3n-2}{2}$, then $|e_f(1) - e_f(2)| = \left|\frac{3n-4}{2} - \frac{3n-2}{2}\right| = 1$. Hence T_n is contra harmonic cardial mean graph.

Case (2): If n is odd, then the number of vertices in the graph T_n is even.
Define

$$f(u_i) = \begin{cases} 1, & 1 \le i \le \left(\frac{n+1}{2}\right) \\ 2, & \left(\frac{n+1}{2}\right) < i \le (n+1) \end{cases}$$

and

$$f(v_i) = \begin{cases} 2, & \left(\frac{n+1}{2}\right) \le i \le (n-1) \\ 1, & \text{otherwise.} \end{cases}$$

$V_f(1) = n$ and $V_f(2) = (n-1)$, then $|V_f(1) - V_f(2)| = |n - (n-1)| = 1$.
Also, $e_f(1) = \frac{3n-3}{2}$ and $e_f(2) = \frac{3n-3}{2}$, then $|e_f(1) - e_f(2)| = \left|\frac{3n-3}{2} - \frac{3n-3}{2}\right| = 0$.
Hence the T_n is contra harmonic cardial mean graph.

□

Example: The following figure shows the contra harmonic mean cordial labeling of T_n

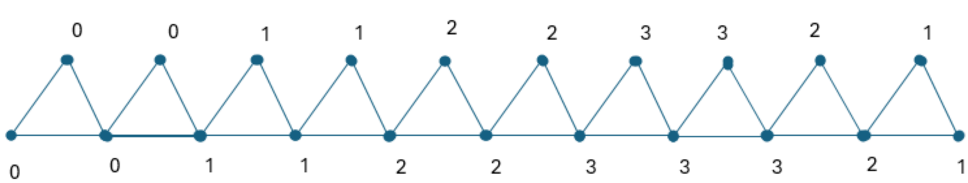

Figure 5: Graph $T_n \bigodot K_1$

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Nithin J. V.
Department Post Graduate Studies in Mathematics,
P.E.S. College of Science, Arts and Commerce,
Mandya - 571 401,
India
E-mail address: nithinjvpatel@gmail.com

and

Veeresha R. G.
Department of Mathematics,
Sri Jayachamarajendra College of Engineering,
JSS Science and Technology University,
Manasagangotri, Mysuru - 570 006,
India
ORCID 0000-0003-1297-0568
E-mail address: veeru.rg@gmail.com

and

Natesha M. K.
Department of Mathematics,
Government First Grade College for Women,
M G Road, Hassan - 573 201,
India
ORCID 0009-0006-4334-4729
E-mail address: nateshgfgc123@gmail.com

and

Shankaralingappa B. M.
Department of Mathematics,
Government First Grade College for Women,
M G Road, Hassan - 573 201,
India
ORCID 0000-0001-8121-2715
E-mail address: shankar.gsch@gmail.com

and

Sridevi M. J.
Department of Mathematics,
Government First Grade College for Women,
M G Road, Hassan - 573 201,
India
E-mail address: thanusri.j@gmail.com