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A Note on Eisenstein Series and Convolution of Sums

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ABSTRACT: The Eisenstein series plays a central role in modern number theory and mathematical analysis, especially in the theory of modular forms. Its applications span arithmetic, combinatorics, and mathematical physics. Eisenstein series play a significant role in mathematical physics, primarily due to their modular and automorphic properties. They arise naturally in string theory, where non-holomorphic Eisenstein series encode S-duality symmetries and appear in the coefficients of higher-order terms in superstring amplitudes. In quantum field theory and conformal field theory, Eisenstein series contribute to modular-invariant partition functions and describe lattice sums associated with compactified dimensions. In this paper, we investigate connections between Borweins' cubic theta functions and modular forms of level 6. Utilizing these relationships, we obtain an explicit representation of an Eisenstein series of level 6. In addition, we deduce several convolution sum identities of the form $\sum_{2i+3j=m} \sigma(i)\sigma(j)$, $\sum_{i+6j=m} \sigma(i)\sigma(j)$ and $\sum_{i+8j=m} \sigma(i)\sigma(j)$ which illustrate the interplay between theta functions and arithmetic functions.

Key Words: Convolution of sums, Eisenstein Series.

Contents

1. Introduction

The Borweins' cubic theta functions $a(q), b(q)$ and $c(q)$ are defined by

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2},$$

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2} \tag{1.1}$$

we assume $\mid q \mid< 1$ and $\omega = exp(2\pi i/3)$. It is easy to see that $a(0) = 1$, $b(0) = 1$, and $c(0) = 0$. The Jacobi's theta function $\varphi(q)$ is defined as

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{1.2}$$

Alaca and Williams [1,2,3] obtained the parametric representations for $mL(q^m) - L(q)$, $M(q^n)$, $b(q^i)$ and $c(q^i)$ for $m \in \{2, 3, 4, 6, 12\}$, $n \in \{1, 2, 3, 6\}$ and $i \in \{1, 2, 4\}$ in terms of parameters p and k, namely

$$p = p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \qquad \text{and} \qquad k = k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{1.3}$$

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Since $\varphi(0) = 1$, it is easy to see that $p(0) = 0$ and $k(0) = 1$. From [1] we have

$$a(q) = (1 + 4p + p^2)k. \tag{1.4}$$

S. Ramanujan, in his second notebook [12], provide the definitions of the Eisenstein series $P(q)$, $Q(q)$ and $R(q)$ as follows:

$$P(q) := 1 - 24\sum_{m=1}^{\infty}\frac{mq^m}{1-q^m}, \tag{1.5}$$

$$Q(q) := 1 + 240\sum_{m=1}^{\infty}\frac{m^3q^m}{1-q^m}, \tag{1.6}$$

$$R(q) := 1 - 504\sum_{m=1}^{\infty}\frac{m^5q^m}{1-q^m}. \tag{1.7}$$

The divisor function $\sigma_k(m)$ is given by

$$\sigma_k(m) = \sum_{d|m} d^k, \qquad m, k \in \mathbb{N}, \tag{1.8}$$

here d runs through the positive divisors of m. If m is not a positive integer, set $\sigma_i(m) = 0$. For convenience, we denote $\sigma(m)$ for $\sigma_1(m)$. For the wonderful work on convolution $\sum_{i+kj=m}\sigma(i)\sigma(j)$ for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 16, 18$ and 24 one can refer [1,2,3,4,5,6,9,10,13,14,15,16,17,18].

2. Result on Eisenstein Series

Theorem 2.1 *We have*

$$3P(q^3) - P(q) - 3P(q^6) + P(q^2) = \frac{1}{\psi^3(q)\phi^3(q^3)}(12q\psi^2(q)\psi^3(q^3)\varphi^2(q)\varphi^3(q^3) + 60q^2\psi(q)\psi^6(q^3) + 48q^3\varphi(q)\psi^9(q^9))).$$

Proof: From [7,11], we have

$$3P(q^3) - P(q) = 2a^2(q), \tag{2.1}$$

where $a(q)$ is as defined as in (1.4). On replacing q to q^2 in (2.1), we have

$$3P(q^6) - P(q^2) = 2a^2(q^2). \tag{2.2}$$

On subtracting (2.1) from (2.2), we obtain

$$3P(q^3) - P(q) - 3P(q^6) + P(q^2) = 2(a^2(q^2) - a^2(q)). \tag{2.3}$$

On simplifying using (1.4), we obtain

$$3P(q^3) - P(q) - 3P(q^6) + P(q^2) = 3k^2(2p + 5p^2 + 2p^3).$$

On using (1.3) in the above, we deduce

$$3P(q^3) - P(q) - 3P(q^6) + P(q^2) = 3\{\frac{\varphi^2(q) - \varphi^2(q^3)}{\varphi^2(q^3)} + \frac{5}{4}\frac{(\varphi^2(q) - \varphi^2(q^3))^2}{\varphi^4(q^3)} + \frac{1}{4}\frac{(\varphi^2(q) - \varphi^2(q^3))^3}{\varphi^6(q^3)}\}\left(\frac{\varphi^6(q^3)}{\varphi^2(q)}\right). \tag{2.4}$$

From [11, p.232, eq. (5.1)], we have

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} = \frac{m+1}{2} \quad \text{and} \quad \left(\frac{\beta^3}{\alpha}\right) = \frac{m-1}{2}, \tag{2.5}$$

where β has degree 3 over α and $m = \dfrac{z_1}{z_3}$, the multiplier. Also from [11, p. 122-123], we have

$$\varphi(q) = \sqrt{z} \tag{2.6}$$

and

$$\psi(q) = \sqrt{\frac{z}{2}} \left(\frac{\alpha}{q}\right)^{1/8}. \tag{2.7}$$

Using (2.6) and (2.7) in (2.5) and after simplifying, we deduce

$$\varphi^2(q) - \varphi^2(q^3) = 4q\frac{\varphi(q)\psi^3(q^3)}{\varphi^3(q^3)\psi(q)}. \tag{2.8}$$

Using (2.8) in (2.4) and then simplifying, we obtain the result. □

3. Convolution Sums

Theorem 3.1 *We have*

$$\begin{aligned}\sum_{2i+3j=n} \sigma(i)\sigma(j) = {} & \frac{20n-11}{240}\sigma(n) - \frac{10n+1}{240}\sigma\left(\frac{n}{2}\right) - \frac{n}{8}\sigma\left(\frac{n}{3}\right) + \frac{3n-1}{24}\sigma\left(\frac{n}{6}\right) \\ & - \frac{19}{432}\sigma_3(n) + \frac{16n-13}{48}\sigma_3\left(\frac{n}{3}\right) + \frac{17n-28}{48}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{1040}\sigma_5(n) \\ & + \frac{1}{1040}\sigma_5\left(\frac{n}{2}\right) + \frac{9}{104}\sigma_5\left(\frac{n}{3}\right) + \frac{9}{104}\sigma_5\left(\frac{n}{6}\right) + \frac{A(n)}{312} + \frac{1}{312}A\left(\frac{n}{2}\right) \\ & - \frac{B(n)}{648} - \frac{1}{27}C(n) - \frac{1}{3}C\left(\frac{n}{3}\right) - \frac{D(n)}{3456},\end{aligned}$$

where

$$\begin{aligned}\sum_{n=1}^{\infty} A(n)q^n &= q(q;q)_\infty^6 (q^3;q^3)_\infty^6, \\ 1 + \sum_{n=1}^{\infty} B(n)q^n &= \frac{(q^3;q^3)_\infty^{12}(q^6;q^6)_\infty^6(q^4;q^4)_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}, \\ \frac{1}{256} + \sum_{n=1}^{\infty} C(n)q^n &= \left(\frac{\varphi^4(q)}{16} + q\psi^4(q^2)\right)^2,\end{aligned}$$

and $D(n)$ is the right hand side of Theorem 2.1.

Proof: On squaring Theorem 2.1 and simplifying using (1.5), we deduce

$$\begin{aligned}& 576\left(\sum_{n=1}^{\infty}\sigma(n)q^n\right)^2 + 576\left(\sum_{n=1}^{\infty}\sigma(n)q^{2n}\right)^2 + 5184\left(\sum_{n=1}^{\infty}\sigma(n)q^{3n}\right)^2 + 5184\left(\sum_{n=1}^{\infty}\sigma(n)q^{6n}\right)^2 \\ & - 1152\left(\sum_{n=1}^{\infty}\sigma(n)q^n\right)\left(\sum_{n=1}^{\infty}\sigma(n)q^{2n}\right) - 3456\left(\sum_{n=1}^{\infty}\sigma(n)q^n\right)\left(\sum_{n=1}^{\infty}\sigma(n)q^{3n}\right) \\ & + 3456\left(\sum_{n=1}^{\infty}\sigma(n)q^{2n}\right)\left(\sum_{n=1}^{\infty}\sigma(n)q^{3n}\right) + 3456\left(\sum_{n=1}^{\infty}\sigma(n)q^n\right)\left(\sum_{n=1}^{\infty}\sigma(n)q^{6n}\right) \\ & - 3456\left(\sum_{n=1}^{\infty}\sigma(n)q^{2n}\right)\left(\sum_{n=1}^{\infty}\sigma(n)q^{6n}\right) - 10368\left(\sum_{n=1}^{\infty}\sigma(n)q^{3n}\right)\left(\sum_{n=1}^{\infty}\sigma(n)q^{6n}\right) = D(n),\end{aligned} \tag{3.1}$$

where $D(n)$ is as defined as in Theorem 3.1. On extracting the coefficients of q^n on both sides of the above, we deduce that

$$\begin{aligned}\sum_{2i+3j=n}\sigma(i)\sigma(j) = &-\frac{1}{6}\sum_{i+j=n}\sigma(i)\sigma(j) - \frac{1}{6}\sum_{i+j=\frac{n}{2}}\sigma(i)\sigma(j) - \frac{3}{2}\sum_{i+j=\frac{n}{3}}\sigma(i)\sigma(j)\\ &-\frac{3}{2}\sum_{i+j=\frac{n}{6}}\sigma(i)\sigma(j) + \frac{1}{3}\sum_{i+2j=n}\sigma(i)\sigma(j) + 3\sum_{i+2j=\frac{n}{3}}\sigma(i)\sigma(j)\\ &+\sum_{i+3j=n}\sigma(i)\sigma(j) + \sum_{i+3j=\frac{n}{2}}\sigma(i)\sigma(j) - \sum_{i+6j=n}\sigma(i)\sigma(j) - \frac{D(n)}{3456.}\end{aligned} \tag{3.2}$$

From ([18], Theorem 4.1), we have

$$\sum_{i+j=n}\sigma(i)\sigma(j) = \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma(n) + \frac{\sigma(n)}{12}, \tag{3.3}$$

$$\sum_{i+2j=n}\sigma(i)\sigma(j) = \frac{5}{48}\sigma_3(n) - \frac{n}{8}\sigma(n) + \frac{5n}{12}\sigma_3\left(\frac{n}{2}\right) - \frac{n}{4}\sigma\left(\frac{n}{2}\right) + \frac{1}{24}\left(\frac{n}{2}\right) + \frac{1}{24}\sigma(n) - \frac{1}{9}C(n) \tag{3.4}$$

$$\sum_{i+3j=n}\sigma(i)\sigma(j) = \frac{1}{1040}\sigma_5(n) + \frac{9}{104}\sigma_5\left(\frac{n}{3}\right) + \frac{1-3n}{24}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{240}\sigma(n) + \frac{A(n)}{312} \tag{3.5}$$

$$\sum_{i+6j=n}\sigma(i)\sigma(j) = \frac{1}{108}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{2}\right) + \frac{1-n}{24}\sigma(n) + \frac{1-6n}{24}\sigma\left(\frac{n}{6}\right) + \frac{B(n)}{648}. \tag{3.6}$$

On employing (3.3), (3.4), (3.5) and (3.6) in (3.2) and simplifying further, we obtain th result. □

Theorem 3.2 *We have*

$$\begin{aligned}\sum_{i+6j=n}\sigma(i)\sigma(j) = &\left(\frac{1-n}{24}\right)\sigma(n) + \frac{n}{4}\sigma\left(\frac{n}{3}\right) + \left(\frac{1-9n}{24}\right)\sigma\left(\frac{n}{6}\right) + \frac{1}{108}\sigma_3(n)\\ &+\frac{1}{27}\sigma_3\left(\frac{n}{2}\right) + \frac{B(n)}{648}\end{aligned}$$

where $B(n)$ is as defined as in Theorem 3.1.

Proof: On extracting the coefficients of q^n in both sides of the (3.1), we have

$$\begin{aligned}\sum_{i+6j=n}\sigma(i)\sigma(j) = &-\frac{1}{6}\sum_{i+j=n}\sigma(i)\sigma(j) - \frac{1}{6}\sum_{i+j=\frac{n}{2}}\sigma(i)\sigma(j) - \frac{3}{2}\sum_{i+j=\frac{n}{3}}\sigma(i)\sigma(j)\\ &-\frac{3}{2}\sum_{i+j=\frac{n}{6}}\sigma(i)\sigma(j) + \sum_{i+2j=n}\sigma(i)\sigma(j) + 3\sum_{i+2j=\frac{n}{3}}\sigma(i)\sigma(j)\\ &+\sum_{i+3j=n}\sigma(i)\sigma(j) + \sum_{i+3j=\frac{n}{2}}\sigma(i)\sigma(j) - \sum_{2i+3j=n}\sigma(i)\sigma(j) - \frac{D(n)}{3456}.\end{aligned}$$

On using (3.3), (3.4), (3.5) and Theorem 3.1 and then simplifying we obtain the required result. □

Theorem 3.3 *We have*

$$\begin{aligned}\sum_{i+8j=n}\sigma(i)\sigma(j) = &-\left(\frac{4-3n}{96}\right)\sigma(n)+\left(\frac{1-27n}{24}\right)\sigma\left(\frac{n}{8}\right)-\frac{25}{48}\sigma_3(n)-\left(\frac{112+35n}{96}\right)\sigma_3\left(\frac{n}{2}\right)\\ &+\frac{63}{128}\sigma_3\left(\frac{n}{3}\right)+\left(\frac{742-245n}{96}\right)\sigma_3\left(\frac{n}{4}\right)-\frac{2835}{64}\sigma_3\left(\frac{n}{6}\right)\\ &+\left(\frac{180-35n}{24}\right)\sigma_3\left(\frac{n}{8}\right)-\frac{3591}{16}\sigma_3\left(\frac{n}{12}\right)-\frac{315}{2}\sigma_3\left(\frac{n}{24}\right)+\frac{7}{72}C(n)\\ &+\frac{49}{36}C\left(\frac{n}{2}\right)+\frac{14}{9}C\left(\frac{n}{4}\right)+\frac{7}{128}E(n)+\frac{7}{32}E\left(\frac{n}{2}\right)+\frac{189}{512}B(n)\\ &+\frac{189}{128}B\left(\frac{n}{2}\right)-\frac{7}{96}F(n)-\frac{7}{24}F\left(\frac{n}{2}\right)-\frac{b(n)}{9216}\end{aligned}$$

where

$$1+\sum_{n=1}^{\infty}E(n)q^n = \frac{(q;q)_\infty^6(q^2;q^2)_\infty^6}{(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2},$$

$$1+\sum_{n=1}^{\infty}F(n)q^n = \frac{(q^2;q^2)_\infty^{24}(q^3;q^3)_\infty^3(q^{12};q^{12})_\infty}{(q;q)_\infty^9(q^4;q^4)_\infty^3(q^6;q^6)_\infty^8},$$

where $B(n)$ and $C(n)$ is as defined as in Theorem 3.1.

Proof: From [8,11], we have

$$P(q)-6P(q^2)+8P(q^4)=3\varphi^4(-q). \tag{3.7}$$

Changing q to q^2 in (3.7), we have

$$P(q^2)-6P(q^4)+8P(q^8)=3\varphi^4(-q^2). \tag{3.8}$$

On subtracting (3.7) from (3.8), we have

$$P(q)-7P(q^2)+14P(q^4)-8P(q^8)=3(\varphi^4(-q)-\varphi^4(-q^2)). \tag{3.9}$$

On squaring (3.9) in the above, we have

$$\begin{aligned}&P^2(q)+49P^2(q^2)+196P^2(q^4)+64P^2(q^8)-14P(q)P(q^2)+28P(q)P(q^4)\\ &-196P(q^2)P(q^4)-16P(q)P(q^8)+112P(q^2)P(q^8)-224P(q^4)P(q^8)\\ &=9(\varphi^4(-q)-\varphi^4(-q^2))^2.\end{aligned} \tag{3.10}$$

Employing (1.5) in (3.10) and simplifying, we obtain

$$\begin{aligned}\sum_{i+8j=n}\sigma(i)\sigma(j) =&\frac{1}{16}\sum_{i+j=n}\sigma(i)\sigma(j)+\frac{49}{16}\sum_{i+j=\frac{n}{2}}\sigma(i)\sigma(j)+\frac{49}{4}\sum_{i+j=\frac{n}{4}}\sigma(i)\sigma(j)\\ &+4\sum_{i+j=\frac{n}{8}}\sigma(i)\sigma(j)-\frac{7}{8}\sum_{i+2j=n}\sigma(i)\sigma(j)-\frac{49}{4}\sum_{i+2j=\frac{n}{2}}\sigma(i)\sigma(j)\\ &-14\sum_{i+2j=\frac{n}{4}}\sigma(i)\sigma(j)+\frac{7}{4}\sum_{i+4j=n}\sigma(i)\sigma(j)+7\sum_{i+4j=\frac{n}{2}}\sigma(i)\sigma(j)-\frac{b(n)}{9216}.\end{aligned}$$

On using (3.3) and (3.4) and from Theorem 4 [16] we have

$$\begin{aligned}\sum_{i+4j=m}\sigma(i)\sigma(j) = &-\frac{25}{96}\sigma_3(m)+\frac{3}{8}\sigma_3\left(\frac{m}{2}\right)+\frac{9}{32}\sigma_3\left(\frac{m}{3}\right)+\frac{5}{6}\sigma_3\left(\frac{m}{4}\right)-\frac{423}{16}\sigma_3\left(\frac{m}{6}\right)\\ &-\frac{45}{2}\sigma_3\left(\frac{m}{12}\right)+\frac{2-3m}{48}\sigma(m)+\frac{1-6m}{24}\sigma\left(\frac{m}{4}\right)+\frac{E(m)}{32}+\frac{27}{128}F(m)-\frac{G(m)}{24},\end{aligned}$$

and then simplifying , we obtain the required result. □

4. Conclusion

In the present investigation, we obtained an relation on Eisenstein series of level 6 in terms of theta functions. By exploiting the modular transformation properties and the theta-functional decompositions of these series, we obtain identities that express certain weighted divisor-sum convolutions in closed form. In particular, our analysis yields three convolution–sum formulas of the types

$$\sum_{2i+3j=m}\sigma(i)\sigma(j), \sum_{i+6j=m}\sigma(i)\sigma(j)\text{and} \sum_{i+8j=m}\sigma(i)\sigma(j)$$

which illustrate the interplay between theta functions and arithmetic functions. These identities highlight the structural interplay between modular forms of half-integral weight, the associated theta functions, and arithmetic functions arising from the coefficients of Eisenstein series. The results further demonstrate how level-structure in modular forms naturally encodes convolution relations among divisor functions.

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