



Diverse Properties of (p, q) -Cosine and (p, q) -Sine Fubini-Type Polynomials and Numbers

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ABSTRACT: In this work, the (p, q) -Sine and (p, q) -Cosine Fubini-type polynomials are introduced and multifarious summation formulae and relationships for these polynomials are derived by utilizing some series manipulation methods. Also, (p, q) -derivative operator rules and (p, q) -integral representations for the (p, q) -Sine and (p, q) -Cosine Fubini-type polynomials are given. Moreover, several correlations related to the both (p, q) -Bernoulli, Euler and Genocchi polynomials and the (p, q) -Stirling numbers of the second kind are developed.

Keywords: Quantum calculus, Fubini type polynomials, parametric (p, q) -Fubini type polynomials, generating functions.

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1. Introduction

In recent years, (p, q) -calculus has been studied and examined widely by many physicists and mathematicians, [4,10,9] and see also the references cited therein. Also, (p, q) -special polynomials, such as (p, q) -Euler, (p, q) -Genocchi, (p, q) -Bernoulli, (p, q) -Frobenius-Euler, have been firstly considered and developed by Duran et al. [2,3], then many authors worked on other (p, q) -special polynomials (see [35,34,38,42,37,36]). Khan et al. [8] introduced (p, q) -Fubini type polynomials and analyzed some of their basic properties. Khan et al. [5] considered (p, q) -extension of Apostol type Frobenius-Eulerian polynomials and derived diverse basic properties and identities covering derivative properties, difference equations, addition theorems, integral representations, recurrence relations, implicit and explicit formulas. Sadjang [12] defined (p, q) -Appell type polynomials and provided some of their characterizations including several algebraic properties. Sadjang et al. [11] introduced (p, q) -generalizations of two bivariate kinds of Bernoulli numbers and polynomials and then analyzed multifarious relations and formulae including connection formulas, recurrence formulas, (p, q) -integral representations and partial (p, q) -differential equations.

Along this work, we make use of the following notations:

$$\mathbb{N} = \{1, 2, 3, \dots\}, \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} \text{ and } \mathbb{R} = (-\infty, \infty).$$

The (p, q) -numbers $[m]_{p,q}$ are defined as follows

$$[m]_{p,q} = \frac{p^m - q^m}{p - q}, \quad 0 < |q| < |p| \leq 1.$$

It can be rewritten that $[m]_{p,q} = p^{m-1}[m]_{q/p}$, where $[m]_{q/p}$ is the q -number in quantum calculus (q -calculus) defined as $[m]_{q/p} = \frac{(q/p)^m - 1}{(q/p) - 1}$. Hence, it is observed that (p, q) -numbers and q -numbers are different, namely, it cannot derive (p, q) -numbers just by changing q by q/p in the definition of q -numbers.

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Again, when $p = 1$, the (p, q) -numbers reduce to the q -numbers, (see [1,13,14,15,16,17,18,19,20,21,32,33,39,40,41,22,23,24,25,26,27,28,29,30,31]).

The (p, q) -extension of the derivative operator of a function g with respect to t is given by

$$D_{p,q}g(t) = D_{p,q;t}g(t) = \frac{g(pt) - g(qt)}{(p-q)t}, (t \neq 0),$$

and $(D_{p,q}g(0)) = g'(0)$, provided that g is differentiable at 0. These operator satisfy the following properties

$$D_{p,q}(g(t)f(t)) = f(qt)D_{p,q}g(t) + g(pt)D_{p,q}f(t),$$

and

$$D_{p,q} \left(\frac{g(t)}{f(t)} \right) = \frac{f(qt)D_{p,q}g(t) - g(qt)D_{p,q}f(t)}{f(qt)f(pt)}.$$

The (p, q) -factorial numbers $[m]_{p,q}!$ and the (p, q) -binomial coefficients $\binom{m}{r}_{p,q}$ are provided by

$$[m]_{p,q}! = [m]_{p,q} \cdots [2]_{p,q} [1]_{p,q} \text{ for } m \in \mathbb{N} \text{ with } [0]_{p,q} = 1$$

and

$$\binom{m}{r}_{p,q} = \frac{[m]_{p,q}!}{[r]_{p,q}! [m-r]_{p,q}!} \quad (m \geq r).$$

The (p, q) -power basis is defined by

$$(t+a)(pt+aq) \cdots (p^{m-2}t + aq^{m-2})(p^{m-1}t + aq^{m-1}) = (t \oplus_{p,q} a)^m \quad (m \geq 1)$$

and also has the following expansion

$$\sum_{r=0}^m \binom{m}{r}_{p,q} p^{\binom{m}{2}} q^{\binom{m-r}{2}} t^r a^{m-r} = (t \oplus_{p,q} a)^m.$$

The (p, q) -exponential functions, $e_{p,q}(t)$ and $E_{p,q}(t)$, are introduced by

$$e_{p,q}(t) = \sum_{m=0}^{\infty} \frac{p^{\binom{m}{2}} t^m}{[m]_{p,q}!} \text{ and } E_{p,q}(t) = \sum_{m=0}^{\infty} \frac{q^{\binom{m}{2}} t^m}{[m]_{p,q}!}, \quad (1.1)$$

have the following relationships

$$e_{p^{-1}q^{-1}}(t) = E_{p,q}(t) \text{ and } e_{p,q}(t)E_{p,q}(-t) = 1, \quad (1.2)$$

and hold the following properties

$$D_{p,q}e_{p,q}(t) = e_{p,q}(pt) \text{ and } D_{p,q}E_{p,q}(t) = E_{p,q}(qt). \quad (1.3)$$

From (1.1), it is observed that

$$e_{p,q}(it) = \sum_{m=0}^{\infty} \frac{p^{\binom{m}{2}} (it)^m}{[m]_{p,q}!} = \sum_{m=0}^{\infty} \frac{(-1)^m p^{\binom{2m}{2}} t^{2m}}{[m]_{p,q}!} + \sum_{m=0}^{\infty} \frac{(-1)^m p^{\binom{2m+1}{2}} t^{2m+1}}{[2m+1]_{p,q}!}. \quad (1.4)$$

By (1.4), the (p, q) -sine function and the (p, q) -cosine function are given (cf. [12]) by

$$\sum_{m=0}^{\infty} \frac{(-1)^m p^{\binom{2m+1}{2}} t^{2m+1}}{[2m+1]_{p,q}!} = \sin_{p,q}(t) \text{ and } \sum_{m=0}^{\infty} \frac{(-1)^m p^{\binom{m}{2}} t^{2m}}{[2m]_{p,q}!} = \cos_{p,q}(t). \quad (1.5)$$

The (p, q) -Bernoulli, (p, q) -Euler and (p, q) -Genocchi polynomials are introduced as follows (see [2,3]):

$$\frac{ze_{p,q}(tz)}{e_{p,q}(z) - 1} = \sum_{m=0}^{\infty} B_m(t : p, q) \frac{z^m}{[m]_{p,q}!}, \quad |z| < 2\pi, \quad (1.6)$$

$$\frac{[2]_{p,q}e_{p,q}(tz)}{e_{p,q}(z) + 1} = \sum_{m=0}^{\infty} E_m(t : p, q) \frac{z^m}{[m]_{p,q}!}, \quad |z| < \pi, \quad (1.7)$$

and

$$\frac{[2]_{p,q}ze_{p,q}(tz)}{e_{p,q}(z) + 1} = \sum_{m=0}^{\infty} G_m(t : p, q) \frac{z^m}{[m]_{p,q}!}, \quad |z| < \pi. \quad (1.8)$$

When $t = 0$, the polynomials given above reduce to their corresponding numbers shown respectively by $B_m(p, q)$, $E_m(p, q)$ and $G_m(p, q)$ for $m \in \mathbb{R}$.

The Taylor series expansions of the functions $e^{tz} \sin(wz)$ and $e^{tz} \cos(wz)$ are presented as given below:

$$\sum_{m=0}^{\infty} S_m(t, w) \frac{z^m}{m!} = e^{tz} \sin wz \quad \text{and} \quad \sum_{m=0}^{\infty} C_m(t, w) \frac{z^m}{m!} = e^{tz} \cos wz, \quad (1.9)$$

where

$$S_m(t, w) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{m}{2r+1} t^{m-2r-1} w^{2r+1} \quad \text{and} \quad C_m(t, w) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{m}{2r} t^{m-2r} w^{2r}, \quad (1.10)$$

where the symbol $\lfloor \cdot \rfloor$ is the greatest integer function.

In the recent studies, Sharma et al. [43,7] considered (p, q) -generalizations of $S_m(t, w)$ and $C_m(t, w)$:

$$\sum_{m=0}^{\infty} S_{m,p,q}(t, w) \frac{z^m}{[m]_{p,q}!} = SIN_{p,q}(wz)e_{p,q}(tz), \quad (1.11)$$

and

$$\sum_{m=0}^{\infty} C_{m,p,q}(t, w) \frac{z^m}{[m]_{p,q}!} = COS_{p,q}(wz)e_{p,q}(tz), \quad (1.12)$$

where

$$S_{m,p,q}(t, w) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{m}{2r+1}_{p,q} p^{\binom{m-2r-1}{2}} t^{m-2r-1} w^{(2r+1)} q^{\binom{2r+1}{2}}, \quad (1.13)$$

and

$$C_{m,p,q}(t, w) = \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{m}{2r}_{p,q} p^{\binom{m-2r}{2}} q^{\binom{2r}{2}} t^{m-2r} w^{2r}. \quad (1.14)$$

For $\lambda \in \mathbb{C}$, the generalized (p, q) -Stirling numbers of the second kind $S_{m,p,q}^{\omega}(\lambda)$ are given by (see [11]):

$$\frac{(\lambda e_{p,q}(t) - 1)^m}{m!} = \sum_{\omega=0}^{\infty} S_{m,p,q}^{\omega}(\lambda) \frac{t^{\omega}}{[\omega]_{p,q}!} \quad m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}. \quad (1.15)$$

Taking $\lambda = 1$, equation (1.15) reduces to the (p, q) -Stirling numbers of the second kind as follows

$$\frac{(e_{p,q}(t) - 1)^m}{m!} = \sum_{\omega=m}^{\infty} S_{2,p,q}(\omega, m) \frac{t^{\omega}}{[\omega]_{p,q}!}. \quad (1.16)$$

The (p, q) -type Geometric polynomials of two variables are defined by (see [6]):

$$\frac{1}{1 - y(e_{p,q}(t) - 1)} e_{p,q}(xt) = \sum_{n=0}^{\infty} \mathbb{F}_{n,p,q}(x; y) \frac{t^n}{[n]_{p,q}!}. \quad (1.17)$$

When $x = 0$, $\mathbb{F}_{n,p,q}(y) = \mathbb{F}_{n,p,q}(0; y)$ are called the (p, q) -type Geometric polynomials and $\mathbb{F}_{n,p,q}(0; 1) = \mathbb{F}_{n,p,q}$ are called the (p, q) -type Geometric numbers.

The Fubini type polynomials of order α are defined by the following generating function [44]

$$\frac{2^\alpha}{(2 - e^t)^{2\alpha}} e^{xt} = \sum_{n=0}^{\infty} a_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad \alpha \in \mathbb{N}_0; \quad |t| < \log(2). \quad (1.18)$$

When $x = 0$, $a_n^{(\alpha)} = a_n^{(\alpha)}(0)$ are called the Fubini type numbers of order α .

In this work, the (p, q) -Sine and (p, q) -Cosine Fubini-type polynomials are introduced and multifarious summation formulae and relationships for these polynomials are derived by utilizing some series manipulation methods. Also, (p, q) -derivative operator rules and (p, q) -integral representations for the (p, q) -Sine and (p, q) -Cosine Fubini-type polynomials are given. Moreover, several correlations related to the both the (p, q) -Stirling numbers of the second kind and the (p, q) -Genocchi, Euler and Bernoulli polynomials are developed.

2. Bivariate Kinds of (p, q) -Cosine and (p, q) -Sine Fubini Type Polynomials

In this section, we consider the (p, q) -Cosine and (p, q) -Sine Fubini type polynomials of complex variable and deduce some identities of these polynomials. First, we present the following definition as.

$$\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) E_{p,q}(ity) = \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(\alpha)}((x + iy)_{p,q}) \frac{t^n}{[n]_{p,q}!}. \quad (2.1)$$

On the other hand, we suppose that

$$e_{p,q}(xt) E_{p,q}(ity) = e_{p,q}(xt) (COS_{p,q}(yt) + iSIN_{p,q}(yt)). \quad (2.2)$$

Thus, by (2.1) and (2.2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(\alpha)}((x + iy)_{p,q}) \frac{t^n}{[n]_{p,q}!} &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) E_{p,q}(ity) \\ &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) (COS_{p,q}(yt) + iSIN_{p,q}(yt)), \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(\alpha)}((x - iy)_{p,q}) \frac{t^n}{[n]_{p,q}!} &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) E_{p,q}(-ity) \\ &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) (COS_{p,q}(yt) - iSIN_{p,q}(yt)). \end{aligned} \quad (2.4)$$

From (2.3) and (2.4), we get

$$\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) COS_{p,q}(yt) = \sum_{n=0}^{\infty} \left(\frac{\mathcal{F}_{n,p,q}^{(\alpha)}((x \oplus iy)_{p,q}) + \mathcal{F}_{n,p,q}^{(\alpha)}((x \ominus iy)_{p,q})}{2} \right) \frac{t^n}{[n]_{p,q}!}, \quad (2.5)$$

and

$$\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) SIN_{p,q}(yt) = \sum_{n=0}^{\infty} \left(\frac{\mathcal{F}_{n,p,q}^{(\alpha)}((x \oplus iy)_{p,q}) - \mathcal{F}_{n,p,q}^{(\alpha)}((x \ominus iy)_{p,q})}{2} \right) \frac{t^n}{[n]_{p,q}!}. \quad (2.6)$$

Definition 2.1 Let $n \geq 0$. We define two parametric kinds of (p, q) -Cosine Fubini type polynomials $\mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y)$ and (p, q) -Sine Fubini type polynomials $\mathcal{F}_{n,p,q}^{(S,\alpha)}(x, y)$, for non negative integer n are defined by

$$\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) \text{COS}_{p,q}(yt) = \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!}, \quad (2.7)$$

and

$$\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) \text{SIN}_{p,q}(yt) = \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(S,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!}, \quad (2.8)$$

respectively.

Note that

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(0, 0) = \mathcal{F}_{n,p,q}^{(\alpha)}, \quad \mathcal{F}_{n,p,q}^{(S,\alpha)}(0, 0) = 0. \quad (n \geq 0).$$

From (2.5)-(2.8), we have

$$\mathcal{F}_{n,q}^{(C,\alpha)}(x, y) = \frac{\mathcal{F}_{n,p,q}^{(\alpha)}((x \oplus iy)_{p,q}) + \mathcal{F}_{n,p,q}^{(\alpha)}((x \ominus iy)_{p,q})}{2}, \quad (2.9)$$

and

$$\mathcal{F}_{n,p,q}^{(S)}(x, y) = \frac{\mathcal{F}_{n,p,q}^{(\alpha)}((x \oplus iy)_{p,q}) - \mathcal{F}_{n,p,q}^{(\alpha)}((x \ominus iy)_{p,q})}{2i}. \quad (2.10)$$

Remark 2.1 For $x = 0$ in (2.7) and (2.8), we get new type of (p, q) -Cosine Fubini type polynomials $\mathcal{F}_{n,q}^{(C,\alpha)}(y)$ and (p, q) -Sine Fubini type polynomials $\mathcal{F}_{n,p,q}^{(S,\alpha)}(y)$ as

$$\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \text{COS}_{p,q}(yt) = \sum_{j=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(y) \frac{t^j}{[j]_{p,q}!}, \quad (2.11)$$

and

$$\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \text{SIN}_{p,q}(yt) = \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(S,\alpha)}(y) \frac{t^n}{[n]_{p,q}!}, \quad (2.12)$$

respectively.

It is clear that

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(0) = \mathcal{F}_{n,p,q}^{(\alpha)}, \quad \mathcal{F}_{n,p,q}^{(S,\alpha)}(0) = 0, \quad n \geq 0.$$

Now, we start some basic properties of these polynomials.

Theorem 2.1 Let n integer. Then

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(y) = \sum_{v=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+v}{2v}_{p,q} (-1)^v q^{\binom{2v}{2}} y^{2v} \mathcal{F}_{n-2v,p,q}^{(\alpha)}, \quad (2.13)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}(y) = \sum_{v=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n+v}{2v+1}_{p,q} (-1)^v q^{\binom{2v+1}{2}} y^{2v+1} \mathcal{F}_{n-2v-1,p,q}^{(\alpha)}. \quad (2.14)$$

Proof: By (2.11) and (2.12), we can derive the following equations

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n]_{p,q}!} &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \text{COS}_{p,q}(yt) = \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(\alpha)} \frac{t^n}{[n]_{p,q}!} \sum_{v=0}^{\infty} (-1)^v q^{(2v-1)v} \eta^{2v} \frac{t^v}{[2v]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{v=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+v}{2v}_{p,q} (-1)^v q^{\binom{2v}{2}} y^{2v} \mathcal{F}_{n-2v,p,q}^{(\alpha)} \right) \frac{t^n}{[n]_{p,q}!}, \quad (2.15) \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(S,\alpha)}(y) \frac{t^n}{[n]_{p,q}!} &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \text{SIN}_{p,q}(yt) \\ &= \sum_{n=0}^{\infty} \left(\sum_{v=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n+v}{2v+1}_{p,q} (-1)^v q^{\binom{2v+1}{2}} y^{2v+1} \mathcal{F}_{n-2v-1,p,q}^{(\alpha)} \right) \frac{t^n}{[n]_{p,q}!}. \end{aligned} \quad (2.16)$$

Therefore, by (2.15) and (2.16), we get (2.13) and (2.14). \square

Theorem 2.2 *Let n integer. Then*

$$\mathcal{F}_{n,p,q}^{(\alpha)}((x \oplus iy)_{p,q}) = \sum_{k=0}^n \binom{n}{k}_{p,q} (x \oplus iy)_{p,q}^k \mathcal{F}_{n-k,p,q} = \sum_{k=0}^n \binom{n}{k}_{p,q} (iy)^k \mathcal{F}_{n-k,p,q}^{(\alpha)}(x), \quad (2.17)$$

and

$$\mathcal{F}_{n,p,q}^{(\alpha)}((x \ominus iy)_{p,q}; z) = \sum_{k=0}^n \binom{n}{k}_{p,q} (x \ominus iy)_{p,q}^k \mathcal{F}_{n-k,p,q} = \sum_{k=0}^n \binom{n}{k}_{p,q} (-1)^k (iy)^k \mathcal{F}_{n-k,p,q}^{(\alpha)}(x). \quad (2.18)$$

Proof: By using (2.3) and (2.4), we obtain (2.17) and (2.18). So we omit the proof. \square

Theorem 2.3 *Let n integer. Then*

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{k,p,q}^{(\alpha)} \mathcal{C}_{n-k,p,q}(x, y), \quad (2.19)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}(x, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{k,p,q}^{(\alpha)} \mathcal{S}_{n-k,p,q}(x, y). \quad (2.20)$$

Proof: Consider

$$\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \right) \left(\sum_{k=0}^{\infty} b_k \frac{t^k}{k!} \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_{n-k} b_k \right) \frac{t^n}{n!}.$$

Now

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!} &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) \text{COS}_{p,q}(yt) \\ &= \left(\sum_{k=0}^{\infty} \mathbb{F}_{k,p,q}^{(\alpha)} \frac{t^k}{[k]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} \mathcal{C}_{n,p,q}(x, y) \frac{t^n}{[n]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{k,p,q}^{(\alpha)} \mathcal{C}_{n-k,p,q}(x, y) \right) \frac{t^n}{[n]_{p,q}!}, \end{aligned}$$

which proves (2.19). The proof of (2.20) is similar. \square

Theorem 2.4 *Let $n \geq 0$. Then*

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(x + r, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{k,p,q}^{(C,\alpha)}(x, y) r^{n-k}, \quad (2.21)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}(x + r, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{k,p,q}^{(S,\alpha)}(x, y) r^{n-k}. \quad (2.22)$$

Proof: By changing x with $x + r$ in (2.7), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x+r, y) \frac{t^n}{[n]_{p,q}!} &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}(xt) \text{COS}_{p,q}(yt) e_{p,q}^{rt} \\ &= \left(\sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!} \right) \left(\sum_{k=0}^{\infty} r^k \frac{t^k}{[k]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{k,p,q}^{(C,\alpha)}(x, y) r^{n-k} \right) \frac{t^n}{[n]_{p,q}!}, \end{aligned}$$

which complete the proof (2.21). The result (2.22) can be similarly proved. \square

Theorem 2.5 *Let n integer. Then*

$$\frac{\partial}{\partial x} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) = [n]_{p,q} \mathcal{F}_{n-1,p,q}^{(C,\alpha)}(px, y), \quad (2.23)$$

$$\frac{\partial}{\partial y} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) = -[n]_{p,q} \mathcal{F}_{n-1,p,q}^{(S,\alpha)}(x, qy), \quad (2.24)$$

and

$$\frac{\partial}{\partial x} \mathcal{F}_{n,p,q}^{(S,\alpha)}(x, y) = [n]_{p,q} \mathcal{F}_{n-1,p,q}^{(S,\alpha)}(px, y), \quad (2.25)$$

$$\frac{\partial}{\partial y} \mathcal{F}_{n,q}^{(S,\alpha)}(x, y) = [n]_q \mathcal{F}_{n-1,q}^{(C,\alpha)}(x, qy). \quad (2.26)$$

Proof: Equation (2.7) yields

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\partial}{\partial x} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!} &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \frac{\partial}{\partial x} e_{p,q}(xt) \text{COS}_{p,q}(yt) = \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(px, y) \frac{t^{n+1}}{[n]_{p,q}!} \\ &= \sum_{n=1}^{\infty} \mathcal{F}_{n-1,p,q}^{(C,\alpha)}(px, y) \frac{t^n}{[(n-1)]_{p,q}!} = \sum_{n=1}^{\infty} [n]_{p,q} \mathcal{F}_{n-1,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!}, \end{aligned}$$

proving (2.23). Other (2.24), (2.25) and (2.26) can be similarly derived. \square

Theorem 2.6 *Let $N \in \mathbb{N}^*$, the following formula holds true. Then*

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(2x, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) x^{n-k}, \quad (2.27)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}(2x, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{n,p,q}^{(S,\alpha)}(x, y) x^{n-k}. \quad (2.28)$$

Proof: By using Definition 2.1, we can easily proof of equations (2.27) and (2.28). We omit the proof. \square

Theorem 2.7 *For $n \geq 0$, we have*

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}((1 \oplus x)_{p,q}, y) = \sum_{r=0}^n \binom{n}{r}_{p,q} \mathcal{F}_{n-r,p,q}^{(C,\alpha)}(x, y), \quad (2.29)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}((1 \oplus x)_{p,q}, y) = \sum_{r=0}^n \binom{n}{r}_{p,q} \mathcal{F}_{n-r,p,q}^{(S,\alpha)}(x, y). \quad (2.30)$$

Proof: Using the generating function (2.7), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}((1 \oplus x)_{p,q}, y) \frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_q!} \\
&= \left(\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \right) (e_{p,q}(t) - 1) e_{p,q}(xt) \text{COS}_{p,q}(yt) \\
&= \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!} \left(\sum_{r=0}^{\infty} \frac{t^r}{[r]_{p,q}!} - 1 \right) \\
&= \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!} \sum_{r=0}^{\infty} \frac{t^r}{[r]_{p,q}!} - \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^n \binom{n}{r}_{p,q} \mathcal{F}_{n-r,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!} - \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) \frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

Finally, equating the coefficients of the like powers of t on both sides, we get (2.29). The proof of (2.30) is similar. \square

Theorem 2.8 *Let n be integer. Then*

$$\mathcal{F}_{n,p,q}^{(C,\alpha+\beta)}(x, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{k,p,q}^{(C,\alpha)} \mathcal{F}_{n-k,p,q}^{(C,\beta)}(x, y) \quad (2.31)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha+\beta)}(x, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{k,p,q}^{(S,\alpha)} \mathcal{F}_{n-k,p,q}^{(S,\beta)}(x, y) \quad (2.32)$$

Proof: By (2.7), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha+\beta)}(x, y) \frac{t^n}{[n]_{p,q}!} &= \frac{2^{\alpha+\beta}}{(2 - e_{p,q}(t))^{2(\alpha+\beta)}} e_{p,q}(xt) \text{COS}_{p,q}(yt) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n \mathcal{F}_{k,p,q}^{(C,\alpha)} \mathcal{F}_{n-k,p,q}^{(C,\beta)}(x, y) \frac{t^k}{[k]_q!} \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{F}_{k,p,q}^{(C,\alpha)} \mathcal{F}_{n-k,p,q}^{(C,\beta)}(x, y) \frac{t^n}{[n]_{p,q}!},
\end{aligned}$$

which proves the result (2.31). The assertion (2.32) can be proved similarly. \square

3. Relationship Between (p, q) -Bernoulli, (p, q) -Euler and (p, q) -Genocchi Polynomials and (p, q) -Stirling Numbers of the Second Kind

In this section, we prove some relationships for two bivariate kind of (p, q) -Cosine Fubini type polynomials and (p, q) -Sine Fubini type polynomials related to (p, q) -Bernoulli polynomials, (p, q) -Euler polynomials and (p, q) -Genocchi polynomials and (p, q) -Stirling numbers of the second kind. We start a following theorem.

Theorem 3.1 *Each of the following relationships holds true:*

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) = \sum_{s=0}^{n+1} \binom{n+1}{s}_{p,q} \left[\sum_{k=0}^s \binom{s}{k}_{p,q} \mathcal{B}_{s-k,p,q}(x) - \mathcal{B}_{s,p,q}(x) \right] \frac{\mathcal{F}_{n+1-s,p,q}^{(C,\alpha)}(y)}{[n+1]_{p,q}}, \quad (3.1)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}(x, y) = \sum_{s=0}^{n+1} \binom{n+1}{s}_{p,q} \left[\sum_{k=0}^s \binom{s}{k}_{p,q} \mathcal{B}_{s-k,p,q}(x) - \mathcal{B}_{s,p,q}(x) \right] \frac{\mathcal{F}_{n+1-s,p,q}^{(S,\alpha)}(y)}{[n+1]_{p,q}}. \quad (3.2)$$

Proof: By using (1.6) and (2.7), we have

$$\begin{aligned}
 \left(\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \right) e_{p,q}(xt) \text{COS}_{p,q}(yt) &= \left(\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \right) \frac{t}{e_{p,q}(t) - 1} \frac{e_{p,q}(t) - 1}{t} e_{p,q}(xt) \text{COS}_{p,q}(yt) \\
 &= \frac{1}{t} \sum_{n=0}^{\infty} \left(\sum_{k=0}^s \binom{s}{k}_{p,q} \mathbb{B}_{s-k,p,q}(x) p^{\binom{k}{s}} \right) \frac{t^s}{[s]_{p,q}!} \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n]_{p,q}!} \\
 &\quad - \frac{1}{t} \sum_{s=0}^{\infty} \mathcal{B}_{s,p,q}(x) \frac{t^s}{[s]_{p,q}!} \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n]_{p,q}!} \\
 &= \frac{1}{t} \sum_{n=0}^{\infty} \left[\sum_{s=0}^n \binom{n}{s}_{p,q} \sum_{k=0}^s \binom{s}{k}_{p,q} \mathcal{B}_{s-k,p,q}(x) \right] \mathcal{F}_{n-s,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n]_{p,q}!} \\
 &\quad - \frac{1}{t} \sum_{n=0}^{\infty} \left[\sum_{s=0}^n \binom{n}{s}_{p,q} \mathcal{B}_{s,p,q}(x) \right] \mathcal{F}_{n-s,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n]_{p,q}!}.
 \end{aligned}$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we arrive at the required result (3.1). The proof of (3.2) is similar. \square

Theorem 3.2 *Each of the following relationships holds true:*

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) = \sum_{s=0}^n \binom{n}{s}_{p,q} \left[\sum_{k=0}^s \binom{s}{k}_{p,q} \mathcal{E}_{s-k,p,q}(x) + \mathcal{E}_{s,p,q}(x) \right] \frac{\mathcal{F}_{n-s,p,q}^{(C,\alpha)}(y)}{[2]_{p,q}}, \quad (3.3)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}(x, y) = \sum_{s=0}^n \binom{n}{s}_{p,q} \left[\sum_{k=0}^s \binom{s}{k}_{p,q} \mathcal{E}_{s-k,p,q}(x) + \mathcal{E}_{s,p,q}(x) \right] \frac{\mathcal{F}_{n-s,p,q}^{(S,\alpha)}(y)}{[2]_{p,q}}. \quad (3.4)$$

Proof: By using definitions (1.7) and (2.7), we have

$$\begin{aligned}
 \left(\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \right) e_{p,q}(xt) \text{COS}_{p,q}(yt) &= \left(\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \right) \frac{[2]_{p,q}}{e_{p,q}(t) + 1} \frac{e_{p,q}(t) + 1}{[2]_{p,q}} e_{p,q}(xt) \text{COS}_{p,q}(yt) \\
 &= \frac{1}{[2]_{p,q}} \left[\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{E}_{n-k,p,q}(x) \right) \frac{t^n}{[n]_{p,q}!} + \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \right] \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n]_{p,q}!} \\
 &= \frac{1}{[2]_{p,q}} \sum_{n=0}^{\infty} \left[\sum_{s=0}^n \binom{n}{s}_{p,q} \sum_{k=0}^s \binom{s}{k}_{p,q} \mathcal{E}_{s-k,p,q}(x) + \sum_{s=0}^n \binom{n}{s}_{p,q} \mathcal{E}_{s,p,q}(x) \right] \mathcal{F}_{n-s,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n]_{p,q}!}.
 \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we arrive at the desired result (3.3). The proof of (3.4) is similar. \square

Theorem 3.3 *Let n be integer. Then*

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) = \sum_{s=0}^n \binom{n+1}{s}_{p,q} \left[\sum_{k=0}^s \binom{s}{k}_{p,q} \mathcal{G}_{s-k,p,q}(x) + \mathcal{G}_{s,p,q}(x) \right] \frac{\mathcal{F}_{n+1-s,p,q}^{(C,\alpha)}(y)}{[2]_{p,q}[n+1]_{p,q}}, \quad (3.5)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}(x, y) = \sum_{s=0}^n \binom{n+1}{s}_{p,q} \left[\sum_{k=0}^s \binom{s}{k}_{p,q} \mathcal{G}_{s-k,p,q}(x) + \mathcal{G}_{s,p,q}(x) \right] \frac{\mathcal{F}_{n+1-s,p,q}^{(S,\alpha)}(y)}{[2]_{p,q}[n+1]_{p,q}}. \quad (3.6)$$

Proof: By (1.8) and (2.7), we have

$$\begin{aligned} \left(\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \right) e_{p,q}(xt) \text{COS}_{p,q}(yt) &= \left(\frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} \right) e_{p,q}(xt) \text{COS}_{p,q}(yt) \frac{[2]_{p,q}t}{e_{p,q}(t) + 1} \frac{e_{p,q}(t) + 1}{[2]_{p,q}t} e_{p,q}(xt) \text{COS}_{p,q}(yt) \\ &= \frac{1}{[2]_{p,q}t} \left[\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{G}_{n-k,p,q}(x) p^{\binom{k}{2}} \right) \frac{t^n}{[n]_{p,q}!} + \sum_{n=0}^{\infty} \mathcal{G}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \right] \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n]_{p,q}!} \\ &= \frac{1}{[2]_{p,q}} \sum_{n=0}^{\infty} \left[\sum_{s=0}^n \binom{n}{s}_{p,q} \sum_{k=0}^s \binom{s}{k}_{p,q} \mathcal{G}_{s-k,p,q}(x) + \sum_{s=0}^n \binom{n}{s}_{p,q} \mathcal{G}_{s,p,q}(x) \right] \mathcal{F}_{n+1-s,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n+1]_{p,q}!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, then we have the asserted result (3.5). The proof of (3.6) is similar. \square

Theorem 3.4 *Let n be integer. Then*

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(x+r, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \sum_{m=0}^k \mathcal{F}_{n-k,p,q}^{(C,\alpha)}(y) x^m S_{2,p,q}(k+r, m+r), \quad (3.7)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}(x+r, y) = \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k}_{p,q} \mathcal{F}_{n-k,p,q}^{(S,\alpha)}(y) x^m S_{2,p,q}(k+r, m+r) \quad (3.8)$$

Proof: Using (1.16) and (2.7), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(x+r, y) \frac{t^n}{[n]_{p,q}!} &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}((r)t) (e_{p,q}(t) - 1 + 1)^x \text{COS}_{p,q}(yt) \\ &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}((r)t) \text{COS}_{p,q}(yt) \sum_{m=0}^{\infty} x^k (e_{p,q}(t) - 1)^m \frac{t^m}{[m]_{p,q}!} \\ &= \frac{2^\alpha}{(2 - e_{p,q}(t))^{2\alpha}} e_{p,q}((r)t) \text{COS}_{p,q}(yt) \sum_{k=0}^{\infty} \sum_{m=0}^k x^k S_{2,p,q}(k, m) \frac{t^k}{[k]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \mathcal{F}_{n,p,q}^{(C,\alpha)}(y) \frac{t^n}{[n]_{p,q}!} \sum_{k=0}^{\infty} \sum_{m=0}^k x^k S_{2,p,q}(k, m) \frac{t^k}{[k]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k}_{p,q} \sum_{m=0}^k \mathcal{F}_{n-k,p,q}^{(C,\alpha)}(y) x^m S_{2,p,q}(k+r, m+r) \frac{t^n}{[n]_{p,q}!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, then we have the asserted result (3.7). The proof of (3.8) is similar. \square

Corollary 3.1 *Let n be integer. Then*

$$\mathcal{F}_{n,p,q}^{(C,\alpha)}(x, y) = \sum_{k=0}^n \binom{n}{k}_{p,q} \sum_{m=0}^k \mathcal{F}_{n-k,p,q}^{(C,\alpha)}(y) x^m S_{2,p,q}(k, m), \quad (3.9)$$

and

$$\mathcal{F}_{n,p,q}^{(S,\alpha)}(x, y) = \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k}_{p,q} \mathcal{F}_{n-k,p,q}^{(S,\alpha)}(y) x^m S_{2,p,q}(k, m). \quad (3.10)$$

4. Conclusion

Utilizing (p, q) -numbers and (p, q) -concepts, Duran et al. [2,3] considered (p, q) -Genocchi polynomials and numbers, (p, q) -Bernoulli polynomials and numbers and (p, q) -Euler polynomials and numbers and provided many properties and formulas for these polynomials. Inspired and motivated by this consideration, many authors have introduced (p, q) -special numbers and polynomials and have described their several identities and properties. In this paper, using the (p, q) -Cosine polynomials and (p, q) -Sine polynomials, we have introduced novel kinds of (p, q) -extensions of Fubini-type polynomials and have acquired multifarious properties and identities by making use of some series manipulation methods. Furthermore, we have computed the q -derivative operator rules for these polynomials. Moreover, we have determined the approximate root movements of the new mentioned polynomials in a complex plane, utilizing the Newton method and illustrating them in figures. The structure of the approximate roots will come out in various ways, depending on the condition of the variables, and new methods and theorems related to this topic need to be created and proven.

Not only can the ideas presented in this paper be utilized for similar polynomials, but these polynomials may also have possible applications in other scientific areas besides the applications described at the end of the paper. We would like to continue to study this line of research in the future.

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