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The Laplacian Minimum Efficient Dominating Energy of a Graph

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ABSTRACT: For a graph G, a subset D of $V(G)$ is called an efficient dominating set for G if for every vertex $v \in V(G)$, there is exactly one $d \in D$ dominating v. The efficient domination number $\gamma_{ED}(G)$ is the minimum cardinality of a efficient dominating set. In this paper we introduce the concept of Laplacian minimum efficient dominating energy $LE_{ED}(G)$ of a graph G and computed Laplacian minimum efficient dominating energies of some standard graphs. Upper and lower bounds for $LE_{ED}(G)$ are established.

Keywords: Dominating set, efficient domination, efficient domination number, efficient domination energy.

Contents

1. Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges, and let $A = (a_{i,j})$ denote its adjacency matrix. The eigenvalues of the graph G are defined as the eigenvalues of its adjacency matrix $A(G)$, and are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. A graph G is said to be singular if at least one of its eigenvalues is zero, which implies that $\det(A) = 0$. Conversely, if none of the eigenvalues are zero, the graph is called nonsingular, in which case $\det(A) > 0$.

A graph G is called k-regular if every vertex in G has degree k. A complete graph, denoted by K_n, is a simple graph in which every pair of distinct vertices is connected by an edge. A star graph, denoted by $K_{1,n-1}$, also referred to as a claw or cherry, is a tree with a single central vertex joined to all other vertices.

A subset $D \subseteq V$ is called a dominating set of G if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. A dominating set D is called minimal if no proper subset of D is also a dominating set. The smallest possible size of a minimal dominating set in G is termed the domination number, denoted by $\gamma(G)$. Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and the edge set E. A vertex set D in G is an efficient dominating for G if for every vertex $v \in V$, there is exactly one $d \in D$ dominating v. For a more detailed treatment of efficient domination, see [4]. In this work, we initiate a study on the Laplacian minimum efficient domination energy of a graph.

The notion of graph energy was introduced by I. Gutman [7]. Although it initially received limited attention, it has since gained widespread interest and has become a topic of active research. Over time,

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analogous energy concepts have also been developed for matrices other than the adjacency matrix. The energy $E(G)$ of a graph G is defined as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

I. Gutman and B. Zhou [8] defined the Laplacian energy of a graph G in the year 2006. Let G be a graph with n vertices and m edges. The Laplacian matrix of the graph G, denoted by $L = (L_{ij})$, is a square matrix of order n. The elements of the Laplacian matrix are defined as

$$L_{ij} = \begin{cases} -1, & \text{if } v_i \, and \, v_j \, are \, adjacent, \\ 0, & \text{if } v_i \, and \, v_j \, are \, not \, adjacent, \\ d_i, & \text{if i=j.} \end{cases}$$

where d_i is the degree of the vertex v_i.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of Laplacian matrix G. Laplacian energy of G is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right|.$$

The basic properties Laplacian energy including various upper and lower bounds have been established in [11], [12], [14], [15] and it has found that remarkable chemical application, high resolution satellite image classification and segmentation using Laplacian graph energy and finding semantic structures in image hierarchies using Laplacian graph energy. The Laplacian energy provides a measure of the overall connectivity and robustness of a network. A low Laplacian energy typically indicates a highly connected network (less fragmentation), where as high Laplacian energy might indicate a more fragmented or less connected network. The Laplacian matrix is used to model molecular structures where atoms are represented as vertices and bonds as edges. In this context, Laplacian energy can be used to estimate the stability and reactivity of molecules. A molecule with a high Laplacian energy might be less stable or more reactive, while lower energy could suggest a more stable structure.

2. The Minimum Efficient Domination Energy of Graphs

Let G be simple graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. A vertex set D in G is an efficient dominating set (EDS) for G if for every vertex $v \in V$, there is exactly one $d \in D$ dominating v. The efficient dominating number is the minimum cardinality taken over all the minimal efficient dominating sets of G. Let EDS be the minimum efficient dominating set of G. The minimum efficient dominating matrix of G is $n \times n$ matrix defined by $A_{ED}(G) = a_{ij}$ where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent}, \\ 1, & \text{if } i = j, v_i \in EDS, \\ 0, & \text{if } otherwise. \end{cases}$$

The characteristic polynomial of $A_{ED}(G)$ is denoted by $f_n(G, \lambda) = det(\lambda I - A_{ED}(G))$. The minimum efficient dominating eigenvalues of the graph G are the eigenvalues of $A_{ED}(G)$. Since $A_{ED}(G)$ is real and symmetric, its eigenvalues are real numbers and are labelled in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The minimum efficient dominating energy of G is defined as

$$E_{ED}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

3. The Laplacian Minimum Efficient Dominating Energy of a Graph

Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G. Then the Laplacian minimum efficient dominating matrix of G is denoted by $LE_{ED}(G)$ and is defined as follows $LE_{ED} = D(G) - A_{ED}(G)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values $LE_{ED}(G)$ arranged in non increasing order. These eigen values are called Laplacian minimum efficient dominating eigen values of G. The Laplacian minimum efficient dominating energy of a graph G is defined as

$$LE_{ED}(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right|,$$

Example 3.1 Consider a graph as shown in Figure 1. The possible minimum efficient dominating sets are: (i) $D_1 = \{v_1, v_6\}$ (ii) $D_2 = \{v_3, v_5\}$

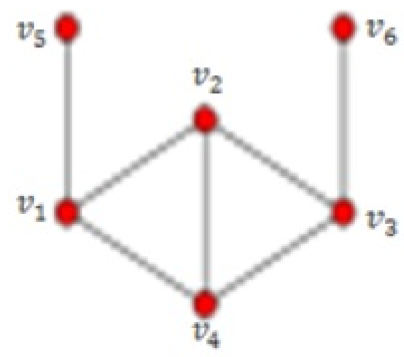

Figure 1. A Simple Graph with 6 Vertices

If the minimum efficient dominating set is $D_1 = \{v_1, v_6\}$, then

$$A_{ED,D_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \text{ and } D(G) = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$LE_{ED}(G) = D(G) - A_{ED,D_1}(G) = \begin{pmatrix} 2 & -1 & 0 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is given by
$f_n(G, \lambda) = \lambda^6 - 12\lambda^5 + 49\lambda^4 - 72\lambda^3 + 9\lambda^2 + 28\lambda = 0.$

$$Spec(G) = \begin{pmatrix} -0.4885 & 0 & 1.1014 & 2.8986 & 4 & 4.4885 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Average degree of the graph $= \frac{2m}{n} = \frac{2\times 6}{6} = \frac{12}{6} = 2$
Hence, Laplacian minimum efficient dominating energy, $LE_{ED}(G) \approx 10.7742$.

4. Laplacian Minimum Efficient Dominating Energy of Standard Graphs

Theorem 4.1 *For $n \geq 3$, the Laplacian minimum efficient dominating energy of a star graph $K_{1,n-1}$ is $\frac{(n-2)^2}{n} + \sqrt{n^2 - 2n + 5}$.*

Proof: Let $K_{1,n-1}$ be a star graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ having the vertex v_1 at the center. The minimum efficient dominating set is $D = \{v_1\}$. Then

$$A_{ED}(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 1 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 \end{pmatrix}.$$

and

$$D(K_{1,n-1}) = \begin{pmatrix} n-1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \end{pmatrix}.$$

$$LE_{ED}(K_{1,n-1}) = D(K_{1,n-1}) - A_{ED}(K_{1,n-1}) = \begin{pmatrix} n-2 & -1 & -1 & ... & -1 & -1 \\ -1 & 1 & 0 & ... & 0 & 0 \\ -1 & 0 & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ -1 & 0 & 0 & ... & 1 & 0 \\ -1 & 0 & 0 & ... & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of $LE_{ED}(K_{1,n-1})$ is given by,
$f_{ED}(K_{1,n-1}, \lambda) = (\lambda - 1)^{n-2}(\lambda^2 - (n-1)\lambda - 1)$
The Laplacian minimum efficient dominating eigen values are:

$$Spec(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{(n-1)+\sqrt{n^2-2n+5}}{2} & \frac{(n-1)-\sqrt{n^2-2n+5}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Average degree of $K_{1,n-1} = \frac{2(n-1)}{n}$.

Hence, the Laplacian minimum efficient dominating energy is

$$\begin{aligned} LE_{ED}(K_{1,n-1}) =& \left|1 - \frac{2(n-1)}{n}\right|(n-2) + \left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2} - \frac{2(n-1)}{n}\right| \\ & + \left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2} - \frac{2(n-1)}{n}\right| \\ =& \left|\frac{(-n+2)}{n}\right|(n-2) + \left|\frac{(n^2-n)+n\sqrt{n^2-2n+5}}{2n}\right| + \\ & \left|\frac{(n^2-n)-n\sqrt{n^2-2n+5}}{2n}\right|. \\ =& \frac{(n-2)^2}{n} + \sqrt{n^2-2n+5}. \end{aligned}$$

Therefore, $LE_{ED}(K_{1,n-1}) = \frac{(n-2)^2}{n} + \sqrt{n^2-2n+5}$. □

Theorem 4.2 *For $n \geq 3$, the Laplacian minimum efficient dominating energy of a friendship graph F_3^n is $\frac{4n^2-2n+1}{2n+1} + \sqrt{4n^2+4}$.*

Proof: Consider a friendship graph F_3^n with vertex set $V = \{v_1, v_2, v_3, ..., v_{2n+1}\}$, where the vertex v_1 is the center vertex of F_3^n. The minimum efficient dominating set of F_3^n is $D = \{v_1\}$. Then

$$A_{ED}(F_3^n) = \begin{pmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 1 & ... & 0 & 0 \\ 1 & 1 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 1 & 0 & 0 & ... & 0 & 1 \\ 1 & 0 & 0 & ... & 1 & 0 \end{pmatrix}.$$

and

$$D(F_3^n) = \begin{pmatrix} 2n & 0 & 0 & ... & 0 & 0 \\ 0 & 2 & 0 & ... & 0 & 0 \\ 0 & 0 & 2 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 2 & 0 \\ 0 & 0 & 0 & ... & 0 & 2 \end{pmatrix}.$$

$$LE_{ED}(F_3^n) = D(F_3^n) - A_{ED}(F_3^n) = \begin{pmatrix} 2n-1 & -1 & -1 & ... & -1 & -1 \\ -1 & 2 & -1 & ... & 0 & 0 \\ -1 & -1 & 2 & ... & -1 & -1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ -1 & 0 & 0 & ... & 2 & -1 \\ -1 & 0 & 0 & ... & -1 & 2 \end{pmatrix}.$$

The characteristic polynomial of $LE_{ED}(F_3^n)$ is given by,
$f_{ED}(F_3^n, \lambda) = (\lambda - 1)^{n-1}(\lambda - 3)^n(\lambda^2 - (2n)\lambda - 1)$
The Laplacian minimum efficient dominating eigen values are:

$$Spec(F_3^n) = \begin{pmatrix} 1 & 3 & \frac{(2n)+\sqrt{4n^2+4}}{2} & \frac{(2n)-\sqrt{4n^2+4}}{2} \\ n-1 & n & 1 & 1 \end{pmatrix}.$$

Average degree of $F_3^n = \frac{6n}{2n+1}$.
Hence, the Laplacian minimum efficient dominating energy is

$$\begin{aligned} LE_{ED}F_3^n =& \left|1 - \frac{6n}{2n+1}\right|(n-1) + \left|3 - \frac{6n}{2n+1}\right|n + \left|\frac{2n+\sqrt{4n^2+4}}{2} - \frac{6n}{2n+1}\right| \\ &+ \left|\frac{2n+\sqrt{4n^2+4}}{2} - \frac{6n}{2n+1}\right| \\ =& \left|\frac{4n-1}{2n+1}\right|(n-1) + \left|\frac{3n}{2n+1}\right|n + \left|\frac{(4n^2+2n)+(2n+1)\sqrt{4n^2+4}-12n}{4n+2}\right| + \\ & \left|\frac{(4n^2+2n)-(2n+1)\sqrt{4n^2+4}-12n}{4n+2}\right| \\ =& \frac{4n^2-2n+1}{2n+1} + \sqrt{4n^2+4}. \end{aligned}$$

Therefore, $LE_{ED}(F_3^n) = \frac{4n^2-2n+1}{2n+1} + \sqrt{4n^2+4}$. □

Theorem 4.3 *For $n > 2$, the Laplacian minimum efficient dominating energy of a complete graph K_n is $2 + \sqrt{n^2 - 2n + 5}$.*

Proof: Let K_n be the complete graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The set $\{v_1\}$ is minimum efficient dominating set. Then the minimmum efficient dominating matrix $A_{ED}(K_n)$ and its characteristics polynomial are as follows.

$$A_{ED}(K_n) = \begin{pmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 1 & ... & 1 & 1 \\ 1 & 1 & 0 & ... & 1 & 1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 1 & 1 & 1 & ... & 0 & 1 \\ 1 & 1 & 1 & ... & 1 & 0 \end{pmatrix}.$$

and

$$D(K_n) = \begin{pmatrix} n-1 & 0 & 0 & ... & 0 & 0 \\ 0 & n-1 & 0 & ... & 0 & 0 \\ 0 & 0 & n-1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & n-1 & 0 \\ 0 & 0 & 0 & ... & 0 & n-1 \end{pmatrix}.$$

$$LE_{ED}(G) = D(K_n) - A_{ED}(K_n) = \begin{pmatrix} n-2 & -1 & -1 & ... & -1 & -1 \\ -1 & n-1 & -1 & ... & -1 & -1 \\ -1 & -1 & n-1 & ... & -1 & -1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ -1 & -1 & -1 & ... & n-1 & -1 \\ -1 & -1 & -1 & ... & -1 & n-1 \end{pmatrix}.$$

The characteristic polynomial of $LE_{ED}(K_n)$ is given by,
$(\lambda - n)^2(\lambda^2 - (n-1)\lambda - 1) = 0$.
The Laplacian minimum efficient dominating eigen values are:

$$Spec(K_n) = \begin{pmatrix} n & \frac{(n-1)+\sqrt{n^2-2n+5}}{2} & \frac{(n-1)-\sqrt{n^2-2n+5}}{2} \\ 2 & 1 & 1 \end{pmatrix}.$$

Average degree of $K_n = \frac{2m}{n} = 2\frac{\frac{n(n-1)}{2}}{n} = n-1$.
Hence, the Laplacian minimum efficient dominating energy of K_n is
$LE_{ED}(K_n) = |n-(n-1)|2 + \left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2} - (n-1)\right| + \left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2} - (n-1)\right|$
Therefore, $LE_{ED}(K_n) = 2 + \sqrt{n^2 - 2n + 5}$. □

5. Properties of Laplacian Minimum Efficient Dominating Energy of Graphs

Theorem 5.1 *If D is a minimum efficient dominating set of a graph G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the Laplacian minimum efficient dominating eigen values of $A_{ED}(G)$ then*

(i) $\sum_{i=1}^{n} \lambda_i = 2 \mid E \mid - \mid D \mid.$

(ii) $\sum_{i=1}^{n} \lambda_i^2 = 2 \mid E \mid + \sum_{i=1}^{n} (d_i - t_i)^2$ *where* $t_i = \begin{cases} 1, & \text{if } v_i \in D, \\ 0, & \text{if } v_i \notin D. \end{cases}$

Proof: **(i)** By definition, the sum of the principal diagonal elements of $LE_{ED}(G)$ is equal to
$\sum_{i=1}^{n} \lambda_i = 2 \mid E \mid - \mid D \mid.$
Also the sum of eigen values of $LE_{ED}(G)$ is trace of $LE_{ED}(G)$.
(ii) The sum of squares of eigen values of $LE_{ED}(G)$ is the trace of $LE_{ED}(G)^2$

Therefore $\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij} l_{ji} = \sum_{i=1}^{n} (l_{ij})^2 + \sum_{j=1}^{n} (l_{ji})^2$

$$= 2\sum_{i<j} (l_{ij})^2 + \sum_{i=1}^{n} (l_{ji})^2$$

$$= 2 \mid E \mid + \sum_{i=1}^{n} (d_i - t_i)^2 \text{ where } t_i = \begin{cases} 1, & \text{if } v_i \in D \\ 0, & \text{if } v_i \notin D \end{cases}$$

$$= 2M, \text{ where } M = \mid E \mid + \frac{1}{2}\left(\sum_{i=1}^{n} (d_i - t_i)^2\right)$$

□

6. Bounds on Laplacian Minimum Efficient Dominating Energy of Graphs

Theorem 6.1 *If G be a graph with n vertices, m edges and D is a minimum efficient dominating set of a graph G. Then $LE_{ED}(G) \leq \sqrt{2Mn} + 2m$.*

Proof: Let G be a graph with n vertices and m edges and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of G. By using Cauchy's - Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \tag{6.1}$$

Put $a_i = 1$, $b_i = \lambda_i$ in equation 6.1 then,

$$\left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \mid \lambda_i \mid^2\right)$$

$$\left(\sum_{i=1}^{n} \mid \lambda_i \mid\right)^2 \leq n\ 2M$$

$$\therefore \left(\sum_{i=1}^{n} \mid \lambda_i \mid\right) \leq \sqrt{2Mn}.$$

By Triangle inequality $\left|\lambda_i - \frac{2m}{n}\right| \leq \mid \lambda_i \mid + \left|\frac{2m}{n}\right| \ \ \forall i = 1, 2, \ldots, n$

$$i.e., \ \left|\lambda_i - \frac{2m}{n}\right| \leq \mid \lambda_i \mid + \frac{2m}{n} \ \ \forall\, i$$

$$\begin{aligned}\Big(\sum_{i=1}^{n}\left|\lambda_i - \frac{2m}{n}\right|\Big) &\leq \Big(\sum_{i=1}^{n}\lambda_i\Big) + \Big(\sum_{i=1}^{n}\frac{2m}{n}\Big)\\ &\leq \sqrt{2Mn} + 2m\\ \therefore LE_{ED}(G) &\leq \sqrt{2Mn} + 2m\end{aligned}$$

□

Theorem 6.2 *Let G be a graph with n vertices and m edges and D be a minimum efficient dominating set of G. Then $LE_{ED}(G) \leq \sqrt{2Mn + 4m(\mid D \mid - m)}$*

Proof: By using Cauchy's - Schwarz inequality

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) \tag{6.2}$$

Put $a_i = 1$, $b_i = \left|\lambda_i - \frac{2m}{n}\right|$ in equation (6.2) then,

$$\Big(\sum_{i=1}^{n}\left|\lambda_i - \frac{2m}{n}\right|\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n}\left|\lambda_i - \frac{2m}{n}\right|^2\Big).$$

$$\begin{aligned} i.e., [LE_{ED}(G)]^2 &= n\Big[\sum_{i=1}^{n}\lambda_i^2 + \sum_{i=1}^{n}\frac{4m^2}{n^2} - \frac{4m}{n}\sum_{i=1}^{n}\lambda_i\Big]\\ &= n\Big[2M + \frac{4m^2}{n^2}.n - \frac{4m}{n}(2m - \mid D \mid)\Big]\\ &= n\Big[2M + \frac{4m^2}{n} - \frac{8m^2}{n} + \frac{4m \mid D \mid}{n}\Big]\\ &= 2Mn + 4m(\mid D \mid - m)\\ \therefore \ LE_{ED}(G) &\leq \sqrt{2Mn + 4m(\mid D \mid - m)}\end{aligned}$$

□

Theorem 6.3 *Let G be a graph with n vertices and m edges and D is a minimum efficient dominating set of G. If $D = \mid detLE_{ED}(G) \mid$ then*

$$LE_{ED}(G) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m.$$

Proof: Consider

$$\begin{aligned}\Big[\sum_{i=1}^{n} \mid \lambda_i \mid\Big]^2 &= \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big).\Big(\sum_{j=1}^{n} \mid \lambda_j \mid\Big)\\ &= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + \sum_{i \neq j} \mid \lambda_i \mid\mid \lambda_j \mid\end{aligned}$$

$$\therefore \sum_{i\neq j} |\lambda_i||\lambda_j| = \Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 - \sum_{i=1}^{n} |\lambda_i|^2 \tag{6.3}$$

Applying Arithmtic and Geometric means for $n(n-1)$ terms, we have

$$\frac{\sum_{i\neq j} |\lambda_i||\lambda_j|}{n(n-1)} \geq \Big[\prod_{i\neq j} |\lambda_i||\lambda_j|\Big]^{\frac{1}{n(n-1)}}$$

$$i.e., \sum_{i\neq j} |\lambda_i||\lambda_j| \geq n(n-1)\Big[\prod_{i\neq j} |\lambda_i||\lambda_j|\Big]^{\frac{1}{n(n-1)}}$$

Using (6.3) we get,

$$\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 - \sum_{i=1}^{n} |\lambda_i|^2 \geq n(n-1)\Big[\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\Big]^{\frac{1}{n(n-1)}}$$

$$\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 - 2M \geq n(n-1)\Big[\prod_{i=1}^{n} |\lambda_i|\Big]^{\frac{2}{n}}$$

$$\Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 \geq 2M + n(n-1)\Big[\prod_{i=1}^{n} |\lambda_i|\Big]^{\frac{2}{n}}$$

$$\therefore \sum_{i=1}^{n} |\lambda_i| \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} \tag{6.4}$$

We know that

$$|\lambda_i| - \Big|\frac{2m}{n}\Big| \leq \Big|\lambda_i - \frac{2m}{n}\Big| \ \forall i$$

$$\sum_{i=1}^{n} |\lambda_i| - \sum_{i=1}^{n} \frac{2m}{n} \leq \sum_{i=1}^{n} \Big|\lambda_i - \frac{2m}{n}\Big|$$

$$i.e., \sum_{i=1}^{n} \Big|\lambda_i\Big| - 2m \leq LE_{ED}(G)$$

$$i.e., LE_{ED}(G) \geq \sum_{i=1}^{n} \Big|\lambda_i\Big| - 2m$$

$$\geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m \quad (from\ 6.4)$$

$$\therefore LE_{ED}(G) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m$$

□

Theorem 6.4 *Let G be a graph with n vertices, m edges and D is a minimum efficient dominating set of a graph G. Then*

$$LE_{ED}(G) \leq \sqrt{n\Big(2m + \sum_{i=1}^{n}(d_i - t_i)^2\Big)}. \tag{6.5}$$

Proof: Let G be a graph with n vertices and m edges and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the Laplacian minimum efficient dominating eigen values of G. By using Cauchy's - Schwarz inequality

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) \tag{6.6}$$

Put $a_i = 1$, $b_i =\mid \lambda_i \mid$ in equation (6.6) then,

$$\big(LE_{ED}(G)\big)^2 = \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n} \mid \lambda_i \mid^2\Big)$$

$$= n\Big(\sum_{i=1}^{n} \mid \lambda_i \mid^2\Big)$$

$$= n\Big(2m + \sum_{i=1}^{n} (d_i - t_i)^2\Big)$$

$$LE_{ED}(G) \leq \sqrt{n\Big(2m + \sum_{i=1}^{n} (d_i - t_i)^2\Big)}.$$

□

This leads to inequality (6.5)

7. Conclusion

In this paper we introduced the notion of Laplacian minimum efficient dominating energy ($LE_{ED}(G)$), extending the interplay between efficient domination and spectral graph theory. By examining graphs that admit minimum efficient dominating sets, we established general bounds for ($LE_{ED}(G)$) and computed exact values for several well-known families of graphs. These results demonstrate how structural properties of a graph influence its Laplacian energy under efficient domination constraints. The bounds developed here provide a foundation for further spectral investigations, and the explicit computations offer a basis for identifying broader classes of graphs with predictable ($LE_{ED}(G)$). Future work may explore algorithmic aspects, characterize extremal graphs with respect to ($LE_{ED}(G)$), and extend the concept to weighted or directed graph settings.

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