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Minimising Delays in M/M/1/K/N Orbiting Queue through Server Vacations and Adjustable Arrival Rates

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ABSTRACT: This paper examines a finite capacity single server orbiting queueing model in which the server takes vacations during inactive periods. The model integrates adjustable arrival rates, developing a realistic representation of real-world systems and considering the mutual dependence of service and arrival processes. The steady-state probability equations are solved recursively and explicit iterative formulas are obtained. Customer arrive according to Poisson process with tunable arrival rates, classified as faster or slower, while blocked customers enter an orbit and attempt service again. The effectiveness of the measures for the systems and specific instances of the model are discussed. Numerical findings have been produced and presented in the form of tables. The results demonstrate that vacations and adjustable arrivals may be proactively managed to reduce delays and improve cost-effective service for both customers and servers.

Keywords: Single server, retrial queue, finite capacity, vacation, interdependent arrival, service rates.

Contents

1. Introduction

Queueing models are essential for the study of service systems, from customer service centers and computer processors to manufacturing processes and communication networks. These models offer an efficient framework for comprehending the formation of waiting lines, the causes of congestion and the most effective ways to use resources. Retrial queueing systems are one of the numerous interpretations that have drawn a lot of interest since they accurately depict scenarios in which blocked consumers arrive after a period of time. In wireless communication systems or telephone networks, for example, calls or packets that cannot be delivered quickly do not permanently exit the system however, they try to be served again after a certain amount of time. The modelling process becomes more realistic with the use of such retry methods. The server's behaviour is another crucial factor to take into account in contemporary service systems. In real-world settings, servers might not always be active all the time. During idle times, they might switch to energy-saving modes, take pauses, or undertake repairs. Server vacation policies are usually used to emulate these behaviours. Server vacations are a useful addition to conventional queueing systems since they have been demonstrated to increase affordability and utilisation of resources while preserving satisfactory service levels.

Arrival rates in real-world systems frequently fluctuate based on system design, customer behaviour, and external circumstances. Managers in call centers, may actively control the volume of calls, traffic management methods in computer systems may modify the arrival of packets. Because arrivals can be

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categorised as quicker or slower based on operational factors, adding configurable arrival rates to queueing models improves their sense of reality.

2. Literature Review

In recent decades, retrial queueing systems have garnered a lot of attention because of their use in computer systems, communication networks, and service operations where blocked consumers retry after arbitrary intervals. Cohen's [7] initial analysis of the classical M/M/1 retrial queue served as the basis for the research of systems with repeated trials. Reynolds [18] discussed the stationary solution of a multiple server model with discouragement. Falin *et* al. [9] subsequently created a detailed structure for retrial queues, emphasising its steady-state analysis and stochastic characteristics. Gross [10] gave queueing theory concepts in Fundamentals of Queueing Theory.

The basic model was expanded in later studies to reflect more realistic service environments by adding interdependent service mechanisms, server vacations, and state-dependent arrivals. In his comprehensive review of retrial queueing systems, Artalejo [2,3,4] covered analytical techniques such embedded Markov chains and more variables. Tadj *et* al. [19] further investigated the combination of retrials with server vacations by analysing systems with different vacation disciplines and obtaining performance metrics like average orbit size and server usage.Doshi [8] created vacation policies, which improve resource usage and energy efficiency by modelling times when the server is unavailable. Jain [13,14,15] studied invert reneging and invert balking have been developed and integrated into the single server queuing system.Zhang *et* al. [21] added improvements to different staffing rules and server asymptotics for a Markov process queueing model with abandoned clients. Kapodistria [16] have examined the single queue for coordinated abandonment.

Recently, the idea of controlled or interdependent arrival rates has been a major area of study. Because the system state determines the effective arrival rate in these systems, congestion and quality of service can be dynamically controlled. Queuing systems with state-dependent input, in which the arrival process varies based on the number of consumers in the system or the server condition, were studied by Yechiali [20] and Krishna Kumar *et* al. [17]. Haight [12]obtained the stable state that the service process is not overburdened by service requirements. Gupta [11] studied interrelationship between queueing models with balking and reneging and machine repair problem with warm spares. Begum *et* al. [5] analysed M/M/c interdependent queueing model with controllable arrival rates. This controllability represents actual situations in cloud or telecommunication systems, where incoming traffic can be prioritised or controlled.

The combined dynamics of user impatience, service unavailability, and demand regulation are captured by models that combine retrials, server vacations, and controllable arrivals. Similar interdependent retrial queues were examined in recent publications by Chakravarthy [6] utilising matrix-analytic and generating function techniques. A queueing model with a control strategy, reasonable rates of arrival and balking in reverse for a single server with limited storage and computing capability were discussed by Antline Nisha *et* al. [1].

3. Model Description

The primary arrival process $[Y_1(t)]$ and the service process $[Y_2(t)]$ of the systems are correlated and follow a bivariate poisson process given by

$$P(Y_1 = y_1, Y_2 = y_2, t) = e^{-(\lambda_w+\mu-\epsilon)t} \sum_{j=0}^{\min(y1,y2)} \frac{(\epsilon t)^j[(\lambda_w + -\epsilon)t]^{y1-j}[(\mu-\epsilon)t]^{y2-j}}{j!(y1-j)!(y2-j)!}$$

where y_1, y_2= 0,1,2........
$0 < \lambda_w, \mu$;
$0 < \epsilon <\min(\lambda_w, \mu), w = a^+, a^-$
with parameters $\lambda_a^+, \lambda_a^-, \mu$ and ϵ as mean faster rate of primary arrivals, mean slower rate of primary arrivals, mean service rate and mean dependence rate (covariance between the primary arrival and service processes)respectively.

Consider a finite source single server finite capacity orbiting queueing system with server vacation in which primary customers arrive according to the poisson flow of rate λ_a^+ and λ_a^-,vacation rate γ, service times are exponentially distributed with rate μ. If a primary customer finds some server free, he instantly occupies it and leaves the system after service. Otherwise, if the server is busy, the arriving customer enters an orbit and repeats his demand after an exponential time with rate θ. Thus the poisson flow of repeated attempts follow the retrial policy where the repetition times of each customer is assumed to be independent and exponentially distributed with intensity θ. N denotes finite potential customers.K be the maximum number of customers allowed in the system.

Let $B(t)$ be the number of busy servers and $N(t)$ be the number of sources of repeated calls. The system state at time t can be described by means of a bivariate process $B(t), N(t), t \geq 0$, where $B(t) = 0, 1, 2$ according as the server is on vacation, idle and busy. The process is called BN process. If the service time is exponential, then $\{B(t), N(t)\}$ is Markovian. Let B and N be the numbers of customers in the service facility and in the orbit respectively in steady state.

The process $\{N(t), B(t)\}\,; t \geq 0$ is a continuous time markov chain defined on the state space
$(n, b)|n = \{0, 1, 2, ...q-1, q, q+1, ...K\}\,, b = \{0, 1, 2\}$.
The figure illustrates the transition states and the state probabilities at time t are defined as follows

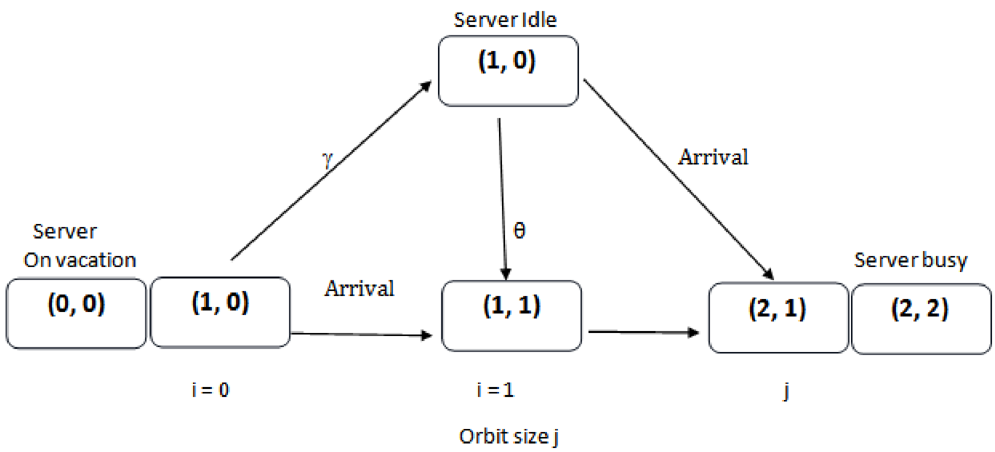

Figure 1: State transition

From state$(0, n)$ transitions when the server on vacation
State $(1, n)$ with the probability that the server completes its vacation during the interval $(t, t + dt)$ of time t and becomes idle when the system is either faster or slower rate of primary arrivals is $\theta dt + o(dt)$.

State $(0, n + 1)$ with the probability that a primary arrival occurs during the interval $(t, t + dt)$ when the system is in faster rate of primary arrivals is $(\lambda_a^+ - \epsilon)dt + o(dt)$ and the system is in slower rate of primary arrivals is $(\lambda_a^- - \epsilon)dt + o(dt)$, arriving customer joins the orbit since the server is on vacation.

No transition occurs and the system remains in the same state $(0, n)$ with the probability that neither vacation completion nor new arrival occurs in $(t, t + dt)$ is $1 - [(\lambda_a^+ - \epsilon)dt] + o(dt)$.

From state $(1, n)$ transitions only to the following states are possible.

State $(2, n)$ with the probability that the new arrival of a primary call finds the server free and starts service suddenly during the interval $(t, t + dt)$ of time t, when the system is in faster rate of primary arrivals is $(\lambda_a^+ - \epsilon)dt + o(dt)$ and the system is in slower rate of primary arrivals is $(\lambda_a^- - \epsilon)dt + o(dt)$.

State $(2, n - 1)$ with the probability that one of the n orbiting customers retries for service and get it during the interval $(t, t + dt)$ of time t, when the system is either in faster or slower rate of primary arrivals is $n\theta dt + o(dt)$.

Again state $(0, n)$ can be reached with transitions only from the following states.

State $(1, n)$ with the probability that a vacation starts during the interval $(t, t + dt)$ is $\gamma dt + o(dt)$.

No transition occurs and the system remains in the same state $(0, n)$ with the probability that neither vacation completion nor new arrival occurs in $(t, t + dt)$ is $1 - [((\lambda_a^- - \epsilon) + n\theta + \gamma)dt] + o(dt)$.

From state $(2, n)$ transitions only to the following states are possible.

State $(2, n)$ with the probability that the service completion of the call in service during the interval $(t, t+dt)$ of time t, when the system is either in faster or slower rate of primary arrivals is $(\mu-\epsilon)dt+o(dt)$.

State $(2, n + 1)$ with the probability that the new arrival of a primary call occurs during the interval $(t, t + dt)$ of time t, when the system is in faster rate of primary arrivals is $(\lambda_a^+ - \epsilon)dt + o(dt)$ and the system is in slower rate of primary arrivals is $(\lambda_a^- - \epsilon)dt + o(dt)$. The arriving customer cannot enter service and therefore enters the orbit, since the server is busy.

State $(0, n)$ with the probability that a server begins vacation after service during the interval $(t, t+dt)$ is $\gamma dt + o(dt)$.

The system remains in the same state and no transition occurs $(0, n)$ with the probability that neither vacation completion nor new arrival occurs in $(t, t + dt)$ is $1 - [((\lambda_a^+ - \epsilon) + n\theta + \gamma)dt] + o(dt)$.

The probability that there is one primary arrival and one service completion during the interval $(t, t + dt)$ of time t, when the system is either in faster or slower rate of primary arrivals is $\epsilon dt + o(dt)$.

4. Steady State Equation

Let $P_{0,n,j}(0)$ and$P_{0,n,j}(1)$ denote the steady state probability that there are n customers in the queue when the system is in the faster rate and slower rate of primary arrivals,j in orbit and the server is in vacation.

Let $P_{1,n,j}(0)$ and $P_{1,n,j}(1)$ denote the steady state probability that there are n customers in the queue when the system is in the faster rate and slower rate of primary arrivals,j in orbit and the server is idle.

Let $P_{2,n,j}(0)$ and $P_{2,n,j}(1)$ denote the steady state probability that there are n customers in the queue when the system is in the faster rate and slower rate of primary arrivals,j in orbit and the server is busy.

We observe $P_{0,n,j}(0), P_{1,n,j}(0)$ and $P_{2,n,j}(0)$ exists when $n = 0, 1, 2, \ldots, q - 1, q; P_{0,n,j}(0), P_{1,n,j}(0), P_{2,n,j}(0), P_{0,n,j}(1), P_{1,n,j}(1), P_{2,n,j}(1)$ exists when $n = q + 1, q + 2, \ldots Q - 1$ and $P_{0,n,j}(1), P_{1,n,j}(1)$ and $P_{2,n,j}(1)$ exists only when $n = Q, Q+1, \ldots$.. Here $i = 0, 1, 2$, $n = 1, 2, 3..q-1, q, q+1, ...K$,$j = 0, 1, 2....N$
The steady state equations are

$$-\big((\lambda_a^+ - \epsilon) + \gamma + j\theta\big)P_{0,0,j}(0) + \gamma P_{1,0,j}(0) + \gamma P_{2,0,j}(0) = 0 \qquad (i = 0, n = 0, j) \tag{4.1}$$

$$-\big((\lambda_a^+ - \epsilon) + j\theta + \gamma\big)P_{1,0,j}(0) + \gamma P_{0,0,j}(0) + (\mu - \epsilon)P_{2,1,j}(0) + (j + 1)\theta P_{1,0,j+1}(0) = 0 \qquad (i = 1, n = 0, j) \tag{4.2}$$

$$-\big((\mu + \lambda_a^+ - 2\epsilon) + j\theta\big)P_{2,0,j}(0) + \big((\lambda_a^+ - \epsilon) + j\theta\big)P_{1,0,j}(0) = 0 \qquad (i = 2, n = 0, j) \tag{4.3}$$

$$\begin{aligned}&-\big((\lambda_a^+ - \epsilon) + \gamma + j\theta\big)P_{0,n,j}(0) + (\lambda_a^+ - \epsilon)P_{0,n-1,j}(0) + (j + 1)\theta P_{0,n-1,j+1}(0)\\&\quad+\gamma P_{1,n,j}(0) + \gamma P_{2,n,j}(0) = 0 \qquad (i = 0, 0 < n < q, j)\end{aligned} \tag{4.4}$$

$$\begin{aligned}&-\big(\lambda_a^+ - \epsilon) + j\theta + \gamma\big)P_{1,n,j}(0) + (\mu - \epsilon)P_{2,n+1,j}(0) + j\theta P_{1,n-1,j}(0)\\&\quad+\gamma P_{0,n,j(0)} + (\lambda_a^+ - \epsilon)P_{1,n-1,j}(0) = 0 \qquad (i = 1, 0 < n < q, j)\end{aligned} \tag{4.5}$$

$$\begin{aligned}&-\big((\mu + \lambda_a^+ - 2\epsilon) + j\theta\big)P_{2,n,j}(0) + (\lambda_a^+ - \epsilon)P_{2,n-1,j}(0) + (j + 1)\theta P_{2,n,j+1}(0)\\&\quad+\big((\lambda_a^+ - \epsilon) + j\theta\big)P_{1,n,j}(0) = 0 \qquad (i = 2, 0 < n < q, j)\end{aligned} \tag{4.6}$$

$$\begin{aligned}&-\big(\gamma + (\lambda_a^+ - \epsilon) + j\theta\big)P_{0,q,j}(0) + (\lambda_a^+ - \epsilon)P_{0,q-1,j}(0) + (j + 1)\theta P_{0,q-1,j+1}(0)\\&\quad+(j + 1)\theta P_{0,q-1,j+1}(1) + \gamma P_{1,q,j}(0) + \gamma P_{2,q,j}(0) = 0 \qquad (i = 0, n = q, j)\end{aligned} \tag{4.7}$$

$$-\big(\lambda_a^+ - \epsilon) + j\theta + \gamma\big)P_{1,q,j}(0) + (\mu - \epsilon)P_{2,q+1,j}(0) + j\theta P_{1,q-1,j}(0) + j\theta P_{1,q-1,j}(1) + \gamma P_{0,q,j}(0) + (\lambda_a^+ - \epsilon)P_{1,q-1,j}(0) = 0 \qquad (i = 1, n = q, j) \tag{4.8}$$

$$-\big(\mu + \lambda_a^+ - 2\epsilon) + j\theta\big)P_{2,q,j}(0) + (\lambda_a^+ - \epsilon)P_{2,q-1,j}(0) + (j+1)\theta P_{2,q-1,j+1}(0) + (j+1)\theta P_{2,q-1,j+1}(1) + \big((\lambda_a^+ - \epsilon) + j\theta\big)P_{1,q,j}(0) = 0 \qquad (i = 2, n = q, j) \tag{4.9}$$

$$-\big((\lambda_a^+ - \epsilon) + \gamma + j\theta\big)P_{0,n,j}(0) + (\lambda_a^+ - \epsilon)P_{0,n-1,j}(0) + (j+1)\theta P_{0,n-1,j+1}(0) + \gamma P_{1,n,j}(0) + \gamma P_{2,n,j}(0) = 0 \qquad (i = 0, q < n < Q-1, j) \tag{4.10}$$

$$-\big(\lambda_a^+ - \epsilon) + j\theta + \gamma\big)P_{1,n,j}(0) + (\mu - \epsilon)P_{2,n+1,j}(0) + j\theta P_{1,n-1,j}(0) + \gamma P_{0,n,j}(0) + (\lambda_a^+ - \epsilon)P_{1,n-1,j}(0) = 0 \qquad (i = 1, q < n < Q-1, j) \tag{4.11}$$

$$-\big((\mu + \lambda_a^+ - 2\epsilon) + j\theta\big)P_{2,n,j}(0) + (\lambda_a^+ - \epsilon)P_{2,n-1,j}(0) + (j+1)\theta P_{2,n,j+1}(0) + \big((\lambda_a^+ - \epsilon) + j\theta\big)P_{1,n,j}(0) = 0 \qquad (i = 2, q < n < Q-1, j) \tag{4.12}$$

$$-\big((\lambda_a^+ - \epsilon) + \gamma + j\theta\big)P_{0,Q-1,j}(0) + (\lambda_a^+ - \epsilon)P_{0,Q-2,j}(0) + (j+1)\theta P_{0,Q-2,j+1}(0) + \gamma P_{1,Q-1,j}(0) + \gamma P_{2,Q-1,j}(0) = 0 \qquad (i = 0, n = Q-1, j) \tag{4.13}$$

$$-\big(\lambda_a^+ - \epsilon) + j\theta + \gamma\big)P_{1,Q-1,j}(0) + j\theta P_{1,Q-2,j}(0) + \gamma P_{0,Q-1,j}(0) + (\lambda_a^+ - \epsilon)P_{1,Q-2,j}(0) = 0 \qquad (i = 1, n = Q-1, j) \tag{4.14}$$

$$-\big((\mu + \lambda_a^+ - 2\epsilon) + j\theta\big)P_{2,Q-1,j}(0) + (\lambda_a^+ - \epsilon)P_{2,Q-2,j}(0) + (j+1)\theta P_{2,Q-1,j+1}(0) + \big((\lambda_a^+ - \epsilon) + j\theta\big)P_{1,Q-1,j}(0) = 0 \qquad (i = 2, n = Q-1, j) \tag{4.15}$$

$$-\big((\lambda_a^- - \epsilon) + \gamma + j\theta\big)P_{0,q+1,j}(1) + (j+1)\theta P_{0,q,j+1}(1) + \gamma P_{1,q+1,j}(1) + \gamma P_{2,q+1,j}(1) = 0 \qquad (i = 0, n = q+1, j) \tag{4.16}$$

$$-\big(\lambda_a^- - \epsilon) + j\theta + \gamma\big)P_{1,q+1,j}(1) + (\mu - \epsilon)P_{2,q+2,j}(1) + j\theta P_{1,q,j}(1) + \gamma P_{0,q+1,j}(1) = 0 \qquad (i = 1, n = q+1, j) \tag{4.17}$$

$$-\big((\mu + \lambda_a^- - 2\epsilon) + j\theta\big)P_{2,q+1,j}(1) + (j+1)\theta P_{2,q+1,j+1}(1) + \big((\lambda_a^- - \epsilon) + j\theta\big)P_{1,q+1,j}(1) = 0 \qquad (i = 2, n = q+1, j) \tag{4.18}$$

$$-\big((\lambda_a^- - \epsilon) + \gamma + j\theta\big)P_{0,n,j}(1) + (\lambda_a^- - \epsilon)P_{0,n-1,j}(1) + (j+1)\theta P_{0,n-1,j+1}(1) + \gamma P_{1,n,j}(1) + \gamma P_{2,n,j}(1) = 0 \qquad (i = 0, q+1 < n < Q, j) \tag{4.19}$$

$$-\big(\lambda_a^- - \epsilon) + j\theta + \gamma\big)P_{1,n,j}(1) + (\mu - \epsilon)P_{2,n+1,j}(1) + j\theta P_{1,n-1,j}(1) + \gamma P_{0,n,j(1)} + (\lambda_a^- - \epsilon)P_{1,n-1,j}(1) = 0 \qquad (i = 1, q+1 < n < Q, j) \tag{4.20}$$

$$-\big((\mu + \lambda_a^- - 2\epsilon) + j\theta\big)P_{2,n,j}(1) + (\lambda_a^- - \epsilon)P_{2,n-1,j}(1) + (j+1)\theta P_{2,n,j+1}(1) + \big((\lambda_a^- - \epsilon) + j\theta\big)P_{1,n,j}(1) = 0 \qquad (i = 2, q+1 < n < Q, j) \tag{4.21}$$

$$-\big((\lambda_a^- - \epsilon) + \gamma + j\theta\big)P_{0,Q,j}(1) + (\lambda_a^- - \epsilon)P_{0,Q-1,j}(1) + (\lambda_a^- - \epsilon)P_{0,Q-1,j}(0)$$
$$+(j+1)\theta P_{0,Q-1,j+1}(1) + \gamma P_{1,Q,j}(1) + \gamma P_{2,Q,j}(1) = 0 \qquad (i=0, n=Q, j) \quad (4.22)$$

$$-\big(\lambda_a^- - \epsilon) + j\theta + \gamma\big)P_{1,Q,j}(1) + (\mu - \epsilon)P_{2,Q+1,j}(1) + j\theta P_{1,Q-1,j}(1) + \gamma P_{0,Q,j(1)}$$
$$+(\lambda_a^- - \epsilon)P_{1,Q-1,j}(1) + (\lambda_a^+ - \epsilon)P_{1,Q-1,j}(0) = 0 \qquad (i=1, n=Q, j) \quad (4.23)$$

$$-\big((\mu + \lambda_a^+ - 2\epsilon) + j\theta\big)P_{2,Q,j}(1) + (\lambda_a^- - \epsilon)P_{2,Q-1,j}(1) + (\lambda_a^+ - \epsilon)P_{2,Q-1,j}(0)$$
$$+(j+1)\theta P_{2,Q,j+1}(1) + \big((\lambda_a^+ - \epsilon) + j\theta\big)P_{1,Q,j}(1) = 0 \qquad (i=2, n=Q, j) \quad (4.24)$$

$$-\big((\lambda_a^- - \epsilon) + \gamma + j\theta\big)P_{0,n,j}(1) + (\lambda_a^- - \epsilon)P_{0,n-1,j}(1) + (j+1)\theta P_{0,n-1,j+1}(1)$$
$$+\gamma P_{1,n,j}(1) + \gamma P_{2,n,j}(1) = 0 \qquad (i=0, n=Q+1...K, j) \quad (4.25)$$

$$-\big(\lambda_a^- - \epsilon) + j\theta + \gamma\big)P_{1,n,j}(1) + (\mu - \epsilon)P_{2,n+1,j}(1) + j\theta P_{1,n-1,j}(1)$$
$$+\gamma P_{0,n,j(1)} + (\lambda_a^- - \epsilon)P_{1,n-1,j}(1) = 0 \qquad (i=1, n=Q+1...K, j) \quad (4.26)$$

$$-\big((\mu + \lambda_a^- - 2\epsilon) + j\theta\big)P_{2,n,j}(1) + (\lambda_a^- - \epsilon)P_{2,n-1,j}(1) + (j+1)\theta P_{2,n,j+1}(1)$$
$$+\big((\lambda_a^- - \epsilon) + j\theta\big)P_{1,n,j}(1) = 0 \qquad (i=2, n=Q+1...K, j) \quad (4.27)$$

From the above equations, we get

$$P_{2,0,j}(0) = \frac{(\lambda_a^+ - \varepsilon + \gamma + j\theta)(\lambda_a^+ - \varepsilon + j\theta)}{\gamma\big(\mu + 2\lambda_a^+ - 3\varepsilon + 2j\theta\big)}P_{0,0,j}(0) \quad (4.28)$$

$$P_{1,0,j}(0) = \frac{\big(\mu + \lambda_a^+ - 2\varepsilon + j\theta\big)\big(\lambda_a^+ - \varepsilon + \gamma + j\theta\big)}{\gamma\big(\mu + 2\lambda_a^+ - 3\varepsilon + 2j\theta\big)}P_{0,0,j}(0) \quad (4.29)$$

$$P_{0,n,j+1}(0) = \frac{(\lambda_a^+ - \epsilon)P_{0,n-1,j+1}(0) + \dfrac{\gamma}{C}\Big((j+1)\theta P_{1,n,j}(0) + (\mu - \epsilon)P_{2,n+1,j+1}(0)\Big) + \gamma P_{2,n,j+1}(0)}{(\lambda_a^+ - \epsilon) + (j+1)\theta + \gamma - \dfrac{\gamma^2}{(\lambda_a^+ - \epsilon) + (j+1)\theta + \gamma}} \quad (4.30)$$

where

$$C = (\lambda_a^+ - \epsilon) + (j+1)\theta + \gamma \quad (4.31)$$

$$P_{1,n,j+1}(0) = \frac{(j+1)\theta P_{1,n,j} + \gamma P_{0,n,j+1}(0) + (\mu - \epsilon)P_{2,n+1,j+1}(0)}{(\lambda_a^+ - \epsilon) + (j+1)\theta + \gamma} \quad (4.32)$$

$$P_{2,n,j+1}(0) = \frac{\big((\mu + \lambda_a^+ - 2\epsilon) + j\theta\big)P_{2,n,j}(0) - (\lambda_a^+ - \epsilon)P_{2,n-1,j}(0) - \big((\lambda_a^+ - \epsilon) + j\theta\big)P_{1,n,j}(0)}{(j+1)\theta} \quad (4.33)$$

$$P_{0,n,j+1}(1) = \frac{(\lambda_a^+ - \epsilon)P_{0,n-1,j+1}(1) + \dfrac{\gamma}{D}\Big((j+1)\theta P_{1,n,j}(1) + (\mu - \epsilon)P_{2,n+1,j+1}(1)\Big) + \gamma P_{2,n,j+1}(1)}{(\lambda_a^- - \epsilon) + (j+1)\theta + \gamma - \dfrac{\gamma^2}{(\lambda_a^- - \epsilon) + (j+1)\theta + \gamma}} \quad (4.34)$$

where

$$D = (\lambda_a^- - \epsilon) + (j+1)\theta + \gamma \tag{4.35}$$

$$P_{1,n,j+1}(1) = \frac{(j+1)\theta P_{1,n,j} + \gamma P_{0,n,j+1}(1) + (\mu - \epsilon)P_{2,n+1,j+1}(1)}{(\lambda_a^- - \epsilon) + (j+1)\theta + \gamma} \tag{4.36}$$

$$P_{2,n,j+1}(1) = \frac{\big((\mu + \lambda_a^- - 2\epsilon) + j\theta\big)P_{2,n,j}(1) - (\lambda_a^- - \epsilon)P_{2,n-1,j}(1) - \big((\lambda_a^- - \epsilon) + j\theta\big)P_{1,n,j}(1)}{(j+1)\theta} \tag{4.37}$$

A level-wise recursion spanning orbit-size j determines the steady-state solution. Let K be the maximum in system count, so there are K+1 values of n per level and let j be the maximum orbit level determined. The unknowns for each level j are the probabilities $P_{0,n,j+1}, P_{1,n,j+1}, P_{2,n,j+1}$ for n=0.....K.

5. Characteristics of the Model

The following system characteristics are considered and their analytical results are derived in this system.

The probability $P(0)$ that the system is in faster rate of primary arrivals with the server vacation, idle and busy.

The probability $P(1)$ that the system is in slower rate of primary arrivals with the server vacation, idle and busy.

The probability $P_{0,0,0}$ that the system is empty.

The expected number of customers in the system Ls_0, When the system is in the faster rate of primary arrivals with the server vacation, idle and busy.

The expected number of customers in the system Ls_1, When the system is in the slower rate of primary arrivals with the server vacation, idle and busy.

The probability that the system is in faster rate of primary arrivals is

$$\begin{aligned}
P(0) &= \sum_{n=0}^{K} P_{0,n,j+1}(0) + \sum_{n=0}^{K} P_{1,n,j+1}(0) + \sum_{n=0}^{K} P_{2,n,j+1}(0) \\
&= \left[\sum_{n=0}^{q} P_{0,n,j+1}(0) + \sum_{n=r+1}^{Q-1} P_{0,n,j+1}(0) + \sum_{n=Q}^{K} P_{0,n,j+1}(0)\right] \\
&\quad + \left[\sum_{n=0}^{q} P_{1,n,j+1}(0) + \sum_{n=r+1}^{Q-1} P_{1,n,j+1}(0) + \sum_{n=Q}^{K} P_{1,n,j+1}(0)\right] \\
&\quad + \left[\sum_{n=0}^{q} P_{2,n,j+1}(0) + \sum_{n=r+1}^{Q-1} P_{2,n,j+1}(0) + \sum_{n=Q}^{K} P_{2,n,j+1}(0)\right]
\end{aligned}$$

Since $P_{0,n,0}$ and $P_{1,n,0}$ exist only when $n = 0, 1, 2, \ldots, q-1, q, q+1, q+2, \ldots, Q-2, Q-1$, we get

$$\begin{aligned}
P(0) &= \left[\sum_{n=0}^{q} P_{0,n,j+1}(0) + \sum_{n=q+1}^{Q-1} P_{0,n,j+1}(0)\right] + \left[\sum_{n=0}^{r} P_{1,n,j+1}(0) + \sum_{n=q+1}^{Q-1} P_{1,n,j+1}(0)\right] \\
&\quad + \left[\sum_{n=0}^{r} P_{2,n,j+1}(0) + \sum_{n=q+1}^{Q-1} P_{2,n,j+1}(0)\right]
\end{aligned} \tag{5.1}$$

The probability that the system is in slower rate of primary arrivals is

$$\begin{aligned} P(1) &= \sum_{n=0}^{K} P_{0,n,j+1}(1) + \sum_{n=0}^{K} P_{1,n,j+1}(1) + \sum_{n=0}^{K} P_{2,n,j+1}(1) \\ &= \left[\sum_{n=0}^{q} P_{0,n,j+1}(1) + \sum_{n=r+1}^{Q-1} P_{0,n,j+1}(1) + \sum_{n=Q}^{K} P_{0,n,j+1}(1) \right] \\ &\quad + \left[\sum_{n=0}^{q} P_{1,n,j+1}(1) + \sum_{n=r+1}^{Q-1} P_{1,n,j+1}(1) + \sum_{n=Q}^{K} P_{1,n,j+1}(1) \right] \\ &\quad + \left[\sum_{n=0}^{q} P_{2,n,j+1}(1) + \sum_{n=r+1}^{Q-1} P_{2,n,j+1}(1) + \sum_{n=Q}^{K} P_{2,n,j+1}(1) \right] \end{aligned}$$

Since $P_{0,n,j+1}(1)$, $P_{1,n,j+1}(1)$ and $P_{2,n,j+1}(1)$ exists only when $n = q+1, q+2, \ldots, Q-2, Q-1, \ldots, K$, we get

$$\begin{aligned} P(1) &= \left[\sum_{n=q+1}^{Q} P_{0,n,j+1}(1) + \sum_{n=Q+1}^{K} P_{0,n,j+1}(1) \right] + \left[\sum_{n=q+1}^{Q} P_{1,n,j+1}(1) + \sum_{n=Q+1}^{K} P_{1,n,j+1}(1) \right] \\ &\quad + \left[\sum_{n=q+1}^{Q} P_{2,n,j+1}(1) + \sum_{n=Q+1}^{K} P_{2,n,j+1}(1) \right] \end{aligned} \tag{5.2}$$

The probability $P_{0,0,0}$ that the system is empty can be calculated from the normalizing condition.

$$P(0) + P(1) = 1 \tag{5.3}$$

Now we calculate the expected number of customers in the system. Let Ls denote the average number of customers in the system, then we have

$$Ls = Ls_0 + Ls_1 \tag{5.4}$$

where

$$\begin{aligned} Ls_0 &= \left[\sum_{n=0}^{q} n\, P_{0,n,j+1}(0) + \sum_{n=q+1}^{Q-1} n\, P_{0,n,j+1}(0) \right] + \left[\sum_{n=0}^{q} n\, P_{1,n,j+1}(0) \right. \\ &\quad \left. + \sum_{n=q+1}^{Q-1} n\, P_{1,n,j+1}(0) \right] + \left[\sum_{n=0}^{q} n\, P_{2,n,j+1}(0) \quad + \sum_{n=q+1}^{Q-1} n\, P_{2,n,j+1}(0) \right] \end{aligned} \tag{5.5}$$

$$\begin{aligned} Ls_1 &= \left[\sum_{n=q+1}^{Q} n\, P_{0,n,j+1}(1) + \sum_{n=Q+1}^{K} n\, P_{0,n,j+1}(1) \right] + \left[\sum_{n=q+1}^{Q} (n)\, P_{1,n,j+1}(1) \right. \\ &\quad \left. + \sum_{n=Q+1}^{K} (n)\, P_{1,n,j+1}(1) \right] + \left[\sum_{n=0}^{q} n\, P_{2,n,j+1}(1) \quad + \sum_{n=q+1}^{Q-1} n\, P_{2,n,j+1}(1) \right] \end{aligned} \tag{5.6}$$

Using Little's formula, the expected waiting time of the customer in the system is calculated as

$$Ws = \frac{Ls}{\overline{\lambda}} \tag{5.7}$$

where $\overline{\lambda} = \lambda_a^+ P(0) + \lambda_a^- P(1)$

6. Numerical Illustrations

For various values of $\gamma, \lambda_0, \lambda_1, \mu, \varepsilon, \theta, r, R, K$ the values of $P_{0,0,0}, P(0), P(1)$, Ls and Ws are computed and tabulated in the tables from (6.1) to (6.3)

Table 6.1

γ	q	Q	K	λ_a^+	λ_a^-	μ	θ	ϵ	$P_{0,0,0}$	P(0)	P(1)
18	3	8	14	3	2	4	2	1	0.1472	0.884	0.106
18	3	8	14	3	2	4	2	0.5	0.0985	0.789	0.201
18	3	8	14	3	2	4	3	0.5	0.1424	0.786	0.204
18	3	8	14	3	2	4	2	0	0.0439	0.729	0.281
18	3	8	14	3	2	5	2	0.5	0.1816	0.963	0.038
18	3	8	14	3	2	6	2	0.5	0.327	0.964	0.035
18	3	8	14	3	3	4	2	0.5	0.0626	0.645	0.345
18	3	8	14	3	3	4	1	0	0.0265	0.282	0.708
18	3	8	14	2	1	4	2	0.5	0.612	0.979	0.011
18	3	8	14	5	4	4	3	0.5	0.00545	0.0657	0.9333
18	3	8	14	4	3	4	2	0.5	0.0134	0.975	0.015
18	3	8	14	5	4	5	3	0.5	0.007265	0.253	0.737
18	3	8	14	4	4	4	2	0	0.00115	0.057	0.942

Table 6.2

γ	q	Q	K	λ_a^+	λ_a^-	μ	θ	ϵ	L_S	W_S
18	3	8	14	3	2	4	2	1	3.4105	1.178
18	3	8	14	3	2	4	2	0.5	3.9339	1.4055
18	3	8	14	3	2	4	3	0.5	4.0151	1.4360
18	3	8	14	3	2	4	2	0	4.4222	1.6260
18	3	8	14	3	2	5	2	0.5	2.9998	1.0128
18	3	8	14	3	2	6	2	0.5	2.4755	0.8349
18	3	8	14	3	3	4	2	0.5	2.6882	0.8961
18	3	8	14	3	3	4	1	0	8.8864	2.9621
18	3	8	14	2	1	4	2	0.5	2.0619	1.0366
18	3	8	14	5	4	4	3	0.5	13.6470	3.3558
18	3	8	14	4	3	4	2	0.5	5.4224	1.361
18	3	8	14	5	4	5	3	0.5	7.6372	1.792
18	3	8	14	4	4	4	2	0	11.9154	2.979

Table 6.3

γ	q	Q	K	λ_a^+	λ_a^-	μ	θ	ϵ	L_S	W_S
18	3	8	10	3	2	4	2	0.5	3.9042	1.3935
18	3	8	12	3	2	4	2	0.5	3.9282	1.4034
18	3	8	15	3	2	4	2	0.5	3.9339	1.4055
18	3	8	20	3	2	4	2	0.5	3.9365	1.4064
18	3	8	22	3	2	4	2	0.5	3.9367	1.4066

7. Conclusion

From the tables (6.1),(6.2) it is observed that when the mean dependence rate increases and the other parameters are kept fixed, $P_{0,0,0}, P(0)$ increase and $P(1), Ls, Ws$ decreases; when the mean service rate increases and the other parameters are kept fixed, $P_{0,0,0}, P(0)$ increase and $P(1), Ls, Ws$ decreases; when the mean arrival rate increases and the other parameters are kept fixed, $P_{0,0,0}, P(0)$ decrease and $P(1), Ls, Ws$ increases; when the value of θ increases and the other parameters are kept fixed, $P_{0,0,0}, P(0)$ decrease and $P(1), Ls, Ws$ increases.
From the table (6.3) it is observed that when the value of K increases and the other parameters are kept fixed Ls and Ws increase.

This model includes the particular case that taking $\theta \to \infty, \gamma = 0$, we get the standard single server finite capacity interdependent queueing model with controllable arrival rates with vacation. When $\epsilon = 0, \gamma = 0$, this model reduces to $M/M/1/K$ retrial queueing model with controllable arrival rates without interdependence and vacation. When λ_a^+ tends to λ_a^-, this model reduces to $M/M/1/K$ interdependence retrial queueing model with vacation. When λ_a^+ tends to λ_a^- and $\epsilon = 0, \gamma = 0$, this model reduces to $M/M/1/K$ retrial queueing model. When λ_a^+ tends to λ_a^-, $\epsilon = 0, \gamma = 0$ and $\theta \to \infty$ this model reduces to standard $M/M/1/K$ queueing model.

The performance evaluation was quantitatively analysed in order to emphasise the role of the suggested single server finite capacity orbiting queue with server vacations and changeable arrival rates. The flexible arrival-rate mechanism λ_a^+ and λ_a^- significantly increases system efficiency, according to the numerical results. Specifically, the slower-arrival mode lowers system congestion and stabilises the queue length, whereas the faster-arrival mode decreases the average waiting time in the orbit. Because of controlled idle-time scheduling, the incorporation of server vacations and vacation termination lowers operating costs.

References

1. Antline Nisha, B. and Thiagarajan, M., *Control Policy, Manageable Arrival Rates, and Reverse Balking in Mutually Dependent Queueing Model*, Contemporary Mathematics, 5 (1), 17-25, (2024)
2. Artalejo, J. R., *Accessible bibliography on retrial queues: Progress in 1990—1999,* Math. Comput. Model., 30(3-4), 1-20, (1999).
3. Artalejo, J. R., *Accessible bibliography on retrial queues: Progress in 2000—2009,* Math. Comput. Model., 51(9-10), 1071-1081, (2010).
4. Artalejo, J. R., *An M/M/1 retrial queue with server vacation and feedback* Applied Mathematical Modelling, 30(7), 531-545, (2006).
5. Begum,M.I. and Maheswari,D., *The M/M/c interdependent queueing model with controllable arrival rates*, Opsearch, 39, No. 2, 89-110,(2002).
6. Chakravarthy, S. R.,*Analysis of a state-dependent M/M/1 retrial queue with Bernoulli vacation schedule* Performance Evaluation 98, 52-68, (2016).
7. Cohen, J. W., *The single server queue with Poisson input and exponential service times*, Mathematical Proceedings of the Cambridge Philosophical Society. 53 (1), 50-59, (1957).
8. Doshi, B. T., *Queueing systems with vacations—A survey* Queueing Systems, 29–66, (1986).
9. Falin, G. I. and Templeton, J. G. C.*Retrial Queues*, Chapman and Hall., (1997).
10. Gross, D. and Harris, C. M., *Fundamentals of Queueing Theory* John Wiley, New York (1974).
11. Gupta, S. M, *Interrelationship between queueing models with balking and reneging and machine repair problem with warm spares* Microelectronics Reliability, 34, 201-209,(1994).
12. Haight, F.A.,*Queuing with balking* Biometrika 44, No. 3-4, 360-369,(1957).
13. Jain, M. and Singh, P., *M/M/m queue with balking, reneging and additional server*, Int. J. Eng., 15, No. 3, 169-178,(2002).
14. Jain, M. and Sharma, G.C., *Finite controllable Markovian queue with balking and reneging*, Nepali Mathematical Sciences Report, 22, No. 1, 113-120, (2004).
15. Jain, N.K, Rakesh Kumar, Bhupenderkumar Som,*An M/M/1/N Queuing system with Reverse Balking*, American Journal of Operation Research, 4 No.2, 17-20, (2014).
16. Kapodistria, S., *The M/M/1 Queue with Synchronized Abandonment*, Queuing Systems, 68, 79-109,(2011).
17. Krishna Kumar, B. and Arivudainambi, D., *Analysis of an M/M/1 retrial queue with state-dependent arrivals and feedback*, International Journal of Operational Research 14(4), 458-472, (2012).
18. Reynolds, J. F.,*The stationary solution of a multi-server model with discouragement*, Operation Research, 16, 64-71, (1968).
19. Tadj, L. and Choudhury, G. *Optimal control of a queueing system with server vacations and impatient customers*, Computers and Industrial Engineering 54(1), 66-76, (2008).
20. Yechiali, U., *Queuing models with retrials*, Queueing Systems, 54, 195-202, (2007).
21. Zhang,B., Van Leeuwaarden,J S., Zwart,B.,*Staffing call centers with impatient customers*, Operation Research, 60(2), 461-474,(2012)

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