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Constant Local Multiset Dimension of Convex Polytopes-R_n and S_n

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ABSTRACT: In graph theory, the metric dimension is a concept whose primary purpose is to uniquely identify each vertex in a graph based on distance with respect to a collection of reference vertices, called resolving sets. Resolving sets are used in communication networks to uniquely identify the position of each node. In robotics and navigation systems, robots can determine their exact position with reference to a set of codes generated by resolving set. Based on the practical requirements of the network problems, there are several variation in metric dimensions. Local multiset dimension (LMD) of a graph is one such variation, in which a local multiset basis generates a distinct multiset of distances for every pair of adjacent vertices with reference to a subset of the vertex set of the graph. Convex polytopes are the two-dimensional geometric shapes that can be used in designing an optimal network configuration. In this paper, we determined the LMD of two convex polytopes and hence found that the local multiset dimension for both the polytopes is constant, irrespective of the order of the graphs.

Key Words: Convex polytopes, local multiset dimension, metric dimension.

Contents

1. Introduction

Throughout the paper, G be a simple connected and undirected graph [12] of order $|V(G)|$ and size $|E(G)|$, where $V(G)$ is vertex set and $E(G)$ is edge set of G. The length of a shortest path between any two vertices u and v in G is denoted by $d(u, v)$. Two vertices $u, v \in V(G)$ are said to be adjacent if $d(u, v) = 1$.
Metric dimension is a major concept in graph theory and it provides a systematic framework to distinguish all vertices of the graph G, using distances to minimum number of reference vertices. The concept of dimension was discussed early by Erods et. al. [8,9]. P. J. Slater presented the concept of metric dimension [18] followed by F. Harary and R. A. Melter in [11]. Slater addressed the cardinality of minimum resolving set as location number of G. Harary and Melter [11] adopted the term metric dimension rather than location number.

A set $F \subseteq V(G)$ is a metric generator of the graph G, if every pair of vertices from $V(G)$ is distinguished by some element of F. Minimum set of metric generator is called a metric basis and the cardinality of the metric basis is called the metric dimension [19]. Metric generators ensure that each vertex of the graph can be uniquely identified by its distance to the metric basis. By virtue of this characteristic, it finds a wide range of applications in network analysis such as communication networks, transportation systems, robotics and navigation systems, social networks, and biological networks where identification of nodes based on the distance is essential. In chemistry, metric dimension is used to analyze and distinguish the molecular structure; each functional group in a chemical compound is mapped as a

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subgraph. It allows the detection of whether two compounds contain the same functional group at the same position, which is essential in analytical studies related to pharmacological activity.

A subset $F = \{f_1, f_2, f_3, ..., f_n\} \subseteq V(G)$ is called a resolving set if $r(u|F) \neq r(v|F)$ for each $u, v \in V(G)$ whenever $u \neq v$, where $r(x|F) = (d(x, f_1), d(x, f_2), d(x, f_3), ..., d(x, f_n))$ is the representation of metric coordinate of x in G with respect to F. The minimum cardinality of a resolving set is the metric dimension [19]. Metric dimension of basic graphs like cycle, path [7] and other family of graphs like unicyclic [10], bicyclic [13], wheel related have been studied. Similarly, various types of convex polytopes also been studied [16,14]. Various forms of metric dimension has been introduced such as local metric dimension, multiset dimension, local multiset dimension, edge metric dimension, partition metric dimension.

Multiset dimension is a variation of metric dimension, which was proposed by Rinovia Simanjuntak et.al., in 2017 [17]. For $u \in V(G)$ and $F \subseteq V(G)$, the multi code of u with respect to F is a multi set of distances between u and vertices of F and is denoted as $r_m(u|F)$.

In 2020 Ridho et.al. [6] introduced the concept of local multiset dimension (LMD), which finds the same applications as metric dimension with an added advantage of reduced computational complexity since it focuses on adjacent vertices. A set $F = \{f_1, f_2, f_3, ..., f_n\} \subseteq V(G)$ is called a local multiset basis(lmsb), if $r_m(u|F) \neq r_m(v|F)$ whenever $uv \in E(G)$, where $r_m(u|F) = \{d(u, f_1), d(u, f_2), d(u, f_3), ..., d(u, f_n)\}$ is the multiset code of u in G with respect to F. The minimum cardinality of a local multiset basis is called the local multiset dimension. Local multiset dimensions have been computed for various graphs [1,2,3,4,5,15].

Convex polytopes are fundamental objects in geometry and optimization, characterized by flat faces, edges, and vertices, and they generalize polygons and polyhedra to arbitrary dimensions. Due to the structure, the polytopes are used in modeling the real-world scenarios.This paper describes local multiset dimension of some family of convex polytopes R_n and S_n.

This paper is organized as the computation of the local multiset dimension of R_n in section 2 and S_n in section 3. Section 4 discusses the conclusion of the findings of these two graphs.

2. Preliminaries

The following results provides the foundational framework.

Theorem 2.1 ([1]). *A graph G has $\mu_l(G) = 1$, if and only if G is bipartite.*

Lemma 2.1 ([15]). *Let G be a graph with lmsb $F = \{f_1, f_2\}$. Then, $d(f_1, f_2) \equiv 0 \pmod 2$.*

Remark 1. *From Lemma 2.1 it follows that if $d(f_1, f_2) \equiv 1 \pmod 2$, then F cannot form lmsb of G.*

Lemma 2.2. *Let G be a graph and $F = \{f_1, f_2\} \subseteq V(G)$. If $d(f_1, f_2) \equiv 0 \pmod 2$ and f_1 , f_2 lies on an induced odd cycle in G then F is not an lmsb of G.*

Proof. Given that $F = \{f_1, f_2\} \subseteq V(G)$ and $d(f_1, f_2) \equiv 0 \pmod 2$. Since f_1 and f_2 lies on an odd cycle, there exist an odd path between f_1 and f_2, thus from the Remark 1, F is not an lmsb of G □

Lemma 2.3. *Let G be a graph and $F = \{f_1, f_2\} \subseteq V(G)$. Suppose f_1 lies on an induced odd cycle C_{2k+1} in G and the vertices $uv \in E(C_{2k+1})$ with $d(u, f_1) = d(v, f_1) = k$. If f_2 is not equidistant to the vertices u and v, then $F = \{f_1, f_2\}$ is an lmsb of G.*

Proof. Given that $F = \{f_1, f_2\} \subseteq V(G)$, f_1 lies on an induced odd cycle C_{2k+1} in G and $uv \in E(C_{2k+1})$ with $d(u, f_1) = d(v, f_1) = k$. Assume to the contrary $d(u, f_2) = d(v, f_2) = a$. Then $r_m(u|F) = r_m(v|F) = \{k, a\}$. Hence $F = \{f_1, f_2\}$ is not an lmsb of G. □

3. Local multiset dimension of R_n

The graph R_n, for $n \geq 3$ is a symmetric convex polytope with the vertex set $V(R_n) = \{u_j, v_j, w_j : 0 \leq j \leq n-1\}$ and the edge set $E(R_n) = \{u_j u_{j+1}, v_j v_{j+1}, w_j w_{j+1}, v_j u_{j+1} : 0 \leq j \leq n-2\} \cup \{u_j v_j, v_j w_j : 0 \leq j \leq n-1\} \cup \{u_{n-1}u_0, v_{n-1}v_0, w_{n-1}w_0, v_{n-1}u_0\}$. Order and size of R_n is $3n$ and $5n$ respectively. The graph R_n contains vertices of degrees 3, 4, and 5 and each is n in number.
In this section, local multiset dimension of R_n is computed and executed a multiset basis for each $n \in \mathbb{Z}^+$.

Lemma 3.1. *For any $n \in \mathbb{Z}^+$ with $n \geq 3$, $\mu_l(R_n) \geq 3$.*

Proof. Let F be the minimum local multiset basis of R_n.

Claim 1. $|F| \neq 1$

Since the convex polytope R_n contains C_3, R_n is not bipartite and hence by Theorem 2.1, $|F| \neq 1$.

Claim 2. $|F| > 2$

Proof of Claim 2 is discussed by the method of contradiction. Suppose $F = \{f_1, f_2\}$ be an lmsb of R_n, then by Remark 1 $d(f_1, f_2) \equiv 0 \pmod 2$. Since the graph is symmetric, without loss of generality we assume $f_1 \in \{u_0, v_0, w_o\}$. In each cycle whenever there are two vertices which are at the same distance from $f1$. Since the structure is symmetric, it is sufficient to prove the claim for any one vertex. For each pair of vertices, a contradiction for different values of n and the corresponding value of j is produced as follows,
$f_1 = u_0$
Case 1:$f_2 = u_j$

- When $n \geq 6$, $2 \leq j \leq 2\lfloor \frac{n-6}{4} \rfloor + 2$, f_2 lies on a C_3 and f_1 is equidistant to the vertices u_{j+1},v_j. From Lemma 2.3 $r_m(u_{j+1}|F) = r_m(v_j|F) = \{j+1, 1\}$.
- When $5 \leq n \equiv 1 \pmod 4$, $j = \lfloor \frac{n}{2} \rfloor$,$f_1$ and f_2 lies on a C_n cycle. From lemma 2.2 $r_m(u_{k+\frac{j}{2}}|F) = r_m(u_{k+\frac{j}{2}+1}|F) = \{\frac{j}{2}, \frac{j}{2}+1\}$
- When $4 \leq n \equiv 0 \pmod 4$, $j = \frac{n}{2}$, f_2 lies on a C_3 cycle and f_1 is equidistant to the vertices v_j,v_{j-1}. From Lemma 2.3 $r_m(v_j|F) = r_m(v_{j-1}|F) = \{j, 1\}$

Case 2: $f_2 = v_j$

- When $1 \leq j \leq 2\lfloor \frac{n-3}{4} \rfloor + 1$,$f_2$ lies on a C_3 cycle and f_1 is equidistant to vertices u_j, v_{j-1}. From Lemma 2.3, $r_m(u_j|F) = r_m(v_{j-1}|F) = \{j, 1\}$

Case 3: $f_2 = w_j$

- When $j = 0$, f_1 and f_2 lies on a C_5. From Lemma 2.2 $r_m(w_{n-1}|F) = r_m(v_{n-1}|F)$=\{1,2\}.
- When $n \geq 5$, $2 \leq j \leq 2\lfloor \frac{n-5}{4} \rfloor + 1$, f_2 lies on a C_5 and f_1 is equidistant to the vertices u_j,v_{j-1}. From lemma 2.3, $r_m(u_j|F) = r_m(v_{j-1}|F) = \{j, 2\}$

$f_1 = v_0$
Case 1:$f_2 = v_j$

- For $n \geq 5$, $2 \leq j \leq 2\lfloor \frac{n-5}{4} \rfloor + 2$ then $r_m(u_{\frac{j}{2}}|F) = r_m(v_{\frac{j}{2}+1}|F) = \{\frac{j}{2}, \frac{j}{2}+1\}$
- For $4 \leq n \equiv 0 \pmod 4$,$j = \frac{n}{2}$ then $r_m(u_j|F) = r_m(u_{j+1}|F) = \{j, j\}$

Case 2:$f_2 = w_j$

- When $n \geq 4$, $1 \leq j \leq 2\lfloor \frac{n-4}{4} \rfloor + 1$, f_2 lies on C_5 and f_1 is equidistant to the vertices v_{j+1} and u_{j+1}. From Lemma 2.3, $r_m(v_{j+1}|F) = r_m(u_{j+1}|F) = \{j+1, 2\}$
- For $3 \leq n \equiv 3 \pmod 4$, $j = \lfloor \frac{n}{2} \rfloor$, $r_m(v_{n-\lceil \frac{j}{2} \rceil}|F) = r_m(w_{n-\lceil \frac{j}{2} \rceil}|F) = \{\lceil \frac{j}{2} \rceil, \lceil \frac{j}{2} \rceil + 1\}$

- For $6 \leq n \equiv 2 \pmod 4$, $j = \frac{n}{2}$ then $r_m(u_j|F) = r_m(u_{j+1}|F) = \{j, 2\}$

$f_1 = w_0$
Case 1:$f_2 = w_j$

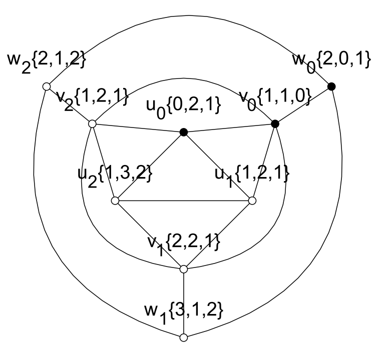
- When $n \geq 5$, $2 \leq j \leq 2\lfloor \frac{n-6}{4} \rfloor + 2$ f_2 lies on C_5 and f_1 is equidistant to the vertices u_{j+1} and v_{j+1}. From Lemma 2.3 $r_m(u_{j+1}|F) = r_m(v_{j+1}|F) = \{j+2, 2\}$
- For $4 \leq n \equiv 0 \pmod 4$, $j = \frac{n}{2}$ then $r_m(u_{j+1}|F) = r_m(u_j|F) = \{j+1, 2\}$
- When $5 \leq n \equiv 1 \pmod 4$, $j = \lfloor \frac{n}{2} \rfloor$, f_1 and f_2 lies on a C_n cycle. From Lemma 2.2 $r_m(w_{k+\frac{j}{2}}|F) = r_m(w_{k+\frac{j}{2}+1}|F) = \{\frac{j}{2}, \frac{j}{2} - 1\}$

It is observed from the above cases, for every pair of $V(R_n)$, atleast there is one pair of adjacent vertices receiving the same multiset code. Therefore $\mu_l(R_n) > 2$ for any n.
From Claim 1 and Claim 2, it completes to show that $\mu_l(R_n) \geq 3$. □

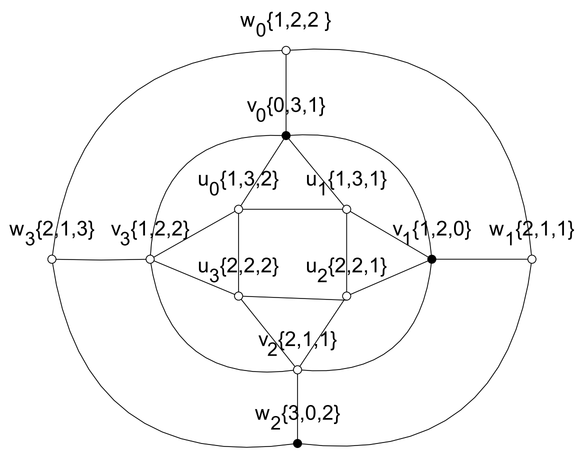
We prove the upper bound $\mu_l(R_n) \leq 3$ in lemma 3.2, where $n \geq 3$.

Lemma 3.2. *For any $n \in \mathbb{Z}^+$ with $n \geq 3$, $\mu_l(R_n) \leq 3$.*

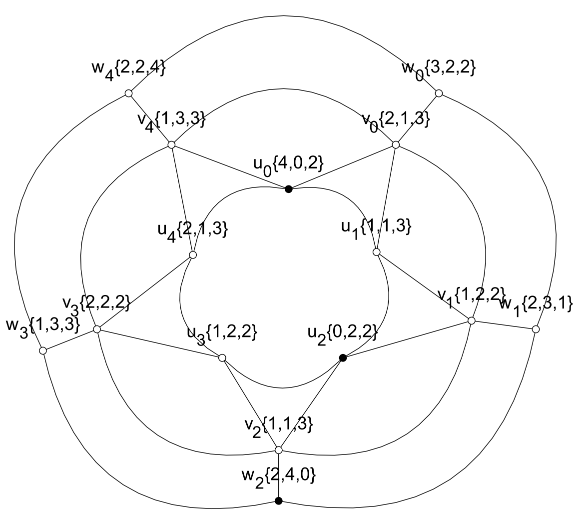
Proof: The proof for $\mu_l(R_n) \leq 3$ for $3 \leq n \leq 7$ is illustrated through the graphs in figure 1.



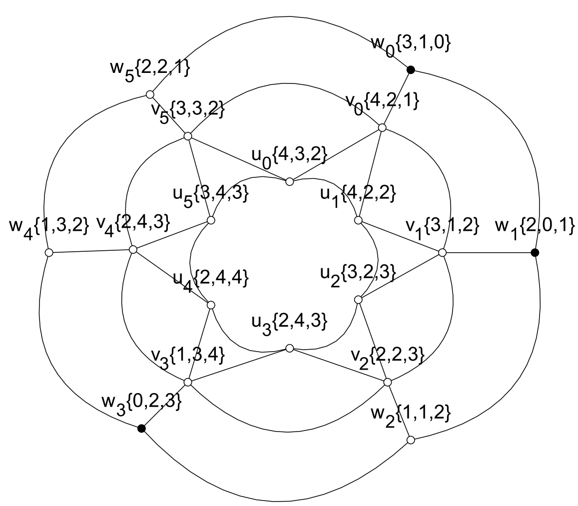
R_3



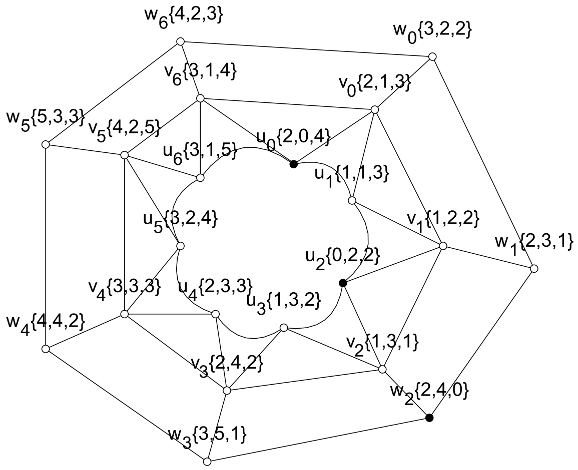
R_4



R_5



R_6



R_7

Figure 1: Illustration of $\mu_l(R_n) \geq 3$ for $R_n(\ 3 \leq n \leq 7)$

Further for $n \geq 8$, depending on the value of n, we have two cases.
Case 1: When n is odd.
Consider $n = 2k+1$,suppose $F = \{u_0, v_{n-1}, w_{n-1}\} \subseteq V(R_n)$. The multiset codes recieved by each vertex with respect to F as follows,
The multiset co-ordinates for the vertices $\{u_j : 0 \leq j \leq n-1\}$

$$r_m(u_j|F) = \begin{cases} \{j,\ j+1,\ j+2\ \} & \text{for } 0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1 \\ \{j,\ n-j,\ n-j+1\ \} & \text{for } j = \lfloor \frac{n}{2} \rfloor \\ \{n-j,\ n-j,\ n-j+1\ \} & \text{otherwise} \end{cases}$$

The multiset code for the vertices $\{v_j : 0 \leq j \leq n-1\}$

$$r_m(v_j|F) = \begin{cases} \{j+1,\ j+1,\ j+2\ \} & \text{for } 0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1 \\ \{j+1,\ n-j-1,\ n-j\ \} & \text{for } j = \lfloor \frac{n}{2} \rfloor \\ \{n-j,\ n-j-1,\ n-j\ \} & \text{otherwise} \end{cases}$$

The multiset code for the vertices $\{w_j : 0 \leq j \leq n-1\}$

$$r_m(w_j|F) = \begin{cases} \{j+2,\ j+2,\ j+1\ \} & \text{for } 0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1 \\ \{j+2,\ n-j,\ n-j-1\ \} & \text{for } j = \lfloor \frac{n}{2} \rfloor \\ \{n-j+1,\ n-j,\ n-j-1\ \} & \text{otherwise} \end{cases}$$

It is noted that there are no two adjacent vertices having same multiset code. Therefore $\mu_l(R_n) \leq 3$ for any odd n.
Case 2 : When n is even.
Consider $n = 2k$, suppose $F = \{u_0, u_{\frac{n}{2}}, u_{\lfloor \frac{n-1}{4} \rfloor}\} \subseteq V(R_n)$. The multiset codes recieved by each vertex with respect to F as follows,
The multiset code for the vertices $\{u_j : 0 \leq j \leq n-1\}$

$$r_m(u_j|F) = \begin{cases} \{j,\ \frac{n}{2}-j,\ \lfloor \frac{n-1}{4} \rfloor - j\} & \text{for } 0 \leq j \leq \lfloor \frac{n-1}{4} \rfloor \\ \{j,\ \frac{n}{2}-j,\ j - \lfloor \frac{n-1}{4} \rfloor\ \} & \text{for } \lceil \frac{n-1}{4} \rceil \leq j \leq \frac{n}{2} \\ \{n-j,\ j-\frac{n}{2},\ j - \lfloor \frac{n-1}{4} \rfloor\ \} & \text{for } \frac{n}{2}+1 \leq j \leq \lfloor \frac{3n-1}{4} \rfloor \\ \{n-j,\ j-\frac{n}{2},\ n-j+\lfloor \frac{n-1}{4} \rfloor\ \}, & \text{otherwise} \end{cases}$$

The multiset code for the vertices $\{v_j : 0 \leq j \leq n-1\}$

$$r_m(v_j|F) = \begin{cases} \{j+1, \frac{n}{2}-j, \lfloor \frac{n-1}{4} \rfloor - j\} & \text{for } 0 \leq j \leq \lfloor \frac{n-1}{4} \rfloor - 1 \\ \{j+1, \frac{n}{2}-j, j-\lfloor \frac{n-1}{4} \rfloor + 1\} & \text{for } \lfloor \frac{n-1}{4} \rfloor \leq j \leq \frac{n}{2} - 1 \\ \{n-j, j-\frac{n}{2}+1, j-\lfloor \frac{n-1}{4} \rfloor + 1\} & \text{for } \frac{n}{2} \leq j \leq \lfloor \frac{3n-1}{4} \rfloor - 1 \\ \{n-j, j-\frac{n}{2}+1, n-j+\lfloor \frac{n-1}{4} \rfloor\} & \text{otherwise} \end{cases}$$

The multiset code for the vertices $\{w_j : 0 \leq j \leq n-1\}$

$$r_m(w_j|F) = \begin{cases} \{j+2, \frac{n}{2}-j+1, \lfloor\frac{n-1}{4}\rfloor - j+1\} & \text{for } 0 \le j \le \lfloor\frac{n-1}{4}\rfloor - 1 \\ \{j+2, \frac{n}{2}-j+1, j-\lfloor\frac{n-1}{4}\rfloor+2\} & \text{for } \lfloor\frac{n-1}{4}\rfloor \le j \le \frac{n}{2}-1 \\ \{n-j+1, j-\frac{n}{2}+2, j-\lfloor\frac{n-1}{4}\rfloor+2\} & \text{for } \frac{n}{2} \le j \le \lfloor\frac{3n-1}{4}\rfloor - 1 \\ \{n-j+1, j-\frac{n}{2}+2, n-j+\lfloor\frac{n-1}{4}\rfloor+1\} & \text{otherwise} \end{cases}$$

It is noted that there are no two adjacent vertices having same multiset code. Therefore $\mu_l(R_n) \le 3$ for any even n.
Therefore $\mu_l(R_n) \le 3$ for any n. □

Theorem 3.1. *For any $n \in \mathbb{Z}^+$ $n \ge 3$, $\mu_l(R_n) = 3$.*

Lemma 3.1 and Lemma 3.2 completes the proof of 3.1.

4. Local multiset dimension of S_n

The graph of the convex polytope S_n is an extended graph of the convex polytope R_n. It is obtained by adding an exterior cycle to the outermost cycle of R_n, such that each vertex of the outermost cycle is connected to the preceeding vertices of the newly added outer cycle. It has a vertex set $V(S_n) = V(R_n) \cup \{w_{j,1} : 0 \le j \le n-1\}$ and edge set $E(S_n) = E(R_n) \cup \{w_{j,1}w_{j+1,1} : 0 \le j \le n-2\} \cup \{w_{j,1}w_j : 0 \le j \le n-1\} \cup \{w_{n-1,1}w_{0,1}\}$. Order and size of S_n is $4n$ and $7n$ respectively. The graph S_n contains vertices of degrees 3,5 each are n in number and degree 4, $2n$ in number.
In this section, local multiset dimension of S_n is computed and executed a multiset basis for each $n \in \mathbb{Z}^+$.

Lemma 4.1. *For any $n \in \mathbb{Z}^+$ with $n \ge 3$, $\mu_l(S_n) \ge 3$.*

Let F be the minimum local multiset basis of S_n.

Claim 3. $|F| \neq 1$

Since the convex polytope S_n contains C_3, S_n is not bipartite and hence by Theorem 2.1, $|F| \neq 1$.

Claim 4. $|F| > 2$

Proof of Claim 4 is discussed by the method of contradiction. Suppose $F = \{f_1, f_2\}$ be a lmsb of S_n, then by Remark 1, $d(f_1, f_2) \equiv 0 \pmod 2$. Since S_n is an extended graph of R_n, it is sufficient to prove the claim for $f_1 = w_{0,1}$ and $f_2 \in \{u_i, v_i, w_i\}$. In each cycle whenever two vertices are at the same distance from $f1$. Since the structure is symmetric, it is sufficient to prove the claim for any one vertex. For each pair of vertices, a contradiction for different values of n and the corresponding value of j is produced as follows,
$f_1 = w_{0,1}$
Case 1: $f_2 = w_{j,1}$

- When $n \ge 6, 2 \le j \le 2\lfloor\frac{n-6}{4}\rfloor + 2$, f_2 lies on C_7 and f_1 is equidistant to u_{j+1} and v_{j+1}. From Lemma 2.3, $r_m(u_{j+1}|F) = r_m(v_{j+1}|F) = \{j+3, 3\}$
- For $4 \le n \equiv 0 \pmod 4$, $j = \frac{n}{2}$ then $r_m(u_j|F) = r_m(u_{j+1}|F) = \{j+2, 3\}$
- When $5 \le n \equiv 1 \pmod 4$, $j = \lfloor\frac{n}{2}\rfloor$ f_1 and f_2 lies on C_n. From Lemma 2.2 $r_m(w_{k+\frac{j}{2}}|F) = r_m(w_{k+\frac{j}{2}+1}|F) = \{\frac{j}{2}, \frac{j}{2}+1\}$

Case 2: $f_2 = w_j$

- When $n \ge 4$, $1 \le j \le 2\lfloor\frac{n-4}{4}\rfloor + 1$ f_2 lies on C_5 and f_1 is equidistant to u_{j+1} and v_{j+1}. From Lemma 2.3 $r_m(u_{j+1}|F) = r_m(v_{j+1}|F) = \{j+3, 2\}$
- For $3 \le n \equiv 3 \pmod 4$, $j = \lfloor\frac{n}{2}\rfloor$ $r_m(w_{k+\lceil\frac{j}{2}\rceil}|F) = r_m(w_{k+\lceil\frac{j}{2}\rceil+1}|F) = \{\lceil\frac{j}{2}\rceil, \lceil\frac{j}{2}\rceil+1\}$
- For $6 \le n \equiv 2 \pmod 4$, $j = \frac{n}{2}$ then $r_m(u_j|F) = r_l(u_{j-1}|F) = \{j+2, 2\}$

Case 3: $f_2 = v_j$

- When $n \geq 3, 0 \leq j \leq 2\lfloor \frac{n-3}{4} \rfloor$, f_2 lies on C_3 and f_1 is equidistant to u_{j+1} and v_{j+1}. From Lemma 2.3 $r_m(u_{j+1}) = r_m(v_{j+1}) = \{j+3, 1\}$

- When $4 \leq n \equiv 0 \pmod 4$, $j = \frac{n}{2}$, f_2 lies on C_3 and f_1 is equidistant to u_{j+1} and u_j. From Lemma 2.3, $r_m(u_{j+1}|F) = r_m(u_j|F) = \{j+2, 1\}$.

- For $5 \leq n \equiv 1 \pmod 4$, $j = \lfloor \frac{n}{2} \rfloor$ then $r_m(w_{\lceil \frac{3j+1}{2} \rceil}|F) = r_m(v_{\lceil \frac{3j+1}{2} \rceil}|F) = \{\frac{j}{2}+1, \frac{j}{2}+2\}$

Case 4: $f_2 = u_j$

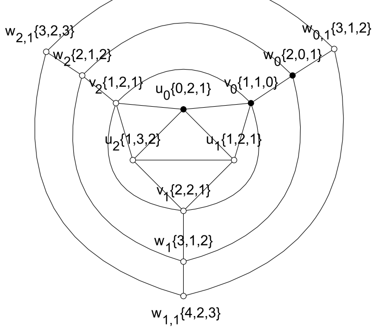
- When $n \geq 3, 2 \leq j \leq 2\lfloor \frac{n-3}{4} \rfloor + 2$, f_2 lies on C_3 and f_1 is equidistant to u_{j-1} and v_{j-1}. From Lemma 2.3, $r_m(u_{j-1}|F) = r_m(v_{j-1}|F) = \{j+1, 1\}$

It is observed from the above cases, for every pair of $V(S_n)$, atleast there is one pair of adjacent vertices receiving the same multiset code. Therefore $\mu_l(S_n) > 2$ for any n.
From Claim 3 and Claim 4, it completes to show that $\mu_l(S_n) \geq 3$.

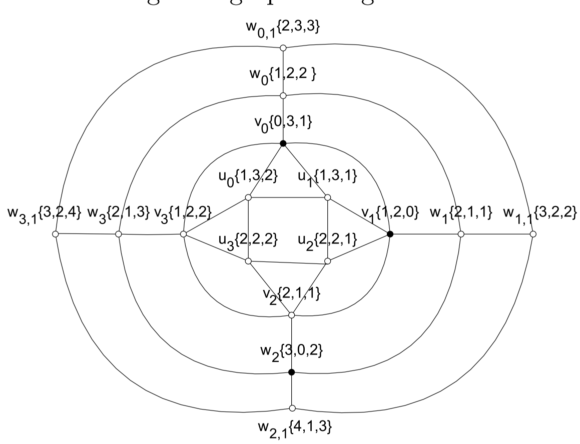
We prove the upper bound $\mu_l(S_n) \leq 3$ in lemma 4.2.

Lemma 4.2. *For any $n \in \mathbb{Z}^+$ with $n \geq 3$, $\mu_l(S_n) \leq 3$.*

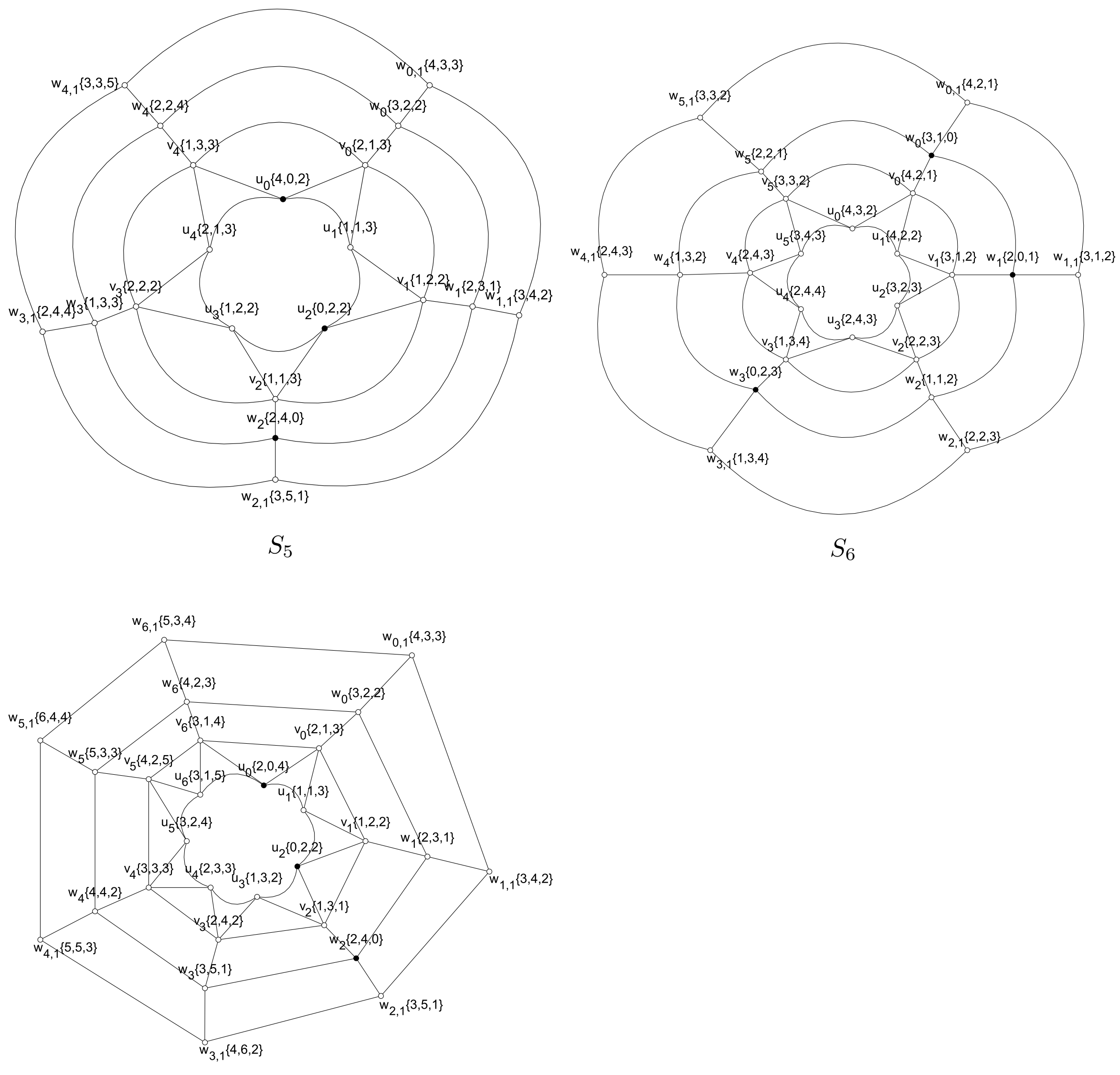
Proof: The proof for $\mu_l(S_n) \leq 3$ for $3 \leq n \leq 7$ is illustrated through the graphs in figure 2.



S_3



S_4



S_5 S_6 S_7

Figure 2: Illustration of $\mu_l(S_n) \geq 3$ for $S_n(\, 3 \leq n \leq 7)$

Further for $n \geq 8$, depending on the value of n, we have two cases.

Case 1: When n is odd.

Consider $n = 2k+1$, suppose $F = \{u_k, w_k, w_{k+4}\} \subseteq V(S_n)$. The multiset codes recieved by each vertex with respect to F remains the same for the vertices u_i, v_i, w_i since the same basis as R_n is considered.

The multiset code for the vertices $\{w_{j,1} : 0 \leq j \leq n-1\}$

$$r_m(w_{j,1}|F) = \begin{cases} \{j+3,\, j+3,\, j+2\ \} & \text{for } 0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1 \\ \{j+3,\, n-j+1,\, n-j\ \} & \text{for } j = \lfloor \frac{n}{2} \rfloor \\ \{n-j+2,\, n-j+1,\, n-j\ \} & \text{otherwise} \end{cases}$$

It is noted that there are no two adjacent vertices having same multiset code. Therefore $\mu_l(S_n) \leq 3$ for any odd n.

Case 2 : When n is even.

Consider $n = 2k$, suppose $F = (w_0, w_{\frac{n}{2}-1}, w_{\frac{n}{2}}) \subseteq V(S_n)$. The multiset codes recieved by each vertex with respect to F remains the same for the vertices u_i, v_i, w_i since the same basis as R_n is considered.

The multiset code for the vertices $\{w_{j,1} : 0 \leq j \leq n-1\}$

$$r_m(w_{j,1}|F) = \begin{cases} \{j+3, \frac{n}{2}-j+2, \lfloor\frac{n-1}{4}\rfloor - j+2\} & \text{for } 0 \le j \le \lfloor\frac{n-1}{4}\rfloor - 1 \\ \{j+3, \frac{n}{2}-j+2, j-\lfloor\frac{n-1}{4}\rfloor+3\} & \text{for } \lfloor\frac{n-1}{4}\rfloor \le j \le \frac{n}{2} - 1 \\ \{n-j+2, j-\frac{n}{2}+3, j-\lfloor\frac{n-1}{4}\rfloor+3\} & \text{for } \frac{n}{2} \le j \le \lfloor\frac{3n-1}{4}\rfloor - 1 \\ \{n-j+2, j-\frac{n}{2}+3, n-j+\lfloor\frac{n-1}{4}\rfloor+2\} & \text{otherwise} \end{cases}$$

It is noted that there are no two adjacent vertices having same multiset code. Therefore $\mu_l(S_n) \le 3$ for any even n.
Therefore $\mu_l(S_n) \le 3$ for any n. □

Theorem 4.1. *For any $n \in \mathbb{Z}^+$ $n \ge 3$, $\mu_l(S_n) = 3$.*

Lemma 4.1 and Lemma 4.2 completes the proof of Theorem 4.1.

5. Conclusion

In this paper, LMD of convex polytopes R_n and S_n is determined to be three. Three vertices are sufficient to uniquely identify all the vertices irrespective of the order of the graph. It can aslo be observed that the same basis resolves both the convex polytopes. Further irrespective of the number of exterior cycles added to the outermost cycle, the LMD remains the same and also the same basis can be used to resolve the obtained convex polytope. Thus, LMD of these convex polytopes remains constant irrespective of the order and also the number of exterior cycles.
The graph of convex polytopes can be modelled as various networks like transportation, computer, electrical, communication, social and so on. Finding the LMD of convex polytopes helps in unique node identification in any network. Further more in detecting the congestion nodes in transportation network, to detect failure in computer network connection. In electrical networks voltage instability, overload of current can be encountered. The local multiset basis is used as monitoring nodes to enhance the efficiency of these networks. It concludes that local multiset dimension of convex polytopes helps in designing and analysing networks of any size.

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