Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–28.
ISSN-0037-8712
doi:10.5269/bspm.80280

Weak Commuting Mappings and its Variants for Generalized ψ- Weak Contraction in Perturbed Metric Spaces

Pawan Kumar, Sanjeev Kumar and Balbir Singh

ABSTRACT: In this paper, we establish several common fixed-point theorems in the setting of perturbed metric spaces, where the usual distance $d(x, y)$ is replaced by the perturbed distance

$$d_\eta(x, y) = d(x, y) + \eta(x, y),$$

with η representing a non-negative perturbation kernel. Working in this generalized framework, we investigate weakly commuting mappings and their pointwise extensions, including R-weakly commuting and reciprocally continuous mappings. Further, we study R-weakly commuting mappings of type (P) under a generalized $*$-weak contraction condition formulated with cubic and quadratic powers of the perturbed distance $d_\eta(x, y)$. These results significantly extend the classical metric-space theorems to situations involving measurement errors and structural perturbations.

Keywords: Perturbed metric space, perturbed distance, weakly commuting mappings, pointwise R-weak commutativity, reciprocally continuous mappings, type-(P) mappings, generalized $\circledast$-weak contraction, cubic and quadratic distance terms, common fixed point, fixed-point theory.

Contents

1. Introduction

The classical Banach Contraction Principle has been extended in numerous directions by modifying continuity assumptions, increasing the number of self-mappings, and introducing suitable control functions and relaxed commutative requirements. Because of its constructive nature, Banach's theorem plays a central role in fixed-point theory, offering an effective iterative process for approximating fixed points of contractive mappings.

With the growing need to model uncertainties arising from measurement errors, noise, and structural distortions, researchers have begun to study contractions in perturbed metric spaces, where the usual metric d is augmented by a perturbation kernel η to form the perturbed distance

$$d_\eta(x, y) = d(x, y) + \eta(x, y).$$

This framework allows the investigation of fixed points under more realistic conditions in which exact distances are difficult to obtain.

A significant milestone in fixed point theory occurred when the notion of commutativity for pairs of mappings was introduced. Later, Sessa [21] refined this idea by defining weak commutativity. A major advancement came in 1986 when Jungck [9] introduced the concept of compatibility, which proved highly

2020 *Mathematics Subject Classification:* 47H10, 54H25, 68U10.
Submitted November 21, 2025. Published June 05, 2026.

effective for deriving common fixed-point results by assuming continuity for at least one mapping in the pair.

The exploration of fixed-point theorems gradually progressed from compatible mappings to non-compatible settings. Pant [18] further expanded this line of research by formulating the concept of R-weakly commuting mappings, capturing weaker interaction conditions between mappings beyond classical compatibility.

The classical Banach Fixed Point Theorem ensures that every contraction mapping on a complete metric space possesses a unique fixed point. In the setting of perturbed metric spaces, this idea naturally extends by working with the perturbed distance

$$d_\eta(x,y) = d(x,y) + \eta(x,y),$$

where $\eta : X \times X \to [0,\infty)$ is a non-negative perturbation kernel. Let (X, d_η) be a complete perturbed metric space and let $T : X \to X$ be a mapping satisfying

$$d_\eta(Tx,Ty) \leq k\, d_\eta(x,y) \quad \text{for all } x,y \in X, \quad 0 \leq k < 1.$$

Then T has a unique fixed point in X. This result reflects the fact that the convergence properties of classical contractions remain valid even when the underlying metric is influenced by perturbations.

In 1969, Boyd and Wong [4] generalized Banach's contraction principle by replacing the constant k with a control function. In the perturbed context, their idea can be formulated as follows: Let (X, d_η) be a complete perturbed metric space and let

$$\circledast : [0,\infty) \to [0,\infty)$$

be a right upper semicontinuous function satisfying $0 \leq \circledast(t) < t$ for all $t > 0$. If the mapping $T : X \to X$ satisfies

$$d_\eta(Tx,Ty) \leq \circledast\big(d_\eta(x,y)\big) \quad \text{for all } x,y \in X,$$

then T admits a unique fixed point. This extension demonstrates that Boyd–Wong-type contractions retain their fixed-point structure even after incorporating perturbation effects.

Motivated by these developments and the increasing relevance of perturbed metrics, the present work first formulates a generalized ψ-weak contraction condition in the context of perturbed metric spaces. This contraction involves quadratic and cubic powers of the perturbed distance $d_\eta(x,y)$, making it more flexible than traditional metric-based contractions. Employing this condition, we establish common fixed-point theorems for weakly commuting mappings, point-wise R-weakly commuting mappings, and their reciprocally continuous counterparts.

Our results extend and unify several existing fixed-point theorems by incorporating perturbation effects, thereby broadening the applicability of fixed-point theory to settings where exact distances are not guaranteed.

2. Perturbed Metric Space

Definition 2.1 (Perturbed Metric Space) *Let (X,d) be a metric space and let $\eta : X \times X \to [0,\infty)$ be a non-negative function called a* perturbation kernel. *The perturbed distance d_η on X is defined by*

$$d_\eta(x,y) = d(x,y) + \eta(x,y), \qquad \textit{for all } x,y \in X.$$

If the function d_η satisfies the axioms of a metric, namely

$$d_\eta(x,y) = 0 \iff x = y,$$

$$d_\eta(x,y) = d_\eta(y,x),$$

$$d_\eta(x,z) \leq d_\eta(x,y) + d_\eta(y,z),$$

then the pair (X, d_η) is called a perturbed metric space. *The perturbation kernel η represents measurement error, environmental noise, or structural uncertainty added to the original metric.*

In particular, a sequence is d_η-Cauchy if and only if it is d-Cauchy, and completeness of (X, d) is equivalent to completeness of (X, d_η).

Symmetry and definiteness are immediate. For the triangle inequality, we observe

$$d_\eta(x, z) \leq d(x, y) + d(y, z) + \eta(x, y) + \eta(y, z)$$

$$\leq d_\eta(x, y) + d_\eta(y, z) + (\alpha - 1)\big(\eta(x, y) + \eta(y, z)\big) + \beta\, d(y, z).$$

Choosing $\alpha < 1$ and applying a standard absorption argument yields

$$d_\eta(x, z) \leq C\big[d_\eta(x, y) + d_\eta(y, z)\big].$$

Renorming d_η by the equivalent metric

$$\widetilde{d}_\eta := \frac{1}{C} d_\eta$$

(or taking $\beta = 0$) gives the triangle inequality exactly. We will work directly with d_η, noting these equivalences.

Example 2.1 Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define

$$\eta(x, y) = \frac{1}{1 + |x| + |y|}.$$

Then

$$d_\eta(x, y) = |x - y| + \frac{1}{1 + |x| + |y|}.$$

Clearly, $\eta(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.

Symmetry holds because $|x| + |y|$ is symmetric in x and y. The triangle inequality holds since the perturbation term is bounded above by 1 and preserves subadditivity.

Hence $(\mathbb{R}, d_\eta)$ is a perturbed metric space.

Definition 2.2 (Commuting Mappings in a Perturbed Metric Space) *Let (X, d) be a metric space and let $\eta : X \times X \to [0, \infty)$ be a non-negative symmetric perturbation kernel. Define the perturbed distance*

$$d_\eta(x, y) = d(x, y) + \eta(x, y), \qquad x, y \in X.$$

Then (X, d_η) is called a perturbed metric space.

Two self-mappings $f, g : X \to X$ on (X, d_η) are said to commute *if*

$$f(gx) = g(fx), \qquad \textit{for every } x \in X.$$

Definition 2.3 (Weak Commutativity in a Perturbed Metric Space) *Let (X, d) be a metric space equipped with a non-negative symmetric perturbation kernel $\eta : X \times X \to [0, \infty)$, and let the perturbed distance be*

$$d_\eta(x, y) = d(x, y) + \eta(x, y), \qquad x, y \in X.$$

Two self-mappings $f, g : X \to X$ are called weakly commuting *on the perturbed metric space (X, d_η) if*

$$d_\eta(fgx, gfx) \leq d_\eta(gx, fx), \qquad \textit{for every } x \in X.$$

This inequality expresses that the deviation between the compositions $f \circ g$ and $g \circ f$ does not exceed the perturbed distance between the individual images $g(x)$ and $f(x)$.

Remark 2.1 (Relation with Commutativity) If two mappings f and g commute in the perturbed metric sense, that is,

$$f(gx) = g(fx), \qquad \text{for all } x \in X,$$

then they automatically satisfy the weak commutativity condition:

$$d_\eta(fgx, gfx) = 0 \leq d_\eta(gx, fx).$$

Thus, commutative mappings always form a subclass of weakly commuting mappings in the perturbed metric framework. However, the converse is not guaranteed; a pair may be weakly commuting without being exactly commuting.

Extension to R-Weak Commutativity in Perturbed Metric Spaces. Pant generalized weak commutativity by introducing the notion of R-weakly commuting mappings in classical metric spaces to enlarge the class of compatible mappings having common fixed points. In the setting of perturbed metric spaces, the same idea extends naturally.

Definition 2.4 (R-Weakly Commuting Mappings) *Let (X, d_η) be a perturbed metric space. A pair of self-mappings $f, g : X \to X$ is said to be R-*weakly commuting *(in the perturbed sense) if there exists a constant $R > 0$ such that*

$$d_\eta(fgx, gfx) \leq R\, d_\eta(fx, gx), \qquad \text{for all } x \in X.$$

This definition permits a controlled degree of non-commutativity depending on the constant R. Unlike commutative or compatible maps, R-weakly commuting mappings in perturbed metric spaces need not be continuous at their fixed points, thereby creating a broader and more flexible class useful in fixed point theory.

Definition 2.5 (Reciprocal Continuity in a Perturbed Metric Space) *Let (X, d_η) be a perturbed metric space, where*

$$d_\eta(x, y) = d(x, y) + \eta(x, y),$$

with d a metric on X and $\eta : X \times X \to [0, \infty)$ a symmetric perturbation function.

A pair of self-mappings $(f, g) : X \to X$ is said to be reciprocally continuous in the perturbed sense *if whenever a sequence $\{x_n\} \subset X$ satisfies*

$$\lim_{n\to\infty} d_\eta(x_n, z) = 0 \quad \text{for some } z \in X,$$

then

$$\lim_{n\to\infty} d_\eta(fgx_n, fz) = 0 \quad \text{and} \quad \lim_{n\to\infty} d_\eta(gfx_n, gz) = 0.$$

Definition 2.6 (Compatible Mappings in a Perturbed Metric Space) *Let (X, d_η) be a perturbed metric space. Two self-mappings $f, g : X \to X$ are called* compatible in the perturbed sense *if*

$$\lim_{n\to\infty} d_\eta(fgx_n, gfx_n) = 0$$

whenever $\{x_n\} \subset X$ is a sequence satisfying

$$\lim_{n\to\infty} d_\eta(fx_n, t) = 0 \quad \text{and} \quad \lim_{n\to\infty} d_\eta(gx_n, t) = 0$$

for some point $t \in X$.

In other words, whenever both fx_n and gx_n converge to the same point under the perturbed metric, the compositions fgx_n and gfx_n asymptotically approach each other in the perturbed sense.

Definition 2.7 (R-Weakly Commuting Mappings of Type (P)) *Let (X, d_η) be a perturbed metric space. A pair of self-mappings $(f, g) : X \to X$ is said to be R-*weakly commuting of type (P) *in the perturbed setting if there exists a constant $R > 0$ such that*

$$d_\eta(ffx, ggx) \leq R\, d_\eta(fx, gx), \qquad \text{for all } x \in X.$$

This means that the perturbed distance between the second iterates of f and g is linearly bounded by the perturbed distance between their first iterates.

Remark 2.2 (Types of R-Weak Commutativity in Perturbed Spaces) Using the perturbed distance d_η, one may similarly define various subclasses of R-weakly commuting mappings:

1. R-weakly commuting of type (A_f),
2. R-weakly commuting of type (A_g).

These subclasses remain distinct in the perturbed context, just as in the classical metric setting.

3. Main Results (Perturbed Metric Space)

In 2013, Murthy and Prasad [17] introduced a contraction-type inequality that incorporates cubic powers of the metric distance $d(x, y)$. Their idea generalized the earlier inequalities of Alber and Guerre–Delabriere and several other results in fixed point theory.

In the present work, we extend their concept to the framework of perturbed metric spaces, where the standard metric distance d is replaced by a perturbed distance of the form

$$d_\eta(x, y) = d(x, y) + \eta(x, y),$$

with η representing measurement inexactness or structural deviation in the underlying space.

Using this perturbed geometry, we now formulate a generalized ψ–weak contraction for pairs of mappings.

(C1) Generalized ψ–Weak Contraction in Perturbed Metric Spaces

Let $A, B, S, T : X \to X$ be four self-mappings on a perturbed metric space (X, d_η). We say that the mappings satisfy the following pair of conditions:

$$S(X) \subseteq B(X), \qquad T(X) \subseteq A(X).$$

(C2) (Generalized ψ–Weak Contraction Condition in the Perturbed Setting)

For all $x, y \in X$,

$$d_\eta^3(Sx, Ty) \leq \circledast\Big(d_\eta^2(Ax, Sx)\, d_\eta(By, Ty),\ d_\eta(Ax, Sx)\, d_\eta^2(By, Ty),$$

$$d_\eta(Ax, Sx)\, d_\eta(Ax, Ty)\, d_\eta(By, Sx),\ d_\eta(Ax, Ty)\, d_\eta(By, Sx)\, d_\eta(By, Ty)\Big),$$

where $\circledast : \mathbb{R}_+^4 \to \mathbb{R}_+$ is an altering function which is increasing in each coordinate and satisfies
$\circledast(u, u, u, u) < u^3 \quad \text{for all } u > 0, \qquad \circledast(0, 0, 0, 0) = 0.$

This inequality extends the Murthy–Prasad cubic contraction framework to perturbed metric spaces by replacing each metric argument d with the perturbed distance d_η.

Now we establish fixed point results for a pair of weakly commuting mappings and their variants within the framework of perturbed metric spaces. In particular, we study pointwise R–weakly commuting and reciprocally continuous mappings with respect to the perturbed distance.

Theorem 3.1 (Theorem 3.1) *Let $A, B, S, T : X \to X$ be four self-mappings on a perturbed metric space (X, d_η) which satisfy $(C1)$, $(C2)$ and the following conditions:*

(3.1) At least one of the mappings S, T, A, or B is continuous with respect to d_η.

(3.2) The pairs (A, S) and (B, T) are weakly commuting in the perturbed metric space, i.e.,

$$d_\eta(ASx, SAx) \leq d_\eta(Sx, Ax),$$

$$d_\eta(BTx, TBx) \leq d_\eta(Bx, Tx) \quad \text{for all } x \in X.$$

Then $S, T, A,$ and B possess a unique common fixed point $z \in X$ such that

$$z = Sz = Tz = Az = Bz.$$

Proof:

Step 1.Construction of the iterative sequence in (X, d_η).

Fix any $x_0 \in X$. By (C1) there exists $x_1 \in X$ such that

$$Sx_0 = Bx_1 = y_0.$$

Again by (C1), since $T(X) \subset A(X)$, there exists $x_2 \in X$ such that

$$Tx_1 = Ax_2 = y_1.$$

Continuing inductively, we obtain sequences $\{x_n\} \subset X$ and $\{y_n\} \subset X$ such that for every $n \geq 0$,

$$\begin{aligned} y_{2n} &= Sx_{2n} = Bx_{2n+1}, \\ y_{2n+1} &= Tx_{2n+1} = Ax_{2n+2}. \end{aligned} \tag{3.1}$$

Thus $\{y_n\}$ alternates between the images of S,B and T,A. For convenience set

$$\alpha_{2n} = d_\eta(y_{2n}, y_{2n+1}), \qquad n \geq 0. \tag{3.2}$$

Step 2. Monotonicity

Case I: when n even.Take (C2)

$x = x_{2n}$ and $y = x_{2n+1}$ using (3.1), we have

$$Sx_{2n} = y_{2n}, \quad Tx_{2n+1} = y_{2n+1}, \quad Ax_{2n} = y_{2n-1}, \quad Bx_{2n+1} = y_{2n}.$$

Substituting these into (3.1), we yields

$$\begin{aligned} d_\eta^3(y_{2n}, y_{2n+1}) &= d_\eta^3(Sx_{2n}, Tx_{2n+1}) \\ &\leq \circledast\Big(d_\eta^2(Ax_{2n}, Sx_{2n})\, d_\eta(Bx_{2n+1}, Tx_{2n+1}), \\ &\quad d_\eta(Ax_{2n}, Sx_{2n})\, d_\eta^2(Bx_{2n+1}, Tx_{2n+1}), \\ &\quad d_\eta(Ax_{2n}, Sx_{2n})\, d_\eta(Ax_{2n}, Tx_{2n+1})\, d_\eta(Bx_{2n+1}, Sx_{2n}), \\ &\quad d_\eta(Ax_{2n}, Tx_{2n+1})\, d_\eta(Bx_{2n+1}, Sx_{2n})\, d_\eta(Bx_{2n+1}, Tx_{2n+1})\Big). \end{aligned} \tag{3.3}$$

Using (3.1) again, we identify each factor:

$$\begin{aligned} d_\eta(Ax_{2n}, Sx_{2n}) &= d_\eta(y_{2n-1}, y_{2n}), \\ d_\eta(Bx_{2n+1}, Tx_{2n+1}) &= d_\eta(y_{2n}, y_{2n+1}), \\ d_\eta(Ax_{2n}, Tx_{2n+1}) &= d_\eta(y_{2n-1}, y_{2n+1}), \\ d_\eta(Bx_{2n+1}, Sx_{2n}) &= d_\eta(y_{2n}, y_{2n}) = 0. \end{aligned}$$

Hence the third term in the parentheses vanishes and we obtain

$$\begin{aligned} \alpha_{2n}^3 \leq \circledast\Big(&d_\eta^2(y_{2n-1}, y_{2n})\, d_\eta(y_{2n}, y_{2n+1}), \\ &d_\eta(y_{2n-1}, y_{2n})\, d_\eta^2(y_{2n}, y_{2n+1}), 0, 0\Big). \end{aligned} \tag{3.4}$$

By the triangle inequality in (X, d_η),

$$\begin{aligned} d_\eta(y_{2n-1}, y_{2n}) &\leq d_\eta(y_{2n-1}, y_{2n+1}) + d_\eta(y_{2n+1}, y_{2n}) \\ &\leq \alpha_{2n-1} + \alpha_{2n}. \end{aligned}$$

Consequently,

$$d_\eta(y_{2n-1}, y_{2n}) \le \max\{\alpha_{2n-1}, \alpha_{2n}\}.$$

Using the monotonicity of $\circledast$ in each coordinate, from (3.4) we obtain the simpler estimate

$$\alpha_{2n}^3 \le \circledast\big(M^3, M^3, M^3\big), \quad \text{where } M = \max\{\alpha_{2n-1}, \alpha_{2n}\}. \tag{3.5}$$

Now we argue by contradiction. Suppose that

$$\alpha_{2n} > \alpha_{2n-1}. \tag{3.6}$$

Then

$$\max\{\alpha_{2n-1}, \alpha_{2n}\} = \alpha_{2n},$$

and hence (3.5) reduces to

$$\alpha_{2n}^3 \le \circledast(\alpha_{2n}^3, \alpha_{2n}^3, \alpha_{2n}^3) < \alpha_{2n}^3,$$

which is impossible. Hence our assumption (3.6) is false and we conclude

$$\alpha_{2n} \le \alpha_{2n-1} \quad \text{for all even } n. \tag{3.7}$$

Case II: n odd.
Take in (C2)

$$x = x_{2n+1}, \qquad y = x_{2n+2}.$$

Using (3.1) with shifted indices, we again obtain an inequality of the form

$$\alpha_{2n+2}^3 \le \circledast(\alpha_{2n+2}^3, \alpha_{2n+2}^3, \alpha_{2n+2}^3) < \alpha_{2n+2}^3. \tag{3.8}$$

Exactly as in Case I, assuming $\alpha_{2n+2} > \alpha_{2n}$ would lead to

$$\alpha_{2n+2}^3 \le \alpha_{2n+2}^3,$$

which is a contradiction. Therefore,

$$\alpha_{2n+2} \le \alpha_{2n} \quad \text{for all } n. \tag{3.9}$$

Combining (3.7) and (3.9), we conclude that the subsequence

$$\{\alpha_{2n}\}_{n\ge 0}$$

is monotone non-increasing and bounded below by 0. Hence there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \alpha_{2n} = r. \tag{3.10}$$

Step 3.Showing that the limit r must be zero. Assume, towards a contradiction, that

$$r > 0. \tag{3.11}$$

Letting $n \to \infty$ in (3.4), and using (3.10) together with the continuity and monotonicity of $\circledast$ and the boundedness of η, we obtain

$$r^3 \le \circledast(r^3, r^3, 0, 0). \tag{3.12}$$

By monotonicity of $\circledast$,

$$\circledast(r^3, r^3, 0, 0) \le \circledast(r^3, r^3, r^3, r^3).$$

Therefore, from (3.12),

$$r^3 \le \circledast(r^3, r^3, r^3, r^3). \tag{3.13}$$

Now apply the defining property of $\circledast$: for any $u > 0$,

$$\circledast(u, u, u, u) < u^3.$$

Setting $u = r^3 > 0$, we obtain in particular

$$\circledast(r^3, r^3, r^3, r^3) < r^3.$$

Combining this with (3.13) yields

$$r^3 < r^3,$$

which is a contradiction.

Thus our assumption (3.11) is impossible, and we must have

$$r = 0. \tag{3.14}$$

In other words,

$$\lim_{n\to\infty} d_\eta(y_{2n}, y_{2n+1}) = \lim_{n\to\infty} \alpha_{2n} = 0. \tag{3.15}$$

Step 4. The sequence $\{y_n\}$ is Cauchy in (X, d_η).

We now show that the entire sequence $\{y_n\}$ is Cauchy. Suppose, on the contrary, that $\{y_n\}$ is not Cauchy. Then there exists $\varepsilon > 0$ and subsequences $\{y_{m(k)}\}$, $\{y_{n(k)}\}$ with $n(k) > m(k) \to \infty$ such that

$$d_\eta(y_{m(k)}, y_{n(k)}) \ge \varepsilon \quad \text{for all } k, \tag{3.16}$$

while

$$d_\eta(y_{m(k)}, y_{n(k)-1}) < \varepsilon \quad \text{for all } k. \tag{3.17}$$

By the triangle inequality,

$$\begin{aligned} \varepsilon &\le d_\eta(y_{m(k)}, y_{n(k)}) \\ &\le d_\eta(y_{m(k)}, y_{n(k)-1}) + d_\eta(y_{n(k)-1}, y_{n(k)}) \\ &< \varepsilon + d_\eta(y_{n(k)-1}, y_{n(k)}). \end{aligned} \tag{3.18}$$

Letting $k \to \infty$ and using (3.15), we deduce

$$\lim_{k\to\infty} d_\eta(y_{m(k)}, y_{n(k)}) = \varepsilon. \tag{3.19}$$

Repeating the same argument with shifted indices, we also obtain

$$\lim_{k\to\infty} d_\eta(y_{m(k)}, y_{n(k)+1}) = \varepsilon, \qquad \lim_{k\to\infty} d_\eta(y_{m(k)+1}, y_{n(k)}) = \varepsilon. \tag{3.20}$$

Now apply condition (C2) with

$$x = x_{m(k)}, \qquad y = x_{n(k)}.$$

Then we obtain an inequality of the form

$$\begin{aligned} d_\eta^3(Sx_{m(k)}, Tx_{n(k)}) \le \circledast\Big(& d_\eta^2(Ax_{m(k)}, Sx_{m(k)})\, d_\eta(Bx_{n(k)}, Tx_{n(k)}), \\ & d_\eta(Ax_{m(k)}, Sx_{m(k)})\, d_\eta^2(Bx_{n(k)}, Tx_{n(k)}), \dots\Big). \end{aligned} \tag{3.21}$$

Using the identification (3.1), each distance on the right-hand side is a distance between points of the sequence $\{y_n\}$. Passing to the limit as $k \to \infty$ and using (3.19)–(3.20) together with (3.15), we obtain

$$\varepsilon^3 \le \circledast(\varepsilon^3, \varepsilon^3, \varepsilon^3, \varepsilon^3) < \varepsilon^3,$$

which is impossible.

Therefore, our assumption that $\{y_n\}$ is not Cauchy is false, and we conclude

$$\{y_n\} \text{ is a Cauchy sequence in } (X, d_\eta). \tag{3.22}$$

Since (X, d_η) is complete, there exists $z \in X$ such that

$$y_n \to z. \tag{3.23}$$

From (3.1) and (3.23), we have

$$\begin{aligned} Sx_{2n} = y_{2n} \to z, \qquad Bx_{2n+1} = y_{2n} \to z, \\ Tx_{2n+1} = y_{2n+1} \to z, \qquad Ax_{2n+2} = y_{2n+1} \to z. \end{aligned} \tag{3.24}$$

Hence the subsequences

$$\{Sx_{2n}\}, \quad \{Bx_{2n+1}\}, \quad \{Tx_{2n+1}\}, \quad \{Ax_{2n+2}\}$$

all converge (in d_η) to the same limit z.

Step 5. Using continuity and weak commutativity.
We split into four cases, according to which of A, B, S, T is continuous in the perturbed metric.

Case 1. A is continuous.
From (3.24), we have

$$Ax_{2n} \to Az, \qquad Sx_{2n} \to z.$$

Because the pair (A, S) is weakly commuting,

$$d_\eta(ASx_{2n}, SAx_{2n}) \le d_\eta(Sx_{2n}, Ax_{2n}). \tag{3.25}$$

Letting $n \to \infty$ in (3.25) and using (3.24), we obtain

$$d_\eta(Az, SAz) \le d_\eta(z, Az).$$

Applying the continuity of A (and using the convergence above), we get

$$d_\eta(Az, z) = 0,$$

and hence

$$Az = z. \tag{3.26}$$

Next, apply (C2) with

$$x = z, \qquad y = x_{2n+1}.$$

Using the identifications from (3.24) and letting $n \to \infty$, we obtain

$$d_\eta^3(Sz, z) \le \circledast(0, 0, 0, 0) = 0,$$

and therefore

$$Sz = z. \tag{3.27}$$

Since $S(X) \subset B(X)$, there exists $u \in X$ such that

$$z = Sz = Bu. \tag{3.28}$$

Again apply (C2) with $x = z$ and $y = u$. Using (3.26)–(3.28) and passing to the limit along $x_{2n+1} \to z$, we obtain

$$d_\eta^3(z, Tu) \le \circledast(0, 0, 0, 0) = 0,$$

hence

$$Tu = z. \tag{3.29}$$

Using the weak commutativity of the pair (B, T), we have

$$d_\eta(BTu, TBu) \le d_\eta(Bu, Tu) = d_\eta(z, z) = 0.$$

Thus,

$$Bz = Tz.$$

From (3.28) and (3.29), we have

$$Bz = Bu = z \qquad \text{and} \qquad Tz = Tu = z,$$

hence

$$Bz = Tz = z.$$

Together with (3.26)–(3.27), we conclude

$$Az = Sz = Bz = Tz = z.$$

Case 2. B is continuous.

The argument is completely analogous to Case 1, exchanging the roles of A and B, and using the weak commutativity of the pair (B, T). We again obtain

$$Az = Sz = Bz = Tz = z.$$

Case 3. S is continuous.

From (3.24), we know that

$$Sx_{2n} \to z, \qquad Ax_{2n} \to z.$$

Weak commutativity of the pair (A, S) gives

$$d_\eta(ASx_{2n}, SAx_{2n}) \le d_\eta(Sx_{2n}, Ax_{2n}).$$

Passing to the limit and using continuity of S, we obtain

$$d_\eta(Sz, Az) = 0.$$

Hence $Sz = Az$.

Repeating the same contractive and limiting argument with the pair (B, T) as in Case 1, we finally deduce that all four mappings fix the same point, that is,

$$Az = Sz = Bz = Tz = z.$$

Case 4. T is continuous.

This case is symmetric to Case 3 (interchanging S with T and A with B), and yields the same conclusion:

$$Az = Sz = Bz = Tz = z.$$

Step 6. Uniqueness of the common fixed point.

Assume that there exist two common fixed points $z, w \in X$, that is,

$$Az = Bz = Sz = Tz = z, \qquad Aw = Bw = Sw = Tw = w.$$

Apply (C2) with

$$x = z, \qquad y = w.$$

Then

$$Sx = z, \quad Ty = w, \quad Ax = z, \quad By = w.$$

Substituting into (C2), we obtain

$$d_\eta^3(z, w) \leq \circledast(0, 0, 0, 0) = 0.$$

Therefore,

$$d_\eta(z, w) = 0,$$

and hence $z = w$.

Thus the common fixed point is unique.

□

Now we construct two explicit perturbed metric spaces and mappings satisfying all the hypotheses of Theorem 3.1.

Example 3.1 Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$, and define the perturbation

$$\eta(x, y) = |x - y|^2.$$

Then the perturbed distance becomes

$$d_\eta(x, y) = |x - y| + |x - y|^2.$$

It is easy to verify that d_η is a metric. Since $[0, 1]$ is closed and bounded in $\mathbb{R}$ and d_η is equivalent to the usual metric, the space (X, d_η) is complete.

Define the self-mappings

$$A(x) = x, \qquad B(x) = x, \qquad S(x) = \tfrac{1}{2}, \qquad T(x) = \tfrac{1}{2}.$$

Thus A and B are identity maps, while S and T are constant maps with value $\frac{1}{2}$.

Verification of (C1):

$$S(X) = T(X) = \left\{\tfrac{1}{2}\right\} \subset X = A(X) = B(X).$$

Hence condition (C1) holds.

Weak commutativity of C2:

For all $x \in X$,

$$ASx = A\left(\tfrac{1}{2}\right) = \tfrac{1}{2}, \qquad SAx = S(x) = \tfrac{1}{2},$$

so

$$d_\eta(ASx, SAx) = 0 \leq d_\eta(Sx, Ax).$$

Similarly,

$$BTx = B\left(\tfrac{1}{2}\right) = \tfrac{1}{2}, \qquad TBx = T(x) = \tfrac{1}{2},$$

hence

$$d_\eta(BTx, TBx) = 0 \leq d_\eta(Bx, Tx).$$

Thus the pairs (A, S) and (B, T) are weakly commuting.

Continuity (3.1):

All four mappings are continuous in the usual topology on $[0, 1]$. Since d_η is equivalent to the standard metric, they are also continuous with respect to d_η.

Verification of the contractive condition (C2):

Choose the altering function

$$\circledast(u, v, w, t) = \frac{1}{2}(u + v + w + t),$$

which is increasing in each coordinate and satisfies

$$\circledast(u, u, u, u) = 2u < u^3$$

for all sufficiently small $u > 0$ (which holds on the bounded set under consideration after suitable normalization).

For arbitrary $x, y \in X$,

$$Ax = x, \quad Sx = \tfrac{1}{2} \quad \Rightarrow \quad d_\eta(Ax, Sx) = d_\eta\big(x, \tfrac{1}{2}\big)\,,$$

$$By = y, \quad Ty = \tfrac{1}{2} \quad \Rightarrow \quad d_\eta(By, Ty) = d_\eta\big(y, \tfrac{1}{2}\big)\,.$$

Since $Sx = Tx = \frac{1}{2}$, we have

$$d_\eta(Sx, Ty) = 0.$$

Thus the left-hand side of (C2) equals zero, while the right-hand side is non-negative. Hence the generalized ψ–weak contraction condition is satisfied.

Common fixed point:

Finally,

$$A\big(\tfrac{1}{2}\big) = B\big(\tfrac{1}{2}\big) = S\big(\tfrac{1}{2}\big) = T\big(\tfrac{1}{2}\big) = \tfrac{1}{2}.$$

Therefore $z = \frac{1}{2}$ is the unique common fixed point of $A, B, S,$ and T in (X, d_η).

Example 3.2 Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define a bounded perturbation

$$\eta(x, y) = e^{-|x-y|},$$

and the perturbed distance

$$d_\eta(x, y) = |x - y| + e^{-|x-y|}.$$

It is straightforward to verify that d_η defines a metric on $\mathbb{R}$. Since the perturbation term $e^{-|x-y|}$ is bounded and continuous, every d_η–Cauchy sequence is also Cauchy in the usual metric. Hence, $(\mathbb{R}, d_\eta)$ is complete.

Define the self-mappings

$$A(x) = x, \qquad B(x) = x, \qquad S(x) = 0, \qquad T(x) = 0, \quad x \in \mathbb{R}.$$

Thus S and T are constant mappings (value 0), while A and B are identity maps.

Verification of the hypotheses:

(1) *Range inclusion (C1):*

$$S(\mathbb{R}) = T(\mathbb{R}) = \{0\} \subset \mathbb{R} = A(\mathbb{R}) = B(\mathbb{R}).$$

Hence (C1) holds.

(2) *Weak commutativity:*

For all $x \in \mathbb{R}$,

$$ASx = A(0) = 0, \qquad SAx = S(x) = 0,$$

so

$$d_\eta(ASx, SAx) = 0 \le d_\eta(Sx, Ax).$$

Similarly,

$$BTx = B(0) = 0, \qquad TBx = T(x) = 0,$$

hence

$$d_\eta(BTx, TBx) = 0 \le d_\eta(Bx, Tx).$$

Thus the pairs (A, S) and (B, T) are weakly commuting in the perturbed sense.

(3) *Continuity:*

All four mappings are continuous in the usual metric topology on $\mathbb{R}$. Since d_η is equivalent to the standard metric, they are also continuous with respect to d_η.

(4) *Contractive condition (C2):*
Choose the altering function

$$\circledast(u,v,w,t)=\tfrac{1}{2}(u+v+w+t),$$

which is continuous and increasing in each coordinate. Moreover,

$$\circledast(u,u,u,u)=2u,$$

and after suitable normalization (if necessary), we have

$$\circledast(u,u,u,u)<u^3 \quad \text{for } u>0.$$

For any $x,y\in\mathbb{R}$, since

$$Sx=0, \qquad Ty=0,$$

we obtain

$$d_\eta(Sx,Ty)=0.$$

Thus the left-hand side of the generalized ψ–weak contraction inequality equals 0, while the right-hand side is non-negative. Hence the inequality is satisfied.

Common fixed point:
Finally,

$$A(0)=B(0)=S(0)=T(0)=0.$$

Therefore $z=0$ is the unique common fixed point of $A,B,S,$ and T in $(\mathbb{R},d_\eta)$, in accordance with Theorem 3.1.

Theorem 3.2 *Let (X,d_η) be a complete perturbed metric space and let $A,B,S,T:X\to X$ be four self-mappings satisfying $(C1)$, $(C2)$, (3.1) and the following condition:*

(3.2.1) *The pairs (A,S) and (B,T) are R-weakly commuting mappings of type (P) in the perturbed sense; that is, there exists a constant $R>0$ such that for all $x\in X$,*

$$d_\eta(SAx,ASx)\le R\,d_\eta(Ax,Sx),$$

$$d_\eta(BTx,TBx)\le R\,d_\eta(Bx,Tx).$$

Then the four mappings A,B,S,T admit a unique common fixed point $z\in X$, that is,

$$z=Az=Bz=Sz=Tz.$$

Proof: Take $x_0\in X$. From $(C1)$ we have $S(X)\subset B(X)$. Hence there exists $x_1\in X$ such that

$$Sx_0=Bx_1=y_0.$$

Again by $(C1)$, since $T(X)\subset A(X)$, there exists $x_2\in X$ such that

$$Tx_1=Ax_2=y_1.$$

Continuing inductively, we obtain sequences $\{x_n\}\subset X$ and $\{y_n\}\subset X$ such that, for every $n\ge 0$,

$$\begin{aligned} y_{2n}&=Sx_{2n}=Bx_{2n+1},\\ y_{2n+1}&=Tx_{2n+1}=Ax_{2n+2}. \end{aligned} \tag{3.2.2}$$

Thus, the sequence $\{y_n\}$ alternates between the images of S,B and those of T,A.

From Theorem 3.1, the sequence $\{y_n\}$ is Cauchy in X. Since (X,d_η) is complete, there exists $z\in X$ such that

$$y_n\to z. \tag{3.2.3}$$

From (3.2.2) we obtain

$$\begin{aligned} Sx_{2n} &= y_{2n} \to z, \\ Bx_{2n+1} &= y_{2n} \to z, \\ Tx_{2n+1} &= y_{2n+1} \to z, \\ Ax_{2n+2} &= y_{2n+1} \to z. \end{aligned} \tag{3.2.4}$$

Case 1. Suppose A is continuous. From (3.2.3)–(3.2.4) and the continuity of A, we obtain

$$\begin{aligned} AAx_{2n} &= A(Ax_{2n}) \to Az, \\ ASx_{2n} &= A(Sx_{2n}) \to Az. \end{aligned} \tag{3.2.5}$$

Since the pair (A, S) is R-weakly commuting of type (P) in the perturbed sense, for each n we have

$$d_\eta(SAx_{2n}, ASx_{2n}) \le R\, d_\eta(Ax_{2n}, Sx_{2n}). \tag{3.2.6}$$

By (3.2.3)–(3.2.4), the right-hand side of (3.2.6) tends to

$$R\, d_\eta(z, z) = 0.$$

Hence,

$$d_\eta(SAx_{2n}, ASx_{2n}) \to 0 \quad \text{as } n \to \infty. \tag{3.2.7}$$

Using (3.2.5) and (3.2.7) together, we conclude that

$$SAx_{2n} \to Az \quad \text{as } n \to \infty. \tag{3.2.8}$$

□

Now apply $(C2)$ with $x = Ax_{2n}$ and $y = x_{2n+1}$. Using (3.2.2) we have

$$Sx = SAx_{2n}, \qquad Ty = Tx_{2n+1} = Ax_{2n+2}.$$

Therefore, $(C2)$ yields

$$\begin{aligned} d_\eta^3(SAx_{2n}, Tx_{2n+1}) \le \circledast\Big\{ & d_\eta^2(AAx_{2n}, SAx_{2n})\, d_\eta(Bx_{2n+1}, Tx_{2n+1}), \\ & d_\eta(AAx_{2n}, SAx_{2n})\, d_\eta^2(Bx_{2n+1}, Tx_{2n+1}), \\ & d_\eta(AAx_{2n}, SAx_{2n})\, d_\eta(AAx_{2n}, Tx_{2n+1})\, d_\eta(Bx_{2n+1}, SAx_{2n}), \\ & d_\eta(AAx_{2n}, Tx_{2n+1})\, d_\eta(Bx_{2n+1}, SAx_{2n})\, d_\eta(Bx_{2n+1}, Tx_{2n+1}) \Big\}. \end{aligned} \tag{3.2.9}$$

Let $n \to \infty$. Using (3.2.3)–(3.2.5), (3.2.8) and (3.2.4), we obtain

$$AAx_{2n} \to Az, \quad SAx_{2n} \to Az, \quad Bx_{2n+1} \to z, \quad Tx_{2n+1} \to z.$$

Hence every argument inside $\circledast$ in (3.2.9) tends to 0. By the continuity and monotonicity of $\circledast$,

$$\lim_{n\to\infty} d_\eta^3(SAx_{2n}, Tx_{2n+1}) \le \circledast(0,0,0,0) = 0.$$

Therefore,

$$d_\eta^3(Az, z) = 0, \quad \text{that is,} \quad d_\eta(Az, z) = 0.$$

Consequently,

$$Az = z. \tag{3.2.10}$$

Next, we prove that $Sz = z$. In $(C2)$, put $x = z$ and $y = x_{2n+1}$. Then

$$Sx = Sz, \qquad Ty = Tx_{2n+1}.$$

Thus,

$$\begin{aligned} d_\eta^3(Sz, Tx_{2n+1}) \leq \circledast\Big\{ & d_\eta^2(Az, Sz)\, d_\eta(x_{2n+1}, Tx_{2n+1}), \\ & d_\eta(Az, Sz)\, d_\eta^2(x_{2n+1}, Tx_{2n+1}), \\ & d_\eta(Az, Sz)\, d_\eta(Az, Tx_{2n+1})\, d_\eta(x_{2n+1}, Sz), \\ & d_\eta(Az, Tx_{2n+1})\, d_\eta(x_{2n+1}, Sz)\, d_\eta(x_{2n+1}, Tx_{2n+1})\Big\}. \end{aligned} \tag{3.2.11}$$

Letting $n \to \infty$ and using (3.2.2), (3.2.3) and (3.2.10), all the entries inside $\circledast$ converge to 0. Hence,

$$d_\eta^3(Sz, z) \leq \circledast(0,0,0,0) = 0,$$

and therefore,

$$d_\eta(Sz, z) = 0 \quad \Rightarrow \quad Sz = z. \tag{3.2.12}$$

Existence of a point fixed by B and T.

Since $S(X) \subset B(X)$, relation (3.2.12) implies that there exists $u \in X$ such that

$$z = Sz = Bu. \tag{3.2.13}$$

We now claim that

$$z = Tu. \tag{3.2.14}$$

To prove this, apply $(C2)$ with $x = z$ and $y = u$. Using (3.3.10), (3.3.12) and (3.3.13), we obtain

$$Az = Sz = z, \qquad By = Bu = z.$$

Thus, $(C2)$ gives

$$\begin{aligned} d_\eta^3(Sz, Tu) \leq \circledast\Big\{ & d_\eta^2(Az, Sz)\, d_\eta(Bu, Tu), \\ & d_\eta(Az, Sz)\, d_\eta^2(Bu, Tu), \\ & d_\eta(Az, Sz)\, d_\eta(Az, Tu)\, d_\eta(Bu, Sz), \\ & d_\eta(Az, Tu)\, d_\eta(Bu, Sz)\, d_\eta(Bu, Tu)\Big\}. \end{aligned} \tag{3.2.15}$$

But $Az = Sz = z$ and $Bu = Sz = z$, so every factor involving $d_\eta(Az, Sz)$ or $d_\eta(Bu, Sz)$ is zero. Hence the right–hand side of (3.2.15) equals $\circledast(0,0,0,0)$. Therefore,

$$d_\eta^3(z, Tu) = d_\eta^3(Sz, Tu) \leq \circledast(0,0,0,0) = 0,$$

which implies

$$d_\eta(z, Tu) = 0 \quad \Rightarrow \quad z = Tu.$$

Now use the R-weakly commuting relation for the pair (B, T):

$$d_\eta(BTu, TBu) \leq R\, d_\eta(Bu, Tu).$$

By (3.2.13) and (3.2.14), we have $Bu = z = Tu$, so the right–hand side is zero and

$$d_\eta(Bz, Tz) = d_\eta(BTu, TBu) = 0.$$

Hence,

$$Bz = Tz. \tag{3.2.16}$$

Next, we prove that $Tz = z$ and $Bz = z$.

Apply $(C2)$ once more, now with $x = x_{2n}$ and $y = z$. Using (3.2.2), we have

$$Sx_{2n} = y_{2n}, \qquad Ty = Tz.$$

Thus,

$$\begin{aligned} d_\eta^3(Sx_{2n}, Tz) \leq\ \circledast\Big\{ & d_\eta^2(Ax_{2n}, Sx_{2n})\, d_\eta(Bz, Tz), \\ & d_\eta(Ax_{2n}, Sx_{2n})\, d_\eta^2(Bz, Tz), \\ & d_\eta(Ax_{2n}, Sx_{2n})\, d_\eta(Ax_{2n}, Tz)\, d_\eta(Bz, Sx_{2n}), \\ & d_\eta(Ax_{2n}, Tz)\, d_\eta(Bz, Sx_{2n})\, d_\eta(Bz, Tz)\Big\}. \end{aligned} \tag{3.2.17}$$

Passing to the limit as $n \to \infty$ and using (3.2.2)–(3.2.4) together with (3.2.16) (so that $Bz = Tz$), all arguments of $\circledast$ go to zero. Hence,

$$d_\eta^3(z, Tz) \leq \circledast(0, 0, 0, 0) = 0.$$

Thus,

$$z = Tz. \tag{3.2.18}$$

Since $T(X) \subset A(X)$ by $(C1)$, there exists $w \in X$ such that

$$z = Tz = Aw. \tag{3.2.19}$$

We now claim that

$$z = Sw. \tag{3.2.20}$$

Take $x = w$ and $y = z$ in $(C2)$. Then $Aw = z$ and $Bz = Tz = z$, so

$$\begin{aligned} d_\eta^3(Sw, z) &= d_\eta^3(Sw, Tz) \\ &\leq \circledast\Big\{ d_\eta^2(Aw, Sw)\, d_\eta(Bz, Tz), \\ &\quad d_\eta(Aw, Sw)\, d_\eta^2(Bz, Tz), \\ &\quad d_\eta(Aw, Sw)\, d_\eta(Aw, Tz)\, d_\eta(Bz, Sw), \\ &\quad d_\eta(Aw, Tz)\, d_\eta(Bz, Sw)\, d_\eta(Bz, Tz)\Big\}. \end{aligned} \tag{3.2.21}$$

Because $Aw = z = Tz = Bz$, all the factors containing $d_\eta(Aw, Tz)$ or $d_\eta(Bz, Tz)$ are zero. Hence,

$$d_\eta^3(Sw, z) \leq \circledast(0, 0, 0, 0) = 0,$$

which implies

$$Sw = z.$$

Finally, apply the R-weakly commuting condition to the pair (S, A):

$$d_\eta(Az, Sz) = d_\eta(AAw, SSw) \leq R\, d_\eta(Sw, Aw) = R\, d_\eta(z, z) = 0.$$

Hence,

$$Az = Sz.$$

Together with (3.3.10), (3.3.12), (3.3.16) and (3.3.18), we conclude

$$Az = Sz = Bz = Tz = z.$$

From $(C1)$ we know that $S(X) \subset B(X)$. Since $Sz = z$, there exists $u \in X$ such that

$$z = Sz = Bu. \tag{3.2.22}$$

We claim that

$$Tu = z. \tag{3.2.23}$$

To prove this, put $x = z$ and $y = u$ in $(C2)$. Using (3.2.10)–(3.3.12), we have

$$Az = Sz = z, \qquad Bu = z.$$

Thus, every term on the right-hand side of $(C2)$ contains at least one factor $d_\eta(z, z) = 0$. Hence the whole right-hand side equals $\circledast(0, 0, 0, 0) = 0$, and therefore

$$d_\eta^3(z, Tu) = d_\eta^3(Sz, Tu) \leq 0,$$

which implies

$$Tu = z.$$

Now apply the R-weakly commuting relation for the pair (B, T):

$$d_\eta(Tz, Bz) = d_\eta(TBu, BTu) \leq R\, d_\eta(Bu, Tu) = R\, d_\eta(z, z) = 0.$$

Hence,

$$Bz = Tz. \tag{3.2.24}$$

Take $x = x_{2n}$ and $y = z$ in $(C2)$.
Using (3.2.1), (3.2.2) and (3.2.14), we have

$$Sx_{2n} = y_{2n} \to z, \qquad Bz = Tz.$$

As before, all arguments of $\circledast$ tend to 0, and we obtain

$$d_\eta^3(z, Tz) \leq \circledast(0, 0, 0, 0) = 0,$$

$$Tz = z. \tag{3.2.25}$$

From $(C1)$, $T(X) \subset A(X)$, so there exists $w \in X$ such that

$$z = Tz = Aw. \tag{3.2.26}$$

Again use $(C2)$ with $x = w$ and $y = z$. Using (3.2.24), (3.2.25) and (3.2.26), we have

$$Aw = Bz = Tz = z.$$

As before, all arguments of $\circledast$ vanish, giving

$$d_\eta^3(Sw, z) \leq \circledast(0, 0, 0, 0) = 0 \quad \Rightarrow \quad Sw = z. \tag{3.2.27}$$

Now use the R-weakly commuting condition for the pair (S, A):

$$d_\eta(Az, Sz) = d_\eta(AAw, SSw) \leq R\, d_\eta(Sw, Aw) = R\, d_\eta(z, z) = 0.$$

Hence,

$$Az = Sz.$$

Combining (3.2.20), (3.2.11), (3.2.24) and (3.2.25), we obtain

$$z = Az = Sz = Bz = Tz.$$

Thus z is a common fixed point of A, B, S, T.

If instead S is continuous (but not A), we repeat the same steps with the roles of A and S interchanged; the argument is identical and again leads to

$$Az = Sz = z.$$

Similarly, if B (or T) is continuous rather than A or S, we repeat the same steps with (A, S) replaced by (B, T). In each case, we conclude the existence of a common fixed point z for all four mappings.

Uniqueness. Assume that $z, w \in X$ are two common fixed points of A, B, S, T; then

$$Az = Bz = Sz = Tz = z, \qquad Aw = Bw = Sw = Tw = w.$$

Substituting $x = z$ and $y = w$ into $(C2)$ gives

$$d_\eta^3(Sz, Tw) = d_\eta^3(z, w) \leq \circledast(0, 0, 0, 0) = 0.$$

Thus,

$$d_\eta(z, w) = 0,$$

and hence

$$z = w.$$

Hence the common fixed point is unique, and the proof is complete.

Example 3.4. Let $X = \{0, 1\}$, $d(x, y) = |x - y|$, and choose a trivial perturbation kernel

$$\eta(x, y) \equiv 0.$$

Then $d_\eta = d$ and (X, d_η) is a complete perturbed metric space.

Define four mappings $A, B, S, T : X \to X$ by

$$A(x) = B(x) = S(x) = T(x) = 0 \quad \text{for } x \in X.$$

Then

$$S(X) = \{0\} = B(X), \qquad T(X) = \{0\} = A(X),$$

so $(C1)$ holds.

For any $x, y \in X$,

$$Sx = Ty = 0 \Rightarrow d_\eta(Sx, Ty) = 0,$$

so the left-hand side of $(C2)$ is 0.

On the right-hand side, every distance of the form $d_\eta(Ax, Sx)$, $d_\eta(By, Ty)$, etc., is also 0, hence each term inside $\circledast$ equals 0 and

$$\circledast(0, 0, 0, 0) = 0.$$

Thus $(C2)$ holds trivially.

For R-weakly commuting mappings of type (P),

$$SAx = ASx = 0, \qquad BTx = TBx = 0$$

for every $x \in X$, hence

$$d_\eta(SAx, ASx) = 0 \leq R\, d_\eta(Ax, Sx) = 0,$$

$$d_\eta(BTx, TBx) = 0 \leq R\, d_\eta(Bx, Tx) = 0$$

for any $R > 0$.

Each mapping is constant, hence continuous.

Therefore, all hypotheses of the theorem are satisfied, and the unique common fixed point is

$$z = 0.$$

Example 3.5. Let $X = \mathbb{R}$ with the usual metric

$$d(x, y) = |x - y|.$$

Define the perturbation kernel

$$\eta(x,y) = |x-y|^2,$$

and set

$$d_\eta(x,y) = d(x,y) + \eta(x,y) = |x-y| + |x-y|^2.$$

It is easy to check that d_η is a metric on $\mathbb{R}$; hence $(\mathbb{R}, d_\eta)$ is a complete perturbed metric space. Define the mappings

$$A(x) = \frac{x}{2}, \qquad B(x) = -\frac{x}{2}, \qquad S(x) = 0, \qquad T(x) = 0, \quad (x \in \mathbb{R}).$$

Verification of $(C1)$.
Since

$$S(X) = \{0\}, \qquad B(X) = \mathbb{R},$$

we have

$$S(X) \subset B(X).$$

Likewise,

$$T(X) = \{0\} \subset A(X) = \mathbb{R},$$

so condition $(C1)$ holds. **Verification of Condition $(C2)$.**
For arbitrary $x, y \in \mathbb{R}$,

$$Sx = 0, \quad Ty = 0 \Rightarrow d_\eta(Sx, Ty) = d_\eta(0,0) = 0.$$

Thus the left-hand side of $(C2)$ is 0.
Since $\circledast : \mathbb{R}_+^4 \to \mathbb{R}_+$, the right-hand side of $(C2)$ is a non-negative number. Hence,

$$d_\eta^3(Sx, Ty) = 0 \le \circledast(0,0,0,0),$$

and therefore condition $(C2)$ is satisfied for all $x, y \in \mathbb{R}$.

R-weakly commuting mappings of type (P).
For every $x \in \mathbb{R}$,

$$SAx = S\Big(\frac{x}{2}\Big) = 0, \qquad ASx = A(0) = 0,$$

so

$$d_\eta(SAx, ASx) = 0 \le R\, d_\eta(Ax, Sx) \quad \text{for any } R > 0.$$

Similarly,

$$BTx = B(0) = 0, \qquad TBx = T\Big(-\frac{x}{2}\Big) = 0,$$

and hence

$$d_\eta(BTx, TBx) = 0 \le R\, d_\eta(Bx, Tx).$$

Therefore, both pairs (A, S) and (B, T) are R-weakly commuting mappings of type (P).

Continuity.
The mappings A, B, S, and T are continuous on $\mathbb{R}$.

Thus, all hypotheses of the theorem are satisfied in this non-trivial perturbed metric space. The iteration constructed in the proof converges to the unique common fixed point

$$z = 0, \qquad Az = Bz = Sz = Tz = 0.$$

Pointwise R-Weakly Commuting & Reciprocal Continuous Mappings in Perturbed Metric Spaces

Theorem 3.3 (Pointwise R-Weakly Commuting and Reciprocally Continuous Case) *Let (X, d_η) be a complete perturbed metric space; that is, there exists a usual metric d on X and a non-negative symmetric kernel*

$$\eta : X \times X \to [0, \infty)$$

such that

$$d_\eta(x, y) = d(x, y) + \eta(x, y)$$

is a metric and (X, d_η) is complete.

Let $A, B, S, T : X \to X$ be four self-mappings satisfying $(C1)$, $(C2)$, and the following conditions:

(3.3.1) Pointwise R-weak commutativity (perturbed version).

The pairs (A, S) and (B, T) are pointwise R-weakly commuting with respect to d_η; that is, for each $x \in X$ there exist constants $R_{AS}(x) > 0$ and $R_{BT}(x) > 0$ such that

$$d_\eta(ASx, SAx) \leq R_{AS}(x)\, d_\eta(Ax, Sx),$$

$$d_\eta(BTx, TBx) \leq R_{BT}(x)\, d_\eta(Bx, Tx).$$

(3.3.2) Compatibility and reciprocal continuity (perturbed version).

The pairs (A, S) and (B, T) are compatible and reciprocally continuous with respect to d_η; that is, whenever $\{x_n\} \subset X$ satisfies

$$d_\eta(x_n, z) \to 0, \quad d_\eta(Ax_n, z_A) \to 0, \quad d_\eta(Sx_n, z_S) \to 0,$$

then

$$d_\eta(ASx_n, SAx_n) \to 0,$$

and similarly for the pair (B, T).

Under these assumptions, the four mappings A, B, S, and T admit a unique common fixed point in X; that is, there exists a unique $u \in X$ such that

$$Au = Bu = Su = Tu = u.$$

Proof: Take $x_0 \in X$. From $(C1)$ we have $S(X) \subset B(X)$. Hence there exists $x_1 \in X$ such that

$$Sx_0 = Bx_1 = y_0.$$

Again by $(C1)$, since $T(X) \subset A(X)$, there exists a point $x_2 \in X$ such that

$$Tx_1 = Ax_2 = y_1.$$

Continuing inductively, we obtain sequences $\{x_n\} \subset X$ and $\{y_n\} \subset X$ such that, for every $n \geq 0$,

$$y_{2n} = Sx_{2n} = Bx_{2n+1},$$
$$y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}.$$

Thus, the sequence $\{y_n\}$ alternates between images of S, B and of T, A.

By hypothesis, we are exactly in the situation of Theorem 3.1 already proved in the perturbed metric setting (same $(C1)$ and $(C2)$, but using only compatibility, not the additional R-weak commutativity). Applying that theorem to the sequence just constructed, we conclude that $\{y_n\}$ is a Cauchy sequence in the complete space (X, d_η).

Hence, there exists a point $z \in X$ such that

$$d_\eta(y_n, z) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Since both $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are subsequences of $\{y_n\}$, we also have

$$d_\eta(y_{2n}, z) \to 0, \qquad d_\eta(y_{2n+1}, z) \to 0.$$

□

Using the definitions of y_{2n} and y_{2n+1}, this means

$$\begin{aligned} d_\eta(Sx_{2n}, z) &\to 0, \\ d_\eta(Bx_{2n+1}, z) &\to 0, \\ d_\eta(Tx_{2n+1}, z) &\to 0, \\ d_\eta(Ax_{2n+2}, z) &\to 0. \end{aligned} \tag{3.3.3}$$

First, we prove that $Bz = Tz$.

From (3.3.3) and the reciprocal continuity of B and T, we get

$$Bx_{2n+1} \to z \ \Rightarrow\ TBx_{2n+1} \to Tz, \qquad Tx_{2n+1} \to z \ \Rightarrow\ BTx_{2n+1} \to Bz,$$

where all limits are taken with respect to d_η.

Because the pair (B, T) is compatible in (X, d_η), we also have

$$d_\eta(BTx_{2n+1}, TBx_{2n+1}) \longrightarrow 0. \tag{3.3.4}$$

Passing to the limit in (3.3.4) and using the above convergences yields

$$d_\eta(Bz, Tz) = 0 \ \Rightarrow\ Bz = Tz. \tag{3.3.5}$$

Next, we relate Sw and Tz using the perturbation contraction. From the inclusion $T(X) \subset A(X)$, there exists some $w \in X$ such that

$$Tz = Aw. \tag{3.3.6}$$

Apply the $\circledast$-weak contraction $(C2)$ with $x = w$ and $y = z$:

$$\begin{aligned} d_\eta^3(Sw, Tz) \le \circledast\{ & d_\eta^2(Aw, Sw)\, d_\eta(Bz, Tz), \\ & d_\eta(Aw, Sw)\, d_\eta^2(Bz, Tz), \\ & d_\eta(Aw, Sw)\, d_\eta(Aw, Tz)\, d_\eta(Bz, Sw), \\ & d_\eta(Aw, Tz)\, d_\eta(Bz, Sw)\, d_\eta(Bz, Tz)\}. \end{aligned} \tag{3.3.7}$$

By (3.3.5) we know $Bz = Tz$, hence all factors involving $d_\eta(Bz, Tz)$ are zero. Also from (3.3.6) we have $Aw = Tz$, so every term where Aw and Tz appear together collapses to zero.

Thus, all four arguments of $\circledast$ in (3.3.7) are zero, and we get

$$d_\eta^3(Sw, Tz) \le *(0, 0, 0, 0) = 0,$$

hence

$$Sw = Tz. \tag{3.3.8}$$

Combining (3.3.5), (3.3.6) and (3.3.8), we obtain

$$Sw = Aw = Tz = Bz. \tag{3.3.9}$$

Pointwise R-weak commutativity of (B, T):

Because (B, T) is pointwise R-weakly commuting, there exists some $R > 0$ (depending on z) such that

$$d_\eta(BTz, TBz) \le R\, d_\eta(Bz, Tz).$$

Using (3.3.5) we have

$$d_\eta(Bz, Tz) = 0,$$

hence

$$d_\eta(BTz, TBz) = 0 \ \Rightarrow\ BTz = TBz. \tag{3.3.10}$$

Now from (3.3.5) we know $Bz = Tz$. Combining this with (3.3.9) and (3.3.10) gives

$$BTz = B(Bz) = B(Tz) = T(Bz) = T(Tz).$$

Thus Tz is a fixed point of both B and T:

$$BTz = Tz, \qquad TBz = Tz. \tag{3.3.11}$$

Since $Tz = Aw = Sw$, the point Tz also lies in the ranges of A and S.

Applying (C2) again to obtain a fixed point of A and S:
Now take $x = w$ and $y = w$ in the contractive condition (C2):

$$\begin{aligned} d_\eta^3(Sw, Tw) \le \circledast\{&d_\eta^2(Aw, Sw)\, d_\eta(Bw, Tw), \\ &d_\eta(Aw, Sw)\, d_\eta^2(Bw, Tw), \\ &d_\eta(Aw, Sw)\, d_\eta(Aw, Tw)\, d_\eta(Bw, Sw), \\ &d_\eta(Aw, Tw)\, d_\eta(Bw, Sw)\, d_\eta(Bw, Tw)\}. \end{aligned} \tag{3.3.12}$$

Recall from (3.3.9) that $Aw = Sw = Tz$. Using this in (3.3.12), every factor containing $d_\eta(Aw, Sw)$ or $d_\eta(Aw, Tw)$ becomes zero; hence all four arguments of $\circledast$ vanish. Consequently,

$$d_\eta^3(Sw, Tw) \le \circledast(0, 0, 0, 0) = 0,$$

so

$$Sw = Tw. \tag{3.3.13}$$

But by (3.3.9) we already know $Sw = Tz$, hence $Tw = Tz$ as well. Therefore Tz is a common fixed point of B and T, and it also coincides with Aw and Sw. Thus Tz is a common fixed point of all four maps:

$$ATz = BTz = STz = TTz = Tz. \tag{3.3.14}$$

Denote

$$u = Tz.$$

Hence

$$Au = Bu = Su = Tu = u.$$

Uniqueness of the common fixed point:
Assume that u and v are two common fixed points of A, B, S, and T in (X, d_η). So

$$Au = Bu = Su = Tu = u, \qquad Av = Bv = Sv = Tv = v.$$

Apply (C_2) with $x = u$ and $y = v$. Using the fixed-point properties, we have

$$Ax = Sx = u, \qquad By = Ty = v,$$

$$Ax = u, \qquad Ty = v.$$

and similarly for the other combinations. Therefore

$$d_\eta(Au, Su) = 0, \qquad d_\eta(Bv, Tv) = 0,$$

$$d_\eta(Au, Tv) = d_\eta(u, v), \qquad d_\eta(By, Sx) = d_\eta(v, u) = d_\eta(u, v).$$

Substituting into (C_2) gives

$$d_\eta^3(Su,Tv) \le \circledast\big(0,0,0,d_\eta(u,v)^3\big)\,.$$

But $Su = u$ and $Tv = v$, so this becomes

$$d_\eta^3(u,v) \le \circledast\big(0,0,0,d_\eta(u,v)^3\big)\,. \tag{3.3.15}$$

Let

$$r = d_\eta(u,v) > 0 \quad \text{(if } r = 0 \text{ then } u = v \text{ and we are done).}$$

Then (3.3.15) reads

$$r^3 \le \circledast(0,0,0,r^3).$$

Monotonicity of $\circledast$ in each coordinate and the special property

$$\circledast(t,t,t,t) < t^3 \quad \text{for every } t > 0$$

give, by taking $t = r^3$,

$$\circledast(0,0,0,r^3) \le \circledast(r^3,r^3,r^3,r^3) < (r^3)^3 = r^9.$$

In particular,

$$r^3 < r^9 \quad \text{for } r > 0,$$

which is impossible when the inequality

$$r^3 \le \circledast(0,0,0,r^3)$$

holds.

The only way (3.3.15) can be satisfied is therefore $r = 0$, that is,

$$d_\eta(u,v) = 0 \;\Rightarrow\; u = v.$$

Thus, the common fixed point is unique, completing the proof.

Example 3.7 Let $X = [0,\infty)$, $d(x,y) = |x-y|$, and

$$\eta(x,y) = \frac{1}{2}|x-y|.$$

Then

$$d_\eta(x,y) = d(x,y) + \eta(x,y) = \frac{3}{2}|x-y|$$

is a metric equivalent to d; since $([0,\infty),d)$ is complete, so is (X,d_η).

Define four mappings $A,B,S,T : X \to X$ by

$$A(x) = B(x) = S(x) = T(x) = 0 \quad \text{for all } x \in X.$$

Condition (C1):

$$S(X) = \{0\} = B(X) \quad \text{and} \quad T(X) = \{0\} = A(X),$$

so (C_1) holds.

Condition (C2):

For any $x,y \in X$,

$$d_\eta(Sx,Ty) = d_\eta(0,0) = 0,$$

hence

$$d_\eta^3(Sx,Ty) = 0.$$

Every distance appearing on the right-hand side of (C_2) also equals 0, so all four arguments of $\circledast$ are 0. Therefore,

$$d_\eta^3(Sx, Ty) = 0 \le \circledast(0,0,0,0) = 0.$$

Pointwise R-weak commutativity:

For any $x \in X$,

$$BTx = TBx = 0, \qquad ASx = SAx = 0.$$

Hence the inequalities in (3.3) hold with any $R > 0$, since both sides are 0.

Compatibility and reciprocal continuity:

All four maps are constant and hence continuous in d_η. Moreover,

$$AS = SA \qquad \text{and} \qquad BT = TB$$

identically, so the compatibility conditions are trivially satisfied.

The theorem then guarantees a unique common fixed point. In this example the fixed point is $u = 0$, and indeed

$$Au = Bu = Su = Tu = 0.$$

Example 3.8 Take the same perturbed space (X, d_η) as in Example 3.7:

$$X = [0, \infty), \qquad d_\eta(x,y) = \frac{3}{2}|x - y|.$$

Define mappings $A, B, S, T : X \to X$ by

$$A(x) = B(x) = \frac{x}{2}, \qquad S(x) = T(x) = 0, \quad \text{for all } x \in X.$$

Condition (C1):

Since

$$S(X) = T(X) = \{0\}$$

and

$$B(0) = A(0) = 0,$$

we have

$$S(X) \subset B(X), \qquad T(X) \subset A(X).$$

Hence (C_1) holds.

Condition (C2):

For any $x, y \in X$,

$$d_\eta(Sx, Ty) = d_\eta(0,0) = 0,$$

so the left-hand side of (C_2) is zero:

$$d_\eta^3(Sx, Ty) = 0.$$

On the right-hand side, since $Sx = Ty = 0$, each quantity

$$d_\eta(Ax, Sx), \quad d_\eta(By, Ty), \quad d_\eta(Ax, Ty), \quad d_\eta(By, Sx)$$

is finite and non-negative. Hence all four arguments of $\circledast$ are finite non-negative numbers. In particular,

$$0 = d_\eta^3(Sx, Ty) \le \circledast(\cdots),$$

so (C_2) holds automatically.

Pointwise R-weak commutativity:

For any $x \in X$,

$$ASx = A(0) = 0, \qquad SAx = S\left(\frac{x}{2}\right) = 0,$$

and similarly,

$$BTx = TBx = 0.$$

Thus,

$$d_\eta(ASx, SAx) = 0, \qquad d_\eta(BTx, TBx) = 0$$

for all x, so (3.3.1) is satisfied for arbitrary $R > 0$.

Compatibility and reciprocal continuity:

The maps A and B are contractions (hence continuous) in d_η, and S, T are constant (hence continuous). Moreover,

$$ASx = SAx = 0, \qquad BTx = TBx = 0, \quad \text{for all } x,$$

so both pairs (A, S) and (B, T) are compatible. Thus condition (3.7) is fulfilled.

By the perturbed theorem proved above, there exists a unique common fixed point. Solving

$$x = Ax = Bx = Sx = Tx,$$

we obtain

$$x = \frac{x}{2}, \qquad x = 0.$$

Hence the unique common fixed point is $u = 0$, and indeed

$$Au = Bu = Su = Tu = 0.$$

4. Conclusion

In this work we have investigated common fixed-point results for weakly commuting mappings and their variants in the framework of perturbed metric spaces. The starting point was the observation that, in many realistic situations, the "true" distance between points of a space is affected by measurement errors, environmental noise or structural disturbances. This phenomenon is modelled by replacing the standard metric d with a perturbed distance

$$d_\eta(x, y) = d(x, y) + \eta(x, y),$$

where the perturbation kernel η captures the additional uncertainty. Working in this setting, we showed that a number of classical concepts from fixed point theory, such as commutativity, weak commutativity, R-weakly commuting mappings, compatibility and reciprocal continuity, can be successfully extended to the perturbed geometry without losing their essential structure.

The central tool in our analysis was a generalized $\circledast$–weak contraction condition formulated directly in terms of the perturbed distance d_η. Unlike the standard linear or Boyd–Wong type contractions, the inequality used here involves quadratic and cubic powers of $d_\eta(x, y)$, in the spirit of Murthy and Prasad's cubic contractions in ordinary metric spaces. By replacing the metric distance $d(x, y)$ with $d_\eta(x, y)$ in a systematic way, we obtained a flexible contractive framework that remains valid even when the underlying metric is subjected to non-trivial perturbations. This allowed us to derive common fixed-point theorems for four self-mappings

$$A, B, S, T : X \to X$$

satisfying suitable range inclusions and interaction conditions.

The paper established, in particular, a common fixed-point theorem for weakly commuting mappings in perturbed metric spaces (Theorem 3.1), which guarantees the existence and uniqueness of a common fixed point of A, B, S, T under the generalized $\circledast$–weak contraction. We then refined this result by incorporating R-weakly commuting mappings of type (P) in the perturbed sense (Theorem 3.3). This variant shows that even when the non-commutativity between the iterates is controlled only up to a multiplicative constant, the fixed-point structure is preserved. Finally, we treated pointwise R-weakly commuting mappings combined with compatibility and reciprocal continuity conditions (Theorem 3.6), which further relaxes the interaction hypotheses and still yields a unique common fixed point.

To illustrate the applicability of our theoretical results, we constructed several explicit examples of perturbed metric spaces and mappings. These examples confirm that the hypotheses of our theorems are not merely abstract but can be realized in concrete situations, ranging from trivial kernels $\eta \equiv 0$ to bounded or polynomial-type perturbations. Overall, the results obtained here unify and extend known fixed point theorems from the classical metric setting to the more general framework of perturbed metric spaces. In particular, the theorems demonstrate that the presence of measurement errors can be incorporated into the fixed-point analysis without destroying the existence and uniqueness properties of common fixed points, provided that the perturbation is encoded through a suitable kernel and the underlying contractions are formulated in terms of d_η.

5. Future Scope

The present study suggests several promising directions for further research in fixed point theory within perturbed metric environments. A natural extension is to consider multi valued and set-valued versions of the results obtained here. In many applications, especially in control theory, optimization and differential inclusions, the evolution of a system is described by maps that associate to each point a whole set of possible images rather than a single point. Formulating and proving common fixed-point theorems for multi valued mappings which are weakly commuting or R-weakly commuting with respect to a perturbed distance d_η would significantly widen the applicability of the theory developed in this paper.

Another line of investigation is to place additional structure on the underlying perturbed metric space. For instance, by combining the perturbed metric with a partial order, one could develop common fixed-point results for weakly commuting mappings in ordered perturbed metric spaces, inspired by the rich theory of fixed points in ordered metric spaces. Likewise, it would be interesting to study versions of the present theorems in fuzzy, intuitionistic fuzzy or probabilistic perturbed metric spaces, where imprecision and randomness appear simultaneously at the level of both distance and membership. Such settings are particularly relevant in decision theory, information sciences and modeling of uncertain systems.

The contraction conditions themselves also admit further generalization. One may replace the current $\circledast$–weak contraction involving cubic and quadratic terms by integral-type or implicit contractive conditions formulated in terms of d_η. This would connect the present work with Branciari-type integral contractions and various implicit relations that have proved useful in the classical metric context. Another possibility is to investigate random or stochastic perturbation kernels η, in which the size of the perturbation is not fixed but governed by a probability law. Extending the results to such stochastic perturbed metric spaces could have applications in numerical analysis, iterative methods under random noise, and models of learning algorithms where distances are observed with random errors.

From an applications perspective, the framework of perturbed metric spaces combined with weak commuting conditions is well suited to problems in nonlinear analysis, image processing, data science and computational mathematics, where distances are rarely measured exactly. For example, iterative schemes arising in the numerical solution of integral or differential equations, iterative reconstructions in imaging (such as CT, MRI or tomography), and iterative optimization algorithms in machine learning often operate in environments with discretization errors or sensor noise. Recasting these algorithms as fixed point problems in a perturbed metric space and employing generalized $\circledast$–weak contractions could yield new convergence criteria that explicitly account for such imperfections.

Finally, there is scope to explore algorithmic aspects of the constructed iterative sequences. The proofs in this paper already provide an explicit Picard-type iteration based on alternating applications of the mappings A, B, S, T. A careful quantitative study of the convergence rate of this sequence, under additional assumptions on the control function $\circledast$ and the perturbation kernel η, may lead to practical error estimates and stopping criteria for numerical implementations.

In summary, the ideas developed here form a flexible platform on which many further extensions-structural, analytical and computational-can be built in the context of fixed-point theory in perturbed metric spaces.

6. Acknowledgment

The authors gratefully acknowledge the valuable mathematical discussions with the faculty and research colleagues of the Department of Applied Sciences at Panipat Institute of Engineering & Technology

(PIET), Haryana, India, and Ram Lal Anand College, University of Delhi, India, for their insightful suggestions on generalized contraction principles and modern extensions in nonlinear analysis.

References

1. I. Altun, D. Turkoglu, and B. E. Rhoades, Fixed point theorems for weakly compatible maps satisfying an implicit relation in metric spaces, *Applied Mathematics Letters*, 19(8), 849–854, 2006.
2. G. Alsahli, P. Shahi, and E. Karapınar, On perturbed-$S\tau$-contractions in perturbed metric spaces, *AIMS Mathematics*, 10(5), 541–557, 2025.
3. H. Alsulami and N. Hussain, Generalized hybrid contractions in perturbed metric spaces with applications, *Journal of Mathematical Analysis and Applications*, 525(1), 127–145, 2025.
4. D. W. Boyd and J. S. W. Wong, On nonlinear contractions, *Proceedings of the American Mathematical Society*, 20, 458–464, 1969.
5. V. Berinde, On general classes of contractive mappings in metric and generalized metric spaces, *Carpathian Journal of Mathematics*, 35(2), 171–184, 2019.
6. A. N. Branga and I. M. Olaru, Some fixed-point results in spaces with perturbed metrics, *Carpathian Journal of Mathematics*, 38(3), 641–654, 2022.
7. L. B. Ćirić, A generalization of Banach's contraction principle, *Proceedings of the American Mathematical Society*, 45, 267–273, 1974.
8. G. Jungck, Commuting mappings and fixed points, *American Mathematical Monthly*, 83, 261–263, 1976.
9. G. Jungck, Compatible mappings and common fixed points, *International Journal of Mathematics and Mathematical Sciences*, 9, 771–779, 1986.
10. G. Jungck, Common fixed points for non-continuous non-self-maps on non-metric spaces, *Far East Journal of Mathematical Sciences*, 4, 199–215, 1996.
11. G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, *Indian Journal of Pure and Applied Mathematics*, 29, 227–238, 1998.
12. G. Jungck and B. E. Rhoades, Fixed point theorems for occasionally weakly compatible mappings, *Fixed Point Theory*, 7(2), 287–296, 2006.
13. M. Jleli and B. Samet, A new generalization of the metric concept and related fixed-point theorems, *Fixed Point Theory and Applications*, 2014:20, 2014.
14. E. Karapınar and B. Samet, Weak contraction conditions in perturbed metric environments, *Fixed Point Theory*, 25(2), 219–232, 2024.
15. M. Leli and B. Samet, On Banach's fixed point theorem in perturbed metric spaces, *Journal of Applied Analysis and Computation*, 15(2), 993–1007, 2025.
16. Z. Liu and X. Li, Extensions of fixed-point theorems in perturbed metric-like spaces, *Mathematics*, 11(3), 544–559, 2023.
17. P. P. Murthy and K. N. V. V. Vara Prasad, Weak contraction condition involving cubic terms of $d(x, y)$ under the fixed-point consideration, *Journal of Mathematics*, 2013, Article ID 967045, 2013.
18. R. P. Pant, Common fixed-point theorems for contractive maps, *Journal of Mathematical Analysis and Applications*, 226, 251–258, 1998.
19. H. K. Pathak, Y. J. Cho, S. M. Kang, and B. S. Lee, Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming, *Le Matematiche*, 50, 15–33, 1995.
20. P. Shahi, E. Karapınar, and S. Radenović, Common fixed-point results via generalized contractive mappings in perturbed metric spaces, *Nonlinear Functional Analysis and Applications*, 29(1), 101–118, 2024.
21. S. Sessa, On a weak commutativity condition of mappings in fixed point consideration, *Publications de l'Institut Mathématique (Beograd)*, 32(46), 146–153, 1982.

Pawan Kumar,
Department of Mathematics,
Ram Lal Anand College, University of Delhi,
India.
E-mail address: pawan.maths@rla.du.ac.in

and

Sanjeev Kumar and Balbir Singh,
Department of Applied Sciences & Humanities,Panipat Institute of Engineering & Technology, India.
E-mail address: drsanjeev.applied@piet.co.in & balbir.applied@piet.co.in