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Operational Methods with Applications in Fractional Calculus

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ABSTRACT: In this study, after introducing some properties of the Laplace and Stieltjes transforms, we consider certain fractional differential equations, evaluation of new integrals and sums of special functions, singular integro-differential equation where the fractional derivative is in the Caputo-Fabrizio sense. Some constructive examples are also provided.

Keywords: Laplace transform, Caputo-Fabrizio fractional derivative, Weyl fractional integral and derivative, Stieltjes transform, Newmann function, Modified Bessel functions, error function, Kelvin function, operational methods, fractional calculus.

Contents

1. Introduction and Notations

In the previously published articles, some properties of the integral transforms in solving fractional equations and boundary value problems, and other applications of integral transforms such as solving integral equations and computing new integrals were shown [1-4, 6]. In this article, the author intends to present new properties of integral transforms such as Laplace and Stieltjes transforms, and to propose and ultimately solve new problems with the help of a new fractional derivative that has been recently in 2015, defined and is more efficient than previous fractional derivatives, called the Caputo-Fabrizio fractional derivative. This work provides a concise exposition of the basic ideas of the theory of integral transforms and special functions and its applications to fractional calculus. Methods in which techniques are used in applications are illustrated and some non-trivial examples are included.

Definition 1.1. Let us assume that the function $\phi(t)$ is of exponential order, then the Laplace transform of $\phi(t)$ is as follows [5,7,10]

$$\mathcal{L}\{\phi(t);s\} = \Phi(s) = \int_0^{+\infty} e^{-st}\phi(t)dt,$$

provided that the above integral is convergent.

Definition 1.2. The inverse Laplace transform of the function $\Phi(s)$ may be expressed explicitly as a contour integral by considering parameter s as a complex variable.

$$\mathcal{L}^{-1}\{\Phi(s);t\} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{ts}\Phi(s)ds,$$

the above complex integration along vertical line $\mathcal{R}e(s) = c$ in the complex s-plane, is known as Fourier-Mellin or Bromwich integral[5,7,10].

Note. In the aobve complex integral $\Phi(s)$ is $o(s^{-k})$ with $k > 1$, this means that $|s^k\Phi(s)| < M$, whenever $|s|$ is sufficiently large and M is a real constant, then the above integral is convergent. It is worth mentioning that the complex inversion formula for the Laplace transform can often be evaluated quite

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readily through use of the theory of residues.
Let us illustrate the Laplace transform technique on some singular integral equations.
Example 1.1. Solve the following singular integral equation with $erfc(.)$ as kernel

$$\int_0^{+\infty} erfc(\frac{\xi}{2\sqrt{t}})\psi(\xi)d\xi = \frac{e^{\lambda^2 t}}{\lambda} erfc(\lambda\sqrt{t}).$$

Solution. We begin by taking the Laplace transform of both sides of the above integral equation, which leads to

$$\mathcal{L}\{\int_0^{+\infty} erfc(\frac{\xi}{2\sqrt{t}})\psi(\xi)d\xi; s\} = \frac{1}{s}\Psi(\sqrt{s}) = \frac{1}{\sqrt{s}(s-\lambda^2)},$$

after simplifying, we obtain

$$\Psi(s) = \frac{s}{s^2-\lambda^2},$$

taking the inverse Laplace transform, leads to

$$\psi(t) = \cosh(\lambda t).$$

The obtained solution satisfies the integral equation, thus we get

$$\int_0^{+\infty} erfc(\frac{\xi}{2\sqrt{t}})\cosh(\lambda\xi)d\xi = \frac{e^{\lambda^2 t}}{\lambda} erfc(\lambda\sqrt{t}),$$

in special case $t = \frac{1}{4}$ we have

$$\int_0^{+\infty} erfc(\xi)\cosh(\lambda\xi)d\xi = \frac{e^{\frac{\lambda^2}{4}}}{\lambda} erfc(\frac{\lambda}{2}).$$

In the above integral equation $erf(.)$ and $erfc(.)$ stand for the error function and complementary error functions respectively, for more detail see [6]. The above example illustrate the basic procedure used in the method of Laplace transform.
Example 1.2. Evaluate $\phi(t) = \mathcal{L}^{-1}[\frac{e^{-as^\alpha}}{s^\beta}]$, with $0 < \alpha, \beta < 1$.
Solution. We first evaluate $\mathcal{L}^{-1}[e^{-as^\alpha}]$ and $\mathcal{L}^{-1}[s^{-\beta}]$, then by convolution theorem we find

$$\phi(t) = \{\mathcal{L}^{-1}[e^{-as^\alpha}]\} * \{\mathcal{L}^{-1}[s^{-\beta}]\},$$

thus we have

$$\mathcal{L}^{-1}[s^{-\beta}]\} = \frac{t^{\beta-1}}{\Gamma(\beta)},$$

and

$$\mathcal{L}^{-1}[e^{-as^\alpha}] = \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\mathcal{I}m[e^{-a(re^{-i\pi\alpha})}]dr = \frac{1}{\pi}\int_0^{+\infty} e^{-tr-a\cos(\pi\alpha)}\sin(ar\sin(\pi\alpha)dr,$$

from where we arrive at

$$\phi(t) = \int_0^t \frac{(t-\xi)^{\beta-1}}{\Gamma(\beta)}[\frac{1}{\pi}\int_0^{+\infty} e^{-\xi r-a\cos(\pi\alpha)}\sin(ar\sin(\pi\alpha)dr]d\xi.$$

Theorem 1.1. Let us assume that $\mathcal{L}\{\phi(t); s\} = \Phi(s)$ then the following identity holds

$$1. \quad \mathcal{L}\{\phi(a(e^{bt}-1)); s\} = \frac{1}{a\Gamma(\frac{s}{b}+1)}\int_0^{+\infty} u^{\frac{s}{b}}e^{-u}\Phi(\frac{u}{a})du, \quad a, b > 0.$$

$$2. \quad \mathcal{L}\{\int_0^t (\frac{t-\xi}{a\xi})^\nu J_{2\nu}(2\sqrt{a\xi(t-\xi)})\phi(\xi)d\xi = \frac{1}{s^{2\nu+1}}\Phi(s+\frac{a}{s}).$$

$$3.\qquad \mathcal{L}\{\int_0^t (\frac{t-\xi}{a\xi})^{\nu} I_{2\nu}(2\sqrt{a\xi(t-\xi)})\phi(\xi)d\xi = \frac{1}{s^{2\nu+1}}\Phi(s-\frac{a}{s}).$$

Proof. Part 1.
By using the definition of the Laplace transform, right-hand side can be written as follows

$$\frac{1}{a\Gamma(\frac{s}{b}+1)}\int_0^{+\infty} u^{\frac{s}{b}} e^{-u}\Phi(\frac{u}{a})du = \frac{1}{a\Gamma(\frac{s}{b}+1)}\int_0^{+\infty} u^{\frac{s}{b}} e^{-u}[\int_0^{+\infty} e^{-(\frac{u}{a})\xi}\phi(\xi)d\xi]du,$$

changing the order of integration and after simplifying we obtain

$$R.H.S = \frac{1}{a\Gamma(\frac{s}{b}+1)}\int_0^{+\infty}\phi(\xi)[\int_0^{+\infty} e^{-(1+\frac{\xi}{a})u}u^{\frac{s}{b}}du]d\xi = \frac{1}{ab}\int_0^{+\infty}\frac{\phi(\xi)}{(1+\frac{\xi}{a})^{\frac{s}{b}+1}}d\xi,$$

on the other hand, on the left-hand side after making a change of variable $a(e^{bt}-1)=w$, it takes the following form

$$\mathcal{L}\{\phi(a(e^{bt}-1));s\} = \int_0^{+\infty} e^{-\ln(1+\frac{w}{a})^{\frac{s}{b}}}\frac{1}{a(1+\frac{w}{a})}\phi(w)dw = \frac{1}{ab}\int_0^{+\infty}\frac{\phi(w)}{(1+\frac{w}{a})^{\frac{s}{b}+1}}dw = R.H.S$$

Part 2.
Left-hand side can be written as follows

$$L.H.S = \int_0^{+\infty} e^{-st}[\int_0^t (\frac{t-\xi}{a\xi})^{\nu} J_{2\nu}(2\sqrt{a\xi(t-\xi)})\phi(\xi)d\xi]dt,$$

changing the order of integration leads to

$$L.H.S = \int_0^{+\infty}\frac{\phi(\xi)}{(a\xi)^{\nu}}[\int_{\xi}^{+\infty} e^{-st}(t-\xi)^{\nu} J_{2\nu}(2\sqrt{a\xi(t-\xi)})dt]d\xi,$$

at this stage making a change of variable $t-\xi=w$, we arrive at

$$L.H.S = \int_0^{+\infty}\frac{\phi(\xi)}{(a\xi)^{\nu}}[\int_0^{+\infty} e^{-s(w+\xi)}w^{\nu} J_{2\nu}(2\sqrt{(a\xi)w})dw]d\xi = ..$$

$$.. = \int_0^{+\infty} e^{-s\xi}\phi(\xi)[\int_0^{+\infty} e^{-sw}(\frac{w}{a\xi})^{\nu} J_{2\nu}(2\sqrt{(a\xi)w})dw]d\xi,$$

after evaluation of the inner integral, we obtain

$$L.H.S = \int_0^{+\infty} e^{-s\xi}\phi(\xi)[\frac{e^{-\frac{a\xi}{s}}}{s^{2\nu+1}}d\xi = \frac{1}{s^{2\nu+1}}\int_0^{+\infty} e^{-\xi(s+\frac{a}{s})}\Phi(\xi)d\xi = \frac{1}{s^{2\nu+1}}\Phi(s+\frac{a}{s}).$$

Part 3. The same as Part 2.
To illustrate the Laplace transform method for a function, let us take the Laplace transform of the function$\psi(t)$ defined as follows.
Example 1.3. Let us assume that $\psi(t) = \int_0^{+\infty} ber(2\sqrt{t\sinh\beta})d\beta$ then we have

$$\mathcal{L}\{\psi(t);s\} = \frac{1}{s}K_0(\frac{1}{s}),$$

where $ber(.)$ and $K_0(.)$ are kelvin function and Macdonald function of order zero respectively[5,8,9].
Solution.

$$\mathcal{L}\{\psi(t);s\} = \int_0^{+\infty} e^{-st}[\int_0^{+\infty} ber(2\sqrt{t\sinh\beta})d\beta]dt,$$

changing the order of integration we get

$$\mathcal{L}\{\psi(t);s\}=\int_0^{+\infty}[e^{-st}ber(2\sqrt{(\sinh\beta)t})dt]d\beta=\int_0^{+\infty}\frac{1}{s}\cos(\frac{\sinh\beta}{s})d\beta,$$

at this point, making a change of variable $\sinh\beta=w$ in the above integral, after simplifying we have

$$\mathcal{L}\{\psi(t);s\}=\frac{1}{s}\int_0^{+\infty}\frac{\cos(\frac{w}{s})}{\sqrt{1+w^2}}dw=\frac{1}{s}K_0(\frac{1}{s}),$$

in the last step we used the fact that $\mathcal{L}\{ber(2\sqrt{at});s\}=\frac{1}{s}\cos(\frac{a}{s})$ and $K_0(t)=\int_0^{+\infty}\frac{\cos(tw)}{\sqrt{1+w^2}}dw$.

Example 1.4. Let us show that the following relation holds

$$\mathcal{L}^{-1}\{\Phi(s);t\}=\phi(t)=\frac{t}{\sqrt{t^2+a^2-b^2}},$$

where

$$\Phi(s)=\int_0^{+\infty}e^{-s\sqrt{\xi^2-2a\xi+b^2}}d\xi.$$

Solution. By using Bromwich integral we get

$$\phi(t)=\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}e^{ts}[\int_0^{+\infty}e^{-s\sqrt{\xi^2-2a\xi+b^2}}d\xi]ds,$$

changing the order of integration, we have

$$\phi(t)=\int_0^{+\infty}[\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}e^{(t-\sqrt{\xi^2-2a\xi+b^2})s}ds]d\xi,$$

the value of the inner integral is $\delta(t-\sqrt{\xi^2-2a\xi+b^2})$, therefore we obtain

$$\phi(t)=\int_0^{+\infty}\delta(t-\sqrt{\xi^2-2a\xi+b^2})d\xi,$$

at this point, making a change of variable $u=t-\sqrt{\xi^2-2a\xi+b^2}$ in the above integral and after simplifying and in view of the elementary properties of the Dirac-delta function, we get finally

$$\phi(t)=\int_{-\infty}^{t-b}\frac{(t-u)\delta(u)}{\sqrt{(t-u)^2+a^2-b^2}}du=\frac{t}{\sqrt{t^2+a^2-b^2}},\quad t>b>a.$$

Remark. The generalized function such as impulse function or Dirac-delta function $\delta(t)$ belongs to space of locally integrable functions i.e, $\mathcal{L}^1_{loc}$, is a very useful concept in a wide variety of physical problems involving the ideas of point sources or impulsive forces. We have the following useful properties of the impulse function or Dirac-delta function $\delta(t)$ as follows

$$1.\quad t\delta(t)=0,$$

$$2.\quad \delta'(t)=-\frac{1}{t}\delta(t),$$

$$3.\quad \delta^{(n)}(t)=(-1)^n(\frac{n!}{t^n}\delta(t).$$

$$4.\quad \frac{d}{dt}sgn(t)=2\delta(t).$$

$$5.\quad \delta[\phi(t)]=\frac{1}{\phi'(\alpha)}\delta(t-\alpha).$$

$$6. \qquad \mathcal{H}(t) = \frac{1}{2}\{1 + \delta(t)\}.$$

In the above relations $sgn(t)$ stands for the signum function and $\phi(t)$ is a monotonic function for which $\phi(\alpha) = 0$.

Example 1.5. By using complex inversion formula, let us show that

$$\mathcal{L}^{-1}\{\int_{\lambda}^{+\infty} \frac{\xi e^{-s\sqrt{\xi^2-\lambda^2}}}{\sqrt{\xi^2-\lambda^2}} I_0(\alpha\xi)d\xi; t\} = I_0(\alpha\sqrt{t^2+\lambda^2}).$$

Soluion. In view of the Bromwich integral, left-hand side can be written as follows

$$L.H.S = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{ts}[\int_{\lambda}^{+\infty} \frac{\xi}{\sqrt{\xi^2-\lambda^2}} I_0(\alpha\xi)d\xi]ds,$$

changing the order of integration leads to

$$L.H.S = \int_{\lambda}^{+\infty} \frac{\xi}{\sqrt{\xi^2-\lambda^2}} I_0(\alpha\xi)[\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{ts}e^{-s\sqrt{\xi^2-\lambda^2}}ds]d\xi,$$

or

$$L.H.S = \int_{\lambda}^{+\infty} \frac{\xi}{\sqrt{\xi^2-\lambda^2}} I_0(\alpha\xi)[\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} e^{(t-\sqrt{\xi^2-\lambda^2})s}ds]d\xi,$$

but the value of the inner integral is $\delta(t-\sqrt{\xi^2-\lambda^2})$, therefore we have

$$L.H.S = \int_{\lambda}^{+\infty} \frac{\xi}{\sqrt{\xi^2-\lambda^2}} I_0(\alpha\xi)\delta(t-\sqrt{\xi^2-\lambda^2})d\xi,$$

at this stage, making a change of variable $t-\sqrt{\xi^2-\lambda^2} = w$, after simplifying we arrive at

$$L.H.S = \int_{-\infty}^{t} \frac{\sqrt{(t-w)^2+\lambda^2}}{t-w} I_0(\alpha\sqrt{(t-w)^2+\lambda^2})\frac{t-w}{\sqrt{(t-w)^2+\lambda^2}}\delta(w)dw = I_0(\alpha\sqrt{t^2+\lambda^2}).$$

Example 1.6. Let us solve the folowing singular integral equation with trigonometric kernel

$$\frac{1}{\sqrt{\pi t}}\int_0^{+\infty} \cos(2\sqrt{t\xi})\phi(\xi)d\xi = \sqrt[4]{\frac{\lambda}{t}} J_{-\frac{1}{2}}(2\sqrt{\lambda t}), \quad \lambda > 0.$$

Solution. Let us assume that $\mathcal{L}\{\phi(t); s\} = \Phi(s)$, then by taking the Laplace transform of the above integral equation, we get [5]

$$\frac{1}{\sqrt{s}}\Phi(\frac{1}{s}) = \frac{e^{-\frac{\lambda}{s}}}{\sqrt{s}},$$

from which we deduce that

$$\Phi(s) = e^{-\lambda s},$$

at this stage, using the above complex inversion formula we obtain

$$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}e^{-\lambda s}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(t-\lambda)s}ds = \delta(t-\lambda).$$

The obtained solution satisfies the above integral equation, therefore we get the following relation

$$\frac{1}{\sqrt{\pi t}}\int_0^{+\infty} \cos(2\sqrt{t\xi})\delta(\xi-\lambda)d\xi = \frac{1}{\sqrt{\pi t}}\cos(2\sqrt{\lambda t}) = \sqrt[4]{\frac{\lambda}{t}} J_{-\frac{1}{2}}(2\sqrt{\lambda t}).$$

Definition 1.3. Fractional integral of the contiuous function $\phi(t)$ of order α is defined as follows[11]

$$\mathcal{J}^{\alpha}[\phi(t)] = \frac{1}{\Gamma(\alpha)}\int_{t}^{+\infty}(t-\xi)^{\alpha}\phi(\xi)d\xi, \quad 0 \le \alpha \le 1.$$

Definition 1.4. The Laplace transform of fractional integral of order α of the given function $\phi(t)$ is defined as follows[11]

$$\mathcal{L}[\mathcal{J}^{\alpha}[\phi(t)]; s] = \frac{\Phi(s)}{s^{\alpha}}.$$

Definition 1.5. Let us assume that the function $\phi(t)$ is of class $\mathcal{C}^1$, then the fractional derivative of order α of the function $\phi(t)$ in the Caputo-Fabrizio sense is defined as follows

$$D_{0,t}^{C.F,\alpha}\phi(t) = \frac{1}{1-\alpha}\int_{0}^{t} e^{-\frac{\alpha(t-\xi)}{1-\alpha}}\phi'(\xi)d\xi, \quad t \ge 0, \quad 0 < \alpha < 1.$$

provided that the above integral is convergent. Let us assume that $\mathcal{L}\{\phi(t); s\} = \Phi(s)$, then the Laplace transform of the fractional operator $D_{0,t}^{C.F,\alpha}$ with $0 < \alpha < 1$ is defined by

$$\mathcal{L}\{D_{0,t}^{C.F,\alpha}\phi(t); s\} = \frac{s\Phi(s) - u_0}{\alpha + (1-\alpha)s}, \qquad s > 0,$$

we have also

$$\lim_{\alpha \to 1} D_{0,t}^{C.F,\alpha}\phi(t) = \phi'(t).$$

Corollary 1.1. Let us assume that $\psi(t) = \int_0^1 \frac{1-e^{-t\xi}}{\xi}d\xi, \quad \psi(0) = 0$, then we have

$$\mathcal{L}\{D_{0,t}^{C.F,\alpha}\psi(t); s\} = \frac{s}{s+1}\ln(\frac{s+1}{s}).$$

Proof. By definition of the Laplace transform, we have

$$\mathcal{L}\{\psi(t); s\} = \int_{0}^{+\infty} e^{-st}[\int_{0}^{1}\frac{1-e^{-t\xi}}{\xi}d\xi]ds,$$

changing the order of integration leads to

$$\mathcal{L}\{\psi(t); s\} = \int_0^1 \frac{1}{\xi}[\int_0^{+\infty} e^{-st}(1-e^{-t\xi})ds]d\xi = \int_0^1 \frac{1}{\xi}(\frac{1}{s} - \frac{1}{s+\xi})ds = \ln(\frac{s+1}{s}),$$

thus we get

$$\mathcal{L}\{D_{0,t}^{C.F,\alpha}\psi(t); s\} = \frac{s}{s+1}\ln(\frac{s+1}{s}).$$

Definition 1.6. Let us assume that the function $\phi(t)$ is of exponential order, then the Stieltjes transform of $\phi(t)$ is as follows[5,7,10]

$$\mathcal{L}[\mathcal{L}[\phi(t)]; u]; s] = \mathcal{S}[\phi(t); s] = \Phi(s) = \int_0^{+\infty}\frac{\phi(t)}{t+s}dt$$

provided that the above integral is convergent. In fact, the second itrate of the Laplace transform is the Stieltjes transform.

Definition 1.7. The inverse Stieltjes transform of the transformed function $\Phi(s)$ is defined as below[5,7]

$$\mathcal{S}^{-1}[\Phi(s); t] = \frac{1}{\pi}\mathcal{I}_m[\lim_{\nu \to \pi^-}\Phi(te^{i\nu})].$$

Lemma 1.1. Let us consider the following Stieltjes-type singular integral equation

$$\int_0^{+\infty} \frac{\psi(t)}{t+s}dt = \sqrt{s}Y_{\frac{1}{3}}(\sqrt{s}),$$

then the above Stieltjes-type singular integral equation has the following formal solution

$$\psi(t) = \frac{\sqrt{3t}}{\pi} I_{\frac{1}{3}}(\sqrt{t}).$$

Proof. The above singular integral equation can be written in terms of the Stieltjes transform as follows

$$\mathcal{S}\{\psi(t); s\} = \sqrt{s}Y_{\frac{1}{3}}(\sqrt{s}),$$

by using the inverse Stieltjes transform we have

$$\psi(t) = \mathcal{S}^{-1}\{\sqrt{s}Y_{\frac{1}{3}}(\sqrt{s}); t\} = \frac{1}{\pi}\mathcal{I}m\{\sqrt{te^{-i\pi}}Y_{\frac{1}{3}}(\sqrt{te^{-i\pi}})\},$$

at this point, let us recall the well known identity for the Bessel function of the second kind (Newmann function $Y_\nu(.)$)as below[8,9]

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - (-1)^\nu J_\nu(x)}{\sin(\pi\nu)},$$

thus we get

$$\psi(t) = \frac{1}{\pi}\mathcal{I}m\{[\sqrt{te^{-i\pi}}]\frac{J_{\frac{1}{3}}(i\sqrt{t})\cos(\frac{\pi}{3}) - (-1)^{\frac{1}{3}}J_{\frac{1}{3}}(i\sqrt{t})}{\sin(\frac{\pi}{3})}\},$$

after simplifying we arrive at

$$\psi(t) = \frac{\sqrt{3t}}{\pi} I_{\frac{1}{3}}(\sqrt{t}),$$

by setting the obtained solution in the above integral equation, we get

$$\frac{1}{\pi}\int_0^{+\infty} \frac{\sqrt{3t}I_{\frac{1}{3}}(\sqrt{t})}{t+s}dt = \sqrt{s}Y_{\frac{1}{3}}(\sqrt{s}),$$

in the above integral, making a change of variable $t = \xi^2$ and change of parameter $s = \tau^2$, after simplifying we arrive at

$$\frac{2}{\pi}\int_0^{+\infty} \frac{\sqrt{3}\xi^2 I_{\frac{1}{3}}(\xi)}{\xi^2+\tau^2}dt = \tau Y_{\frac{1}{3}}(\tau).$$

Bessel functions, of which there are several varieties, occur in so many areas of applications in applied mathematics, mathematical physics and engineering that they are considered to be the most important special functions beyond the elementary ones studies in calculus. In the sequel, we give some properties of the Bessel functions.

Lemma 1.2. The following integral representation for the modified Bessel function of the second kind holds

$$\int_0^{+\infty} \frac{1}{\sqrt{\xi(\xi+t)}}e^{-\xi(\xi+t)}d\xi = \frac{1}{2}e^{\frac{t^2}{8}}K_0(\frac{t^2}{8}),$$

Proof. The left-hand side can be written as follows

$$\int_0^{+\infty} \frac{1}{\sqrt{\xi(\xi+t)}}e^{-\xi(\xi+t)}d\xi = \int_0^{+\infty} \frac{1}{\sqrt{(\xi+\frac{t}{2})^2-\frac{t^2}{4}}}e^{-(\xi+\frac{t}{2})^2+\frac{t^2}{4}}d\xi,$$

by making a change of variable $\xi + \frac{t}{2} = \frac{t}{2}\cosh\tau$ in the above integral we arrive at

$$L.H.S = \int_0^{+\infty} e^{-\frac{t^2}{4}\cosh^2(\tau)+\frac{t^2}{4}}d\tau = e^{\frac{t^2}{4}}\int_0^{+\infty} e^{-\frac{t^2}{8}(\cosh(2\tau)+1)}d\tau = e^{\frac{t^2}{8}}\int_0^{+\infty} e^{-\frac{t^2}{8}\cosh(2\tau)}d\tau = ..$$

$$.. = \frac{1}{2}e^{\frac{t^2}{8}}\int_0^{+\infty} e^{-\frac{t^2}{8}\cosh(w)}dw = \frac{1}{2}e^{\frac{t^2}{8}}K_0(\frac{t^2}{8}).$$

in the last step we make a change of variable $2\tau = w$ and using well-known integral representation for the modified Bessel function of the second kind of order zero, $K_0(t) = \int_0^{+\infty} e^{-t\cosh(\nu)}d\nu$.

Lemma 1.3. We have the following identity

$$\frac{1}{2}\int_0^{+\infty} e^{-(x+\frac{1}{x})}dx = K_1(2).$$

Proof. Let us start with the following integral representation for the modified Bessel function of the second kind of order zero,

$$K_0(2\sqrt{ab}) = \int_0^{+\infty} e^{-(ax+\frac{b}{x})}\frac{dx}{2x},$$

by taking derivative of the above relation with respect to a, after simplifying we have

$$\frac{d}{da}\{K_0(2\sqrt{ab})\} = -\sqrt{\frac{b}{a}}K_1(2\sqrt{ab}) = -\frac{1}{2}\int_0^{+\infty} e^{-(ax+\frac{b}{x})}dx,$$

or,

$$K_1(2\sqrt{ab}) = \frac{1}{2}\sqrt{\frac{b}{a}}\int_0^{+\infty} e^{-(ax+\frac{b}{x})}dx,$$

by choosing $a = b = 1$ after simplifying we arrive at

$$K_1(2) = \frac{1}{2}\int_0^{+\infty} e^{-(x+\frac{1}{x})}dx.$$

Lemma 1.4. The following integral relation involving the Bessel functions of the first and the second kinds holds

$$\int_0^{+\infty} \frac{\sqrt{t}I_0(2a\sqrt{t})I_0(2b\sqrt{t})dt}{\pi(t+s)} = \sqrt{s}J_0(2a\sqrt{s})J_0(2b\sqrt{s}).$$

Proof. Let us first evaluate $\mathcal{S}^{-1}[\sqrt{s}J_0(2a\sqrt{s})J_0(2b\sqrt{s});t]$, thus we have

$$\mathcal{S}^{-1}[\sqrt{s}J_0(2a\sqrt{s})J_0(2b\sqrt{s});t] = \frac{1}{\pi}\mathcal{I}_m[\lim_{\nu\to\pi^-}\sqrt{te^{i\nu}}J_0(2a\sqrt{te^{i\nu}})J_0(2b\sqrt{te^{i\nu}})],$$

after simplifying we obtain

$$\mathcal{S}^{-1}[\sqrt{s}J_0(2a\sqrt{s})J_0(2b\sqrt{s});t] = \frac{1}{\pi}\mathcal{I}_m[i\sqrt{t}J_0(2ai\sqrt{t})J_0(2bi\sqrt{t})] = \frac{1}{\pi}\mathcal{I}_m[i\sqrt{t}I_0(2a\sqrt{t})I_0(2b\sqrt{t})],$$

finally

$$\mathcal{S}^{-1}[\sqrt{s}J_0(2a\sqrt{s})J_0(2b\sqrt{s});t] = \frac{\sqrt{t}I_0(2a\sqrt{t})I_0(2b\sqrt{t})}{\pi}.$$

The following corollary is an immediate consequence of the above example .

Corollary 1.2.

$$\int_0^{+\infty} \frac{\sqrt{t}\Pi_{k=1}^n I_0(2a_k\sqrt{t})dt}{\pi(t+s)} = \sqrt{s}\Pi_{k=1}^n J_0(2a_k\sqrt{s}).$$

We conclude that many other infinite integrals involving the Bessel functions of this type can be evaluated in this manner by applying the above example and corollary considered here.

One of the principal uses of the Laplace transform is in the solution of certain fractional differential equations. We will briefly illustrate some of the examples of such problems where the Laplace transform is an effective tool.

Lemma 1.5. Let us consider the following fractional differential equation with given initial condition

$$\frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}\phi(t)=D_{0,t}^{C,\frac{1}{2}}\phi(t),\quad \phi(0)=u_0.$$

then the above fractional differential equation has a solution $\phi(t)=u_0\mathcal{H}(t)$.

Proof. By taking the Laplace transform of the above initial value problem term-wise and using initial condition, we get

$$\frac{s\Phi(s)-u_0}{s+1}=\sqrt{s}\Phi(s)-\frac{u_0}{\sqrt{s}}$$

from where after simplifying, we obtain

$$\Phi(s)=\frac{\sqrt{s}-(s+1)}{\sqrt{s}(s-s\sqrt{s}-\sqrt{s})}u_0=\frac{u_0}{s}$$

taking the inverse Laplace transform leads to

$$\phi(t)=u_0\mathcal{H}(t)$$

Example 1.7. Let us solve the following fractional differential equation, where fractional derivative is in the Caputo-Fabrizio sense

$$\frac{1}{2}D_{0,t}^{C.F,\frac{1}{2}}\phi(t)=K_{\pm\frac{1}{2}}(t),\quad \phi(0)=u_0=0$$

in the above equation $K_{\pm\frac{1}{2}}(t)=\sqrt{\frac{\pi}{2t}}e^{-t}$, stands for the modified Bessel function or Macdonald function of order $\pm\frac{1}{2}$.

Solution. By taking the Laplace transform of the above fractional differential equation and using initial condition, we arrive at

$$\frac{s\Phi(s)-u_0}{s+1}=\frac{s\Phi(s)}{s+1}=\frac{\sqrt{\pi}}{\sqrt{2(s+1)}}$$

from which we deduce that

$$\Phi(s)=\frac{\pi}{\sqrt{2}}.\frac{s+1}{s\sqrt{s+1}}=\frac{\pi}{\sqrt{2}}[\frac{1}{\sqrt{s+1}}+\frac{1}{s\sqrt{s+1}}]$$

at this stage, taking the inverse Laplace transform yields

$$\phi(t)=\frac{\pi}{\sqrt{2}}[\frac{e^{-t}\sqrt{t}}{\sqrt{\pi}}+erf(\sqrt{t})]$$

it is easy to verify that $\phi(0)=0$.

1.1. Weyl Fractional Integral and Derivative with Applications

Let us define the Weyl fractional integral and derivative as follows[11]

1. $$\mathcal{W}_{+}^{-\alpha}\phi(x):=\frac{d^{-\alpha}}{dx^{-\alpha}}\phi(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}\phi(\xi)d\xi,$$

2. $$\mathcal{W}_{+}^{\alpha}\phi(x):=\frac{d^{\alpha}}{dx^{\alpha}}\phi(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^{x}(x-\xi)^{n-\alpha-1}\phi(\xi)d\xi,$$

with $\quad n-1<\alpha\leq n.$

Lemma 1.6. Let us assume that $\mathcal{F}\{\phi(x);w\}=\Phi(w)$ then we have

$$\mathcal{F}\{\mathcal{W}_{+}^{-\alpha}\phi(x);w\}=(-iw)^{-\alpha}\Phi(w)=\frac{e^{\frac{i\pi\alpha}{2}}}{w^{\alpha}}\Phi(w).$$

Proof. We have the following chain of relations

$$\mathcal{F}\{\mathcal{W}_{+}^{-\alpha}\phi(x);w\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{iwx}[\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}\phi(\xi)d\xi]dx=..$$

$$..=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{+\infty}\phi(\xi)[\frac{1}{\sqrt{2\pi}}\int_{\xi}^{+\infty}(x-\xi)^{\alpha-1}e^{iwx}dx]d\xi,$$

at this stage making a change of variable $x-\xi=\eta$ in the inner integral, yields

$$\mathcal{F}\{\mathcal{W}_{+}^{-\alpha}\phi(x);w\}=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{+\infty}\phi(\xi)[\frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}\eta^{\alpha-1}e^{iw(\eta+\xi)}d\eta]d\xi=..$$

$$..=\frac{1}{\Gamma(\alpha)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{iw\xi}\phi(\xi)d\xi[\int_{0}^{+\infty}\eta^{\alpha-1}e^{iw\eta}d\eta],$$

or

$$\mathcal{F}\{\mathcal{W}_{+}^{-\alpha}\phi(x);w\}=\frac{1}{\Gamma(\alpha)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{iw\xi}\phi(\xi)d\xi[\frac{\Gamma(\alpha)}{(-iw)^{\alpha}}]=(-iw)^{-\alpha}\Phi(w).$$

In the sequel we evaluate Weyl fractional integral of some elementary functions

Example 1.8. The following relations hold

$$1.\quad \mathcal{F}\{\mathcal{W}_{+}^{-\alpha}\mathcal{H}'(x-\lambda);w\}=\mathcal{F}\{\mathcal{W}_{+}^{-\alpha}\delta(x-\lambda);w\}=\frac{e^{iw\lambda}}{\sqrt{2\pi}}(-iw)^{-\alpha}.$$

$$2.\quad \mathcal{F}\{\mathcal{W}_{+}^{-\alpha}sgn(x);w\}=\sqrt{\frac{2}{\pi}}(-iw)^{-(\alpha+1)}.$$

$$3.\quad \mathcal{F}\{\mathcal{W}_{+}^{-\alpha}\mathcal{H}(x);w\}=(-iw)^{-\alpha}\sqrt{\frac{\pi}{2}}[\delta(w)+\frac{i}{\pi w}].$$

$$4.\quad \mathcal{F}\{\mathcal{W}_{+}^{-\alpha}Ai(\frac{x}{\sqrt[3]{3})});w\}=(-iw)^{-\alpha}e^{\frac{iw^3}{3}}.$$

Solution.

Part 1. We have $\mathcal{H}'(x-\lambda)=\delta(x-\lambda)$ thus we get $\mathcal{F}[\delta(x-\lambda);w]=\frac{1}{\sqrt{2\pi}}e^{iw\lambda}$, in view of the above Example 1.8. we arrive at

$$\mathcal{F}\{\mathcal{W}_{+}^{-\alpha}\delta(x-\lambda);w\}=\frac{e^{iw\lambda}}{\sqrt{2\pi}}(-iw)^{-\alpha}.$$

Part 2. By definition of the signum function, $sgn(x)$ we have $\mathcal{F}[sgn(x);w]=\sqrt{\frac{2}{\pi}}\frac{i}{w}=\sqrt{\frac{2}{\pi}}(-iw)^{-1}$, upon using Example 1.8., we arrive at

$$\mathcal{F}\{\mathcal{W}_{+}^{-\alpha}sgn(x);w\}=\sqrt{\frac{2}{\pi}}(-iw)^{-(\alpha+1)}.$$

Part 3. Let us recall the elementary identity $\mathcal{H}(x)=\frac{1}{2}[1+sgn(x)]$, from which we deduce that $\mathcal{F}[\mathcal{H}(x);w]=\sqrt{\frac{\pi}{2}}[\delta(w)+(-iw\pi)^{-1}$, thus we get

$$\mathcal{F}\{\mathcal{W}_{+}^{-\alpha}\mathcal{H}(x);w\}=(-iw)^{-\alpha}\sqrt{\frac{\pi}{2}}[\delta(w)+\frac{i}{\pi w}].$$

Part 4. By using the fact that $\mathcal{F}[Ai(\frac{x}{\sqrt[3]{3})});w]=e^{\frac{iw^3}{3}}$, and using Example 1.8., we arrive at

$$\mathcal{F}\{\mathcal{W}_{+}^{-\alpha}Ai(\frac{x}{\sqrt[3]{3})});w\}=(-iw)^{-\alpha}e^{\frac{iw^3}{3}}.$$

2. Main Results

Solution For Fractional Singular Integro-Differential Equation (FSIE) With Trigonometric Kernel. Before concluding this work, we choose to address further results which presumably provide an important confirmation of the usefulness of the operational methods that we have discussed so far.

Theorem 2.1. Let us consider the following fractional singular integral equation, where fractional derivative is in the Caputo-Fabrizio sense

$$\frac{1}{2} D_{0,t}^{C.F,\frac{1}{2}} \phi(t) = \frac{1}{\sqrt{\pi t}} \int_0^{+\infty} \cos(2\sqrt{t\xi})\phi(\xi) d\xi, \qquad \phi(0) = 1$$

the above fractional singular integral equation has the following formal solution,

$$\phi(t) = \int_0^t \frac{\sqrt{(t-\xi)}}{\sqrt{\pi}} e^{\frac{\xi}{2}} \sin(\frac{\sqrt{3}\xi}{2}) d\xi - \frac{\sqrt{t}}{\sqrt{\pi}} - e^{\frac{t}{2}} \sin(\frac{\sqrt{3}t}{2})$$

Proof. By taking the Laplace transform of the above FSIE term wise and using initial condition we obtain

$$\frac{s\Phi(s) - 1}{s+1} = \frac{1}{\sqrt{s}} \Phi(\frac{1}{s}) \qquad (2.1.)$$

in the above relation(2.1.) let us replace s by $\frac{1}{s}$ to obtain

$$\frac{\frac{1}{s}\Phi(\frac{1}{s}) - 1}{\frac{1}{s} + 1} = \sqrt{s}\Phi(s) \qquad (2.2.)$$

from (2.2.) we get

$$\Phi(\frac{1}{s}) = s + (s+1)\sqrt{s}\Phi(s) \qquad (2.3.)$$

by setting (2.3.) in (2.1.) we obtain

$$\frac{s\Phi(s) - 1}{s+1} = \frac{1}{\sqrt{s}}[s + (s+1)\sqrt{s}\Phi(s)], \qquad (2.4.)$$

from (2.4.) we obtain $\Phi(s)$ as follows

$$\Phi(s) = \frac{1}{\sqrt{s}(s^2+s+1)} - \frac{1}{\sqrt{s}} - \frac{1}{s^2+s+1}$$

taking the inverse Laplace transform of the above relation term-wise, we get the final solution to FSIE as follows

$$\phi(t) = \frac{1}{\sqrt{\pi}} \int_0^t \sqrt{(t-\xi)} e^{\frac{\xi}{2}} \sin(\frac{\sqrt{3}\xi}{2}) d\xi - \sqrt{\frac{t}{\pi}} - e^{\frac{t}{2}} \sin(\frac{\sqrt{3}t}{2})$$

The Laplace transform is used in a variety of applications, in fact the most common usage of the Laplace transform is in the evaluation of certain infinite sums involving the special functions.

Lemma 2.1. The following identity holds true

$$\psi(t) = \sum_{k=1}^{+\infty} (-1)^{k-1} D_{0,t}^{C,\frac{1}{2}} [e^{ak+a^2 t} erfc(a\sqrt{t} + \frac{k}{2\sqrt{t}})] = ..$$

$$.. = -\frac{1}{2\pi} \int_0^{+\infty} \frac{e^{-tr}}{a^2+r} \frac{a\sin(\sqrt{r}) + \sqrt{r}\cos(\sqrt{r}) + \sqrt{r}}{1+\cos\sqrt{r}} dr.$$

Proof. By taking the Laplace transform of the left hand side and in view of the table of the Laplace transforms, we have

$$\mathcal{L}\{\psi(t); s\} = \Psi(s) = \sum_{k=1}^{+\infty} (-1)^{k-1} \mathcal{L}\{D_{0,t}^{C,\frac{1}{2}} [e^{ak+a^2 t} erfc(a\sqrt{t} + \frac{k}{2\sqrt{t}})]; s\} = \sum_{k=1}^{+\infty} \sqrt{s}[\frac{(-1)^{k-1} e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})}],$$

or

$$\mathcal{L}\{\psi(t);s\} = \Psi(s) = \frac{1}{(a+\sqrt{s})(1+e^{\sqrt{s}})},$$

at this point, since $s = 0$ is a branch point of the transformed function $\Psi(s)$, in view of the Titchmarsh inversion theorem we have[5]

$$\psi(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\mathcal{I}m\{lim_{\beta\to\pi^-}\Psi(re^{-i\beta})\}dr,$$

from which we deduce that

$$\psi(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\mathcal{I}m\{lim_{\beta\to\pi^-}\frac{1}{(a+\sqrt{re^{-i\beta}})(1+e^{\sqrt{re^{-i\beta}}})},\}dr = ..$$

$$.. = \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\mathcal{I}m[\frac{1}{(a+i\sqrt{r})(1+e^{i\sqrt{r}})}]dr,$$

finally, we may easily obtain the following result

$$\psi(t) = -\frac{1}{2\pi}\int_0^{+\infty}\frac{e^{-tr}}{a^2+r}\frac{a\sin(\sqrt{r})+\sqrt{r}\cos(\sqrt{r})+\sqrt{r}}{1+\cos\sqrt{r}}dr.$$

Let us consider the special case $a = 0$, we get the following result

$$\psi(t) = \sum_{k=1}^{+\infty}(-1)^{k-1}D_{0,t}^{C,\frac{1}{2}}erfc(\frac{k}{2\sqrt{t}}) = -\frac{1}{2\pi}\int_0^{+\infty}\frac{e^{-tr}}{r}\frac{\sqrt{r}\cos(\sqrt{r})+\sqrt{r}}{1+\cos(\sqrt{r})}dr = -\frac{1}{2\sqrt{\pi t}}.$$

Lemma 2.2. The following identity holds

$$\int_0^{+\infty}\frac{J_0(a\sqrt{t})J_0(b\sqrt{t})}{t+s}dt = 2I_0(a\sqrt{s})K_0(b\sqrt{s}).$$

Proof. Let us consider the following singular integral equation of Stieltjes-type

$$\int_0^{+\infty}\frac{\psi(t)}{t+s}dt = 2I_0(a\sqrt{s})K_0(b\sqrt{s}),$$

after solving the above integral equation, we show that $\psi(t) = J_0(a\sqrt{t})J_0(b\sqrt{t})$, by using inversion formula for the Stieltjes transform we get

$$\psi(t) = \frac{2}{\pi}\mathcal{I}m[\lim_{\beta\to\pi^-}[2I_0(a\sqrt{te^{-i\beta}})K_0(b\sqrt{te^{-i\beta}})] = \frac{2}{\pi}\mathcal{I}m\{I_0(ia\sqrt{t})K_0(ib\sqrt{t})\},$$

at this stage let us recall some identites for the Bessel functions

$$I_0(iz) = J_0(z), \qquad K_0(z) = \frac{i\pi}{2}H_0^1(iz) = \frac{i\pi}{2}\{J_0(iz)+iY_0(iz)\},$$

from which we deduce that

$$\psi(t) = \frac{2}{\pi}\mathcal{I}m\{J_0(a\sqrt{t})[J_0(-b\sqrt{t})+iY_0(-b\sqrt{t})]\frac{i\pi}{2}\} = J_0(a\sqrt{t})\mathcal{I}m\{i[J_0(-b\sqrt{t})+iY_0(-b\sqrt{t})]\} = ..$$

$$.. = J_0(a\sqrt{t})J_0(-b\sqrt{t}) = J_0(a\sqrt{t})J_0(b\sqrt{t}),$$

in the special case $a = b$, we have the following identity

$$\int_0^{+\infty}\frac{J_0^2(a\sqrt{t})}{t+s}dt = 2I_0(a\sqrt{s})K_0(a\sqrt{s}).$$

An immediate consequence of the above lemma is given as follows,
Corollary 2.1. The following identity holds true

$$\int_0^{+\infty} \frac{\xi^{\frac{\nu}{2}} J_\nu(2\sqrt{t\xi})}{\xi + s} d\xi = 2s^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(2\sqrt{ts}).$$

Proof. By following the same procedure as previous lemma.
Lemma 2.3. Let us consider the following singular integral equation with trigonometric kernel

$$\int_0^{+\infty} \frac{\sin(2\sqrt{t\xi})\phi(\xi)}{\sqrt{\pi\xi}} d\xi = \cos(\lambda t),$$

then the above integral equation has the following formal solution in terms of the Caputo fractional derivative of half order,

$$\phi(t) = D_{0,t}^{C,\frac{1}{2}}\{-\cos(\frac{t}{\lambda})\}.$$

Proof. Taking the Laplace transform of the above singular integral equation, we arrive at

$$\mathcal{L}\{\int_0^{+\infty} \frac{\sin(2\sqrt{t\xi})\phi(\xi)}{\sqrt{\pi\xi}} d\xi; s\} = \frac{1}{s\sqrt{s}}\Phi(\frac{1}{s}) = \frac{s}{s^2 + \lambda^2},$$

at this point, changing $s \to \frac{1}{s}$ yields

$$s\sqrt{s}\Phi(s) = \frac{\frac{1}{s}}{\frac{1}{s^2} + \lambda^2} = \frac{s}{1 + \lambda^2 s^2},$$

after simplifying we get

$$\Phi(s) = \frac{1}{\lambda\sqrt{s}} \frac{\frac{1}{\lambda}}{s^2 + (\frac{1}{\lambda})^2},$$

taking the inverse Laplace transform and in view of the convolution theorem, we obtain

$$\phi(t) = \frac{1}{\lambda}\int_0^t \frac{1}{\sqrt{\pi(t-\xi)}} \sin(\frac{\xi}{\lambda}) d\xi = D_{0,t}^{C,\frac{1}{2}}[-\cos(\frac{\xi}{\lambda})]',$$

the obtained solution satisfies the integral equation, thus we have

$$\int_0^{+\infty} \frac{\sin(2\sqrt{t\xi}) D_{0,\xi}^{C,\frac{1}{2}}[-\cos(\frac{w}{\lambda})]',}{\sqrt{\pi\xi}} d\xi = \cos(\lambda t),$$

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