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Existence of Zeros for a Class of Operators

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ABSTRACT: Let E be a real Banach space. In this note, we establish conditions guaranteeing the existence of zeros of an affine operator $A : E \to E$. We then extend these results to a class of operators that can be approximated by affine operators. Furthermore, in the finite-dimensional setting, we provide sufficient conditions for the existence of zeros of continuously differentiable operators, without appealing to any min–max theorem. Our approach relies on the introduction of a functional parameter associated with convex subsets of both the Banach space and its dual.

Keywords: Banach space, affine operator, continuously differentiable operator.

Contents

1. Introduction

E denotes a real Banach space and E^* its topological dual. The norm and the duality pairing on $E^* \times E$ are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively; for simplicity, we also use $\|\cdot\|$ for the norm on both E and E^*.

For any subset $X \subseteq E$, we denote by $\overline{\text{conv}}(X)$ the closure of the convex hull of X, i.e., the smallest closed convex set containing X. For any nonempty subset $B \subseteq E$ and any $x \in E$,

$$\text{dist}(x, B) = \inf\{\|x - y\| : y \in B\}$$

denotes the distance from x to B.

Let $W \subseteq E^*$ be an open convex set containing the closed unit ball of E^*. For subsets $X \subseteq E$ and $Y \subseteq E^*$, we define

$$\delta_{X,Y} := \inf_{\psi \in \mathcal{C}_{X,Y}} \left(\sup_{(x,y)\in X\times Y} \psi(x,y) - \inf_{(x,y)\in X\times Y} \psi(x,y) \right), \tag{1.1}$$

where $\mathcal{C}_{X,Y}$ denotes the class of all non-constant functions $\psi : X \times Y \to \mathbb{R}$ that are convex in $x \in X$ and concave in $y \in Y$.

The parameter $\delta_{X,Y}$ serves as a unifying tool connecting geometric properties of the sets X and Y (such as boundedness, convexity, or the presence of unbounded directions) with analytical features of convex–concave functionals. In particular, the behavior of $\delta_{X,Y}$ under topological or metric perturbations of X and Y allows one to detect critical threshold phenomena that are essential in applications to minimax problems, variational inequalities, and stability analysis in infinite-dimensional Banach spaces.

The geometric and variational tools used in this work belong to a classical framework that combines functional analysis, convex analysis, and minimax theory. Fundamental references in functional and convex analysis include [2,4,13,11,6,15]. Minimax principles originating from the seminal works of Fan [7], Sion [12], and Kneser [8] provide key duality mechanisms for separation-type results in vector spaces. In addition, the study of zeros of nonlinear and affine operators is closely related to degree theory, fixed-point arguments, and stability properties of differentiable mappings [14,5,9,10,1,3]. Our contribution

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integrates these ideas to obtain new separation criteria for the existence (or nonexistence) of zeros of affine, approximable, and continuously differentiable vector-valued operators.

The paper is organized as follows. In Section 2, we establish separation-type criteria for the existence of zeros of affine operators, relying on the functional parameter $\delta_{X,Y}$ and on geometric minimax arguments. Within the same section, we also extend these results to operators that admit good affine approximations on suitable subsets of the domain. In Section 3, we develop a differentiable counterpart of these criteria in the finite-dimensional setting, where the existence of zeros is derived from Lipschitz estimates of the derivative, without invoking any minimax theorem. Finally, illustrative examples are provided to highlight the applicability of our approach and to clarify the role played by the distance parameter $\delta_{X,Y}$.

2. Zeros of Affine Operators in Banach Spaces

In this note, we investigate an affine operator $\Phi : E \to E$. By combining the functional parameter $\delta_{X,Y}$ with the classical minimax theorem (Theorem 2.1; see, for instance, [8]), we obtain necessary and sufficient conditions for the existence of zeros of affine operators in Banach spaces. We then extend these results to a class of operators that can be approximated by affine operators.

Theorem 2.1 *Let E be a vector space, F a locally convex topological vector space, $X \subseteq E$ a convex set, and $Y \subset F$ a compact convex set. Let*

$$h : X \times Y \to \mathbb{R}$$

be a function that is convex in $x \in X$ and upper semicontinuous and concave in $y \in Y$. Then the following equality holds:

$$\sup_{y\in Y} \inf_{x\in X} h(x,y) = \inf_{x\in X} \sup_{y\in Y} h(x,y).$$

The next theorem is the main result of this note. It provides a nonlinear separation-type condition ensuring that the origin does not belong to the closure of the convex hull of the image of a set under an affine operator.

Theorem 2.2 *Let $\Omega \subseteq E$ be an open convex set, and let $\Phi : \Omega \to E$ be an affine operator. Let $V \subseteq \Omega$ be a set such that*

$$\eta := \inf_{x\in V} \|\Phi(x)\| > 0.$$

Then, for every set $X \subset V$ and every set $Y \subseteq W$ such that $\delta_{X,Y} < \eta$, one has

$$0 \notin \overline{\mathrm{conv}}(\Phi(X)).$$

Proof: By Remark 2.1 below, we may fix a set $X \subset V$, a set $Y \subset W$, and a function

$$\psi : X \times Y \to \mathbb{R}$$

that is convex in x, upper semicontinuous with respect to the weak* topology in y, and concave in y, such that

$$\sup_{(x,y)\in X\times Y} \psi(x,y) - \inf_{(x,y)\in X\times Y} \psi(x,y) < \eta. \tag{2.1}$$

Set $Y = B_{E^*} := \{y \in E^* : \|y\| \le 1\}$. Define the functions

$$f : \Omega \times B_{E^*} \to \mathbb{R}, \qquad f(x,y) = \langle y, \Phi(x)\rangle,$$

and

$$g : X \times B_{E^*} \to \mathbb{R}, \qquad g(x,y) = f(x,y) + \psi(x,y).$$

We first show that g satisfies the hypotheses of Theorem 2.1. For each $y \in B_{E^*}$, both $x \mapsto \psi(x,y)$ and $x \mapsto f(x,y)$ are convex; hence $x \mapsto g(x,y)$ is also convex. Since B_{E^*} is weak* compact and, for each $x \in X$, $y \mapsto \psi(x,y)$ is upper semicontinuous and concave, it follows that $y \mapsto g(x,y)$ is upper semicontinuous and concave as well. Thus Theorem 2.1 yields

$$\sup_{y\in B_{E^*}} \inf_{x\in X} g(x,y) = \inf_{x\in X} \sup_{y\in B_{E^*}} g(x,y). \tag{2.2}$$

We now prove that

$$\inf_{x\in X} \sup_{y\in B_{E^*}} f(x,y) - \sup_{y\in B_{E^*}} \inf_{x\in X} f(x,y) \le \sup_{(x,y)\in X\times B_{E^*}} \psi(x,y) - \inf_{(x,y)\in X\times B_{E^*}} \psi(x,y). \tag{2.3}$$

Indeed, for every $x \in X$ and $y \in B_{E^*}$, we have

$$\inf_{x\in X} g(x,y) \le \inf_{x\in X} f(x,y) + \sup_{(x,y)\in X\times B_{E^*}} \psi(x,y),$$

and

$$\sup_{y\in B_{E^*}} g(x,y) \ge \sup_{y\in B_{E^*}} f(x,y) + \inf_{(x,y)\in X\times B_{E^*}} \psi(x,y).$$

Taking the supremum in y in the first inequality and the infimum in x in the second inequality, and using (2.2), we obtain (2.3).

Since

$$\inf_{x\in X} \sup_{y\in B_{E^*}} f(x,y) = \inf_{x\in X} \sup_{\|y\|\le 1} \langle y, \Phi(x)\rangle = \inf_{x\in X} \|\Phi(x)\|,$$

inequality (2.3), together with (2.1) and $\eta \le \inf_{x\in X} \|\Phi(x)\|$, yields

$$\sup_{y\in B_{E^*}} \inf_{x\in X} \langle y, \Phi(x)\rangle > 0.$$

Hence there exist $y_0 \in B_{E^*}$ and $\alpha > 0$ such that

$$\langle y_0, \Phi(x)\rangle \ge \alpha \quad \text{for all } x \in X.$$

Define

$$C = \{z \in E : \langle y_0, z\rangle \ge \alpha\}.$$

The set C is closed and convex, $0 \notin C$, and

$$\overline{\operatorname{conv}}(\Phi(X)) \subseteq C.$$

Thus $0 \notin \overline{\operatorname{conv}}(\Phi(X))$, as claimed. □

Remark 2.1 *In the proof of Theorem 2.2, fix any $x_0 \in X$ and consider the function*

$$\psi(x,y) = \|x\| - \frac{\eta}{2}\|y\|.$$

For the particular choice $X = \{x_0\}$ and $Y = \{y \in E^* : \|y\| \le 1\}$, one has

$$\sup_{(x,y)\in X\times Y} \psi(x,y) - \inf_{(x,y)\in X\times Y} \psi(x,y) = \frac{\eta}{2} < \eta.$$

Thus, for these sets X and Y, we obtain $\delta_{X,Y} \le \frac{\eta}{2} < \eta$.

Based on Theorem 2.2, we now establish the following result.

Theorem 2.3 *Let $\Omega \subseteq E$ be an open convex set, and let $\Phi : \Omega \to E$ be an affine operator. Let $V \subset \Omega$ be such that $\Phi(V)$ is closed. Then the following statements are equivalent:*

i) $0 \in \Phi(V)$*;*

ii) For every $\varepsilon > 0$*, there exist sets* $X \subset V$ *and* $Y \subset W$ *such that* $0 \in \overline{\mathrm{conv}}(\Phi(X))$ *and* $\delta_{X,Y} < \varepsilon$*.*

Proof: Assume first that $0 \in \Phi(V)$. Then there exists $x_0 \in V$ such that $\Phi(x_0) = 0$. For the singleton sets $X = \{x_0\}$ and $Y = \{0\}$, one clearly has

$$\delta_{X,Y} = 0.$$

Thus condition (ii) is satisfied.

Conversely, assume that (ii) holds. We show that $0 \in \Phi(V)$. Suppose, for contradiction, that $0 \notin \Phi(V)$. Since $\Phi(V)$ is closed, we must have

$$\inf_{x \in V} \|\Phi(x)\| > 0.$$

By assumption (ii), there exist sets $X \subset V$ and $Y \subset W$ such that

$$\delta_{X,Y} < \inf_{x \in V} \|\Phi(x)\| \qquad \text{and} \qquad 0 \in \overline{\mathrm{conv}}(\Phi(X)).$$

However, Theorem 2.2 asserts that whenever $\delta_{X,Y} < \inf_{x \in V} \|\Phi(x)\|$, one necessarily has

$$0 \notin \overline{\mathrm{conv}}(\Phi(X)),$$

yielding a contradiction. Thus $0 \in \Phi(V)$, and the proof is complete. □

We now present an example illustrating the situation described in Theorem 2.3, in which the operator Φ fails to satisfy the conditions of this Theorem.

Example 2.1 *Let* $E = \ell^p$ *with* $1 < p < \infty$ *and* $p \neq 2$*, endowed with the usual norm*

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Since $p \neq 2$*, the space* E *is a Banach space which is not a Hilbert space. Consider the convex set*

$$V = \{x \in \ell^p : \|x\|_p \leq 1\},$$

and define the operator

$$\Phi : V \to \ell^p, \qquad \Phi(x) = e_1 + \varepsilon x,$$

where $e_1 = (1, 0, 0, \dots)$ *and* $\varepsilon > 0$ *is a fixed constant.*

For every $x \in V$*,*

$$\|\Phi(x)\|_p \geq \|e_1\|_p - \|\varepsilon x\|_p \geq 1 - \varepsilon > 0,$$

so Φ *has no zeros on* V*. The closure of the convex hull of the image of any subset* $X \subset V$ *under* Φ *satisfies*

$$\overline{\mathrm{conv}}(\Phi(X)) \subset \overline{\mathrm{conv}}\big(\{\, e_1 + \varepsilon x : x \in X \,\}\big).$$

Since every element in the image has its first coordinate greater than or equal to $1 - \varepsilon > 0$*, it follows that*

$$0 \notin \overline{\mathrm{conv}}(\Phi(X)).$$

Hence, condition ii) *fails to be satisfied.*

We now extend the preceding results to a class of operators that can be approximated on V by affine operators.

Theorem 2.4 *Let $\Omega \subseteq E$ be an open convex set, and let $\Phi : \Omega \to E$ be any operator. Let $V \subseteq \Omega$ be such that*

$$\eta := \inf_{x \in V} \|\Phi(x)\| > 0.$$

Assume that there exists an affine operator $A : \Omega \to E$ and a constant $\delta \geq 0$ such that

$$\sup_{x \in V} \|\Phi(x) - A(x)\| \leq \delta.$$

Then, for every set $X \subset V$ and every set $Y \subset W$ satisfying $\delta_{X,Y} < \eta - \delta$, one has

$$0 \notin \overline{\operatorname{conv}}(A(X)) \quad \text{and} \quad \operatorname{dist}\big(0, \overline{\operatorname{conv}}(\Phi(X))\big) \geq \operatorname{dist}\big(0, \overline{\operatorname{conv}}(A(X))\big) - \delta.$$

Moreover, if

$$\operatorname{dist}\big(0, \overline{\operatorname{conv}}(A(X))\big) > \delta,$$

then

$$0 \notin \overline{\operatorname{conv}}(\Phi(X)).$$

Proof: For every $x \in V$,

$$\|A(x)\| \geq \|\Phi(x)\| - \|\Phi(x) - A(x)\| \geq \eta - \delta.$$

Hence

$$\eta_A := \inf_{x \in X} \|A(x)\| \geq \eta - \delta > 0.$$

Since A is affine, Theorem 2.2 applies to A. Because $\delta_{X,Y} < \eta - \delta \leq \eta_A$, we obtain

$$0 \notin \overline{\operatorname{conv}}(A(X)).$$

Next, let $z = \sum_i \lambda_i \Phi(x_i) \in \operatorname{conv}(\Phi(X))$, with $x_i \in X$, $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$. Define

$$z_A := \sum_i \lambda_i A(x_i) \in \operatorname{conv}(A(X)).$$

Then

$$\|z_A - z\| = \Big\| \sum_i \lambda_i \big(A(x_i) - \Phi(x_i)\big) \Big\| \leq \sum_i \lambda_i \|A(x_i) - \Phi(x_i)\| \leq \delta.$$

Thus

$$\operatorname{conv}(\Phi(X)) \subseteq \operatorname{conv}(A(X)) + B(0, \delta),$$

and therefore

$$\overline{\operatorname{conv}}(\Phi(X)) \subseteq \overline{\operatorname{conv}}(A(X)) + B(0, \delta).$$

Consequently, for every $z \in \overline{\operatorname{conv}}(\Phi(X))$ there exist $a \in \overline{\operatorname{conv}}(A(X))$ and $b \in B(0, \delta)$ such that $z = a + b$. Hence,

$$\|z\| \geq \|a\| - \|b\| \geq \operatorname{dist}\big(0, \overline{\operatorname{conv}}(A(X))\big) - \delta.$$

Taking the infimum over all such z gives

$$\operatorname{dist}\big(0, \overline{\operatorname{conv}}(\Phi(X))\big) \geq \operatorname{dist}\big(0, \overline{\operatorname{conv}}(A(X))\big) - \delta.$$

If $\operatorname{dist}(0, \overline{\operatorname{conv}}(A(X))) > \delta$, then the right-hand side is strictly positive, and therefore $0 \notin \overline{\operatorname{conv}}(\Phi(X))$. □

As a consequence of Theorem 2.4, we derive the following result.

Theorem 2.5 *Let $\Omega \subseteq E$ be an open convex set and let $\Phi : \Omega \to E$ be an operator. Assume that $V \subset \Omega$ is such that $\Phi(V)$ is closed and convex. Suppose that for every $\varepsilon > 0$ and every $X \subset V$, there exists an affine operator $A : \Omega \to E$ such that*

$$\frac{1}{2} \inf_{x \in V} \|\Phi(x)\| \ \leq \ \inf_{x \in V} \|A(x)\| \qquad \textit{and} \qquad \sup_{u \in X} \|\Phi(u) - A(u)\| < \varepsilon.$$

Then the following assertions are equivalent:

i) $0 \in \Phi(V)$*;*

ii) For every $\varepsilon > 0$, there exist sets $X \subset V$ and $Y \subset W$ such that

$$0 \in \overline{\mathrm{conv}}\big(\Phi(X)\big) \qquad \textit{and} \qquad \delta_{X,Y} < \varepsilon.$$

Proof: Assume first that $0 \in \Phi(V)$. Then there exists $\overline{x} \in V$ such that $\Phi(\overline{x}) = 0$. For the singleton sets $X = \{\overline{x}\}$ and $Y = \{0\}$, we have $\delta_{X,Y} = 0$. Thus condition (ii) holds.

Conversely, assume that (ii) holds. Suppose, for contradiction, that $0 \notin \Phi(V)$. Since $\Phi(V)$ is closed, it follows that

$$\eta := \inf_{x \in V} \|\Phi(x)\| > 0.$$

Let

$$d := \mathrm{dist}(0, \Phi(V)) > 0.$$

Choose

$$0 < \varepsilon < \min\left\{\frac{d}{2}, \frac{\eta}{2}\right\}.$$

By assumption (ii), there exist sets $X \subset V$ and $Y \subset W$ such that

$$0 \in \overline{\mathrm{conv}}\big(\Phi(X)\big) \qquad \text{and} \qquad \delta_{X,Y} < \varepsilon.$$

For this X and ε, the hypothesis provides an affine operator $A : \Omega \to E$ such that

$$\frac{\eta}{2} \leq \inf_{x \in V} \|A(x)\| =: \eta_A \qquad \text{and} \qquad \sup_{u \in X} \|\Phi(u) - A(u)\| < \varepsilon.$$

As in the proof of Theorem 2.4, the approximation $\sup_{u \in X} \|\Phi(u) - A(u)\| < \varepsilon$ implies that

$$\mathrm{dist}\big(0, \overline{\mathrm{conv}}(A(X))\big) \geq \mathrm{dist}\big(0, \overline{\mathrm{conv}}(\Phi(X))\big) - \varepsilon.$$

Since $\Phi(V)$ is closed and convex, we have

$$\overline{\mathrm{conv}}\big(\Phi(X)\big) \subseteq \Phi(V),$$

and therefore

$$\mathrm{dist}\big(0, \overline{\mathrm{conv}}(\Phi(X))\big) \geq d.$$

Hence,

$$\mathrm{dist}\big(0, \overline{\mathrm{conv}}(A(X))\big) \geq d - \varepsilon > \frac{d}{2} > \varepsilon.$$

However, by Theorem 2.4, the assumptions

$$\delta_{X,Y} < \varepsilon \qquad \text{and} \qquad \eta_A \geq \frac{\eta}{2} > \varepsilon$$

would imply that

$$0 \notin \overline{\mathrm{conv}}(\Phi(X)),$$

contradicting the choice of X in (ii).

Thus our assumption was false, and $0 \in \Phi(V)$. □

3. Zeros of Continuously Differentiable Vector-Valued Operators

In this section, we provide sufficient conditions for the existence of zeros of continuously differentiable operators in the finite-dimensional case, without using Theorem 2.1. Throughout, for each bounded set $Y \subset W$ we set

$$M_Y := \sup_{y \in Y} \|y\|.$$

For each bounded convex set $X \subset V$, we denote

$$\operatorname{diam}(X) := \sup_{(a,b) \in X^2} \|a - b\|,$$

and for each continuously differentiable operator Φ, we let L_X be the Lipschitz constant of its derivative on $\overline{X}$; that is,

$$\|D\Phi(x) - D\Phi(u)\|_{\mathcal{L}(E)} \leq L_X \|x - u\|, \qquad \forall x, u \in \overline{X}.$$

In contrast with Theorem 2.2, the following result is proved without invoking any minimax theorem.

Theorem 3.1 *Let E be a finite-dimensional space, and let Ω be an open convex subset of E. Let $\Phi \in C^1(\Omega, E)$, and let $V \subset \Omega$ satisfy*

$$\eta := \inf_{x \in V} \|\Phi(x)\| > 0.$$

Assume that for each bounded set $X \subset V$ and each bounded set $Y \subset W$ containing the unit ball one has

$$\|D\Phi(u)\|_{\mathcal{L}(E)} \leq \frac{\delta_{X,Y}}{M_Y \operatorname{diam}(X)}, \qquad \forall u \in \overline{X}.$$

Then for each bounded set $X \subset V$ and each bounded set $Y \subset W$ containing the unit ball and satisfying

$$L_X M_Y \operatorname{diam}(X)^2 + \delta_{X,Y} \leq \frac{\eta}{2},$$

one has

$$0 \notin \operatorname{conv}(\Phi(X)).$$

Proof: Since $D\Phi$ is continuous on the compact set $\overline{X}$, there exists $L_X > 0$ such that

$$\|D\Phi(x) - D\Phi(u)\|_{\mathcal{L}(E)} \leq L_X \|x - u\|, \qquad \forall x, u \in \overline{X}. \tag{3.1}$$

Assume, by contradiction, that $0 \in \operatorname{conv}(\Phi(X))$. Then there exist $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\sum_{i=1}^n \lambda_i \Phi(x_i) = 0. \tag{3.2}$$

Fix $y \in Y$, $u \in \overline{X}$, and $i \in \{1, \ldots, n\}$. Consider the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) = \langle y, \Phi(u + t(x_i - u)) \rangle.$$

Since $\Phi \in C^1$, the function g is also C^1, with derivative

$$g'(t) = \langle y, D\Phi(u + t(x_i - u))(x_i - u) \rangle.$$

Thus,

$$\langle y, \Phi(x_i) \rangle - \langle y, \Phi(u) \rangle = \int_0^1 g'(t)\, dt = \int_0^1 \langle y, D\Phi(u + t(x_i - u))(x_i - u) \rangle\, dt.$$

Using (3.1), we obtain

$$\big| \langle y, \Phi(x_i) \rangle - \langle y, \Phi(u) \rangle - \langle y, D\Phi(u)(x_i - u) \rangle \big| \leq \|y\|\, L_X \|x_i - u\|^2.$$

Since $\|y\| \le M_Y$, the right-hand side is bounded by $L_X M_Y \operatorname{diam}(X)^2$. Hence

$$\langle y, \Phi(x_i)\rangle \ge \langle y, \Phi(u)\rangle + \langle y, D\Phi(u)(x_i - u)\rangle - L_X M_Y \operatorname{diam}(X)^2.$$

Furthermore,

$$|\langle y, D\Phi(u)(x_i - u)\rangle| \le \|y\| \, \|D\Phi(u)\| \, \|x_i - u\| \le M_Y \|D\Phi(u)\|_{\mathcal{L}(E)} \operatorname{diam}(X),$$

and therefore

$$\langle y, \Phi(x_i)\rangle \ge \langle y, \Phi(u)\rangle - M_Y \|D\Phi(u)\|_{\mathcal{L}(E)} \operatorname{diam}(X) - L_X M_Y \operatorname{diam}(X)^2.$$

Multiplying by λ_i and summing over $i = 1, \dots, n$, we obtain

$$\sum_{i=1}^{n} \lambda_i \langle y, \Phi(x_i)\rangle \ge \langle y, \Phi(u)\rangle - L_X M_Y \operatorname{diam}(X)^2 - \delta_{X,Y},$$

where we used the assumption $\|D\Phi(u)\|_{\mathcal{L}(E)} \le \delta_{X,Y}/(M_Y \operatorname{diam}(X))$.

Since the left-hand side equals 0 by (3.2), choose $y \in Y$ with $\|y\| = 1$ and $\langle y, \Phi(u)\rangle = \|\Phi(u)\|$. Then

$$0 \ge \|\Phi(u)\| - L_X M_Y \operatorname{diam}(X)^2 - \delta_{X,Y}.$$

Hence

$$\|\Phi(u)\| \le L_X M_Y \operatorname{diam}(X)^2 + \delta_{X,Y}.$$

But $\|\Phi(u)\| \ge \eta$, so we obtain

$$\eta \le L_X M_Y \operatorname{diam}(X)^2 + \delta_{X,Y},$$

which contradicts the hypothesis

$$L_X M_Y \operatorname{diam}(X)^2 + \delta_{X,Y} \le \frac{\eta}{2}.$$

Therefore $0 \notin \operatorname{conv}(\Phi(X))$. □

Theorem 3.2 *Let E be a finite-dimensional space, and let Ω be an open convex subset of E. Let $\Phi \in C^1(\Omega, E)$, and assume that $V \subset \Omega$ is such that $\Phi(V)$ is closed.*

Suppose that the following two conditions hold:

i) For each bounded set $X \subset V$ and each bounded set $Y \subset W$ containing the unit ball,

$$\|D\Phi(u)\|_{\mathcal{L}(E)} \le \frac{\delta_{X,Y}}{M_Y \operatorname{diam}(X)}, \qquad \forall u \in \overline{X};$$

ii) For every $\varepsilon > 0$, there exist a bounded set $X \subset V$ and a bounded set $Y \subset W$ containing the unit ball such that

$$0 \in \operatorname{conv}(\Phi(X)) \qquad \textit{and} \qquad L_X M_Y \operatorname{diam}(X)^2 + \delta_{X,Y} \le \varepsilon.$$

Then $0 \in \Phi(V)$.

Proof: Assume, for contradiction, that $0 \notin \Phi(V)$. Since $\Phi(V)$ is closed, we may set

$$\eta := \inf_{x \in V} \|\Phi(x)\| > 0.$$

By condition (ii), applied to $\varepsilon = \eta/2$, there exist a bounded set $X \subset V$ and a bounded set $Y \subset W$ containing the unit ball such that

$$0 \in \operatorname{conv}(\Phi(X)) \qquad \text{and} \qquad L_X M_Y \operatorname{diam}(X)^2 + \delta_{X,Y} \le \frac{\eta}{2}.$$

On the other hand, condition (i) ensures that the hypotheses of Theorem 3.1 are satisfied for these sets X and Y. Therefore Theorem 3.1 implies that

$$0 \notin \operatorname{conv}(\Phi(X)),$$

which contradicts the above choice of X.

Hence our assumption was false, and we conclude that $0 \in \Phi(V)$. □

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