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G-Convergences of Submethods

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ABSTRACT: Classical topological notions such as openness, closedness, and continuity can be expressed in sequential form, particularly in the case of first-countable Hausdorff spaces. Motivated by this idea, a generalized convergence method known as the G-method, has been introduced. Different studies related to the G-method have been widely explored in recent literature, including studies on G-continuity, G-compactness, and other related topological properties. In this study, the theory of the G-method defined on a set X is extended by introducing the concept of G-submethod G_Y, induced on a non-empty subset $Y \subseteq X$. We examine the behaviours of G-open and G-closed subsets under these submethods. Several characterizations are provided to determine when G_Y-closedness and G_Y-openness are preserved within submethods. In addition, we also investigate the preservation of topological properties such as G-compactness, and separation axioms G-T_0, G-T_1, and G-Hausdorff for G-submethods.

Keywords: G-convergence, G_Y-convergence, G_s-closed subset.

Contents

1. Introduction

Convergence is a central notion in mathematics, serving as a bridge between analysis and topology. The classical concept, based on the limits of sequences, has been generalized in many directions, through filters, nets, statistical, and matrix methods to describe limiting behavior in broader contexts. There are different convegences such as A-continuity [1], the statistical convergence in topological spaces [2]. The concept of G-convergence has been studied by Connor and Grosse-Erdmann [3], they established the framework for G-methods and G-continuity, by studying how real functions can preserve G-limits under specific regularity conditions. Further many studies have been done by some authors enriched this theory by investigating G-continuity [4,5,6], G-connectedness [7,8,9], G-compactness [10,11] , and G-openness, revealing deep relationships between G-methods and topological structures. Motivated by this research, in our article [12] we study G-closed and G-open sets, as well as their sequential counterparts, and established conditions for their equivalence. In a recent study [13], separation axioms (T_i for $i = 0, 1, 2, 3, 4$) under G-convergence have been explored, describing them via G-open and G-closed sets, and supported the theory with illustrative counterexamples. Further development in this direction led to the introduction of G-sequential convergence, a refinement that distinguishes sequential behaviour from G-convergence. It was observed that while the collection of G-open subsets does not constitute a topology, the family of G-sequentially open subsets indeed forms one, called the *G-sequential topology* [14]. The latter paper has been recently taken into account in [15] to establish the relationships between G-convergence and G-sequentially convergence. Some of these concepts are extended to neutrosophic topological spaces in [16,17].

This paper aims to extend the concept of G-convergence in a set X to the submethod case. In mathematical settings, the study of a subset $Y \subseteq X$ requires analyzing sequences contained entirely in

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Y while preserving the limit defined on X. This leads to the concept of a *submethod* G_Y, obtained by restricting the original method G to the subset Y.

We define and study G_Y-convergence, G_Y-closed and G_Y-open subsets, and analyze how these notions interact with the corresponding G-method concepts.

The framework developed in this work is expected to serve as a foundation for future studies, particularly for extending the analysis to G-sequential submethods, which will be the subject of subsequent research.

We acknowledge that the results of the paper are some outcomes of the PhD thesis of the second author [18].

2. Preliminaries

In this section, we recall certain basic notions and establish the notations that will be used throughout the paper. Unless otherwise stated, X denotes a nonempty set and $s(X)$ is used for the set of the sequences $\mathbf{a} = (a_n)$ with the terms in X.

Definition 2.1 *A* method G *on a set* X *is defined to be a map*

$$G\colon c_G(X) \longrightarrow X,$$

where $c_G(X)$*, which is a nonempty collection of the sequences in* X*, is called the domain of* G*. For a sequence* $\mathbf{a} = (a_n) \in c_G(X)$*, the element* $G(\mathbf{a}) = x \in X$ *is called the* G-limit *of* $\mathbf{a}$ *or the sequence* $\mathbf{a}$ *is* G*-convergent to* x*. Such a* G*-method defined on* X *is sometimes denoted by* (X, G)*.*

Definition 2.2 *A method* G *defined on a set* X *is* regular *if any convergent sequence* $\mathbf{a} = (a_n)$ *with* $\lim a_n = \ell$ *belongs to the domain* $c_G(X)$ *of* G *and satisfies* $G(\mathbf{a}) = \ell$*.*

Definition 2.3 *Let* G *be a method on* X*. A sequence* $\mathbf{a} = (a_n)$ *in* X *is* G-convergent *to a point* $\ell \in X$*, if* $\mathbf{a} \in c_G(X)$ *and* $G(\mathbf{a}) = \ell$*.*

Definition 2.4 *For a subset* $B \subseteq X$*, the* G-hull *of a subset* B *is defined as*

$$[B]^G = \{\ell \in X : \textit{there exists a sequence } (a_n) \textit{ in } B \textit{ such that } G(a_n) = \ell\}.$$

A set B *is called* G-closed *if* $[B]^G \subseteq B$*. The complement of a* G*-closed set is called* G-open*.*

3. G-Convergences via Induced Submethods

In this section, we extend the notion of G-convergence by introducing the concept of G*-submethods*, which enables the study of G-convergence restricted to subsets of a set X together with a method G given on it. The induced submethod framework helps to study the different properties related to G-convergence within subsets such as G-hull and G-open structures.

Definition 3.1 *Let* Y *be a nonempty subset of a set* X*, and* G *be a method on* X*. The* submethod *of* G *on* Y*, denoted by* G_Y*, is defined by* $G_Y(\mathbf{a}) = G(\mathbf{a})$ *for* $\mathbf{a} \in c_{G_Y}(Y)$ *where the domain of* G_Y *is*

$$c_{G_Y}(Y) = \{\mathbf{a} = (a_n) \in c_G(X) \cap s(Y) \ : \ G(\mathbf{a}) \in Y\},$$

and $s(Y)$ *denotes the set of all sequences taking values in* Y*.*

Definition 3.2 *A sequence* $\mathbf{a} = (a_n)$ *in* Y *is* G_Y-convergent *to* $\ell \in Y$*, if* $\mathbf{a} \in c_{G_Y}(Y)$ *and* $G_Y(\mathbf{a}) = \ell$*.*

It is clear from the definition that every G_Y-convergent sequence is G-convergent, as $G_Y(\mathbf{a}) = G(\mathbf{a})$ for all $\mathbf{a} \in c_{G_Y}(Y)$. However, the converse is not necessarily true, a sequence may be G-convergent in X without being G_Y-convergent in Y, because its G-limit may not belong to the subset Y.

Example 3.1 *Let X be a set and $Y \subseteq X$ a nonempty subset. Consider a method G on X defined by $G(\mathbf{a}) = a_1$ for any sequence $\mathbf{a} = (a_n)$. Then the method G is defined for all sequences in $s(X)$, so* $\mathrm{dom}(G) = s(X)$ *and hence $c_G(X) = s(X)$.*

For the subset $Y \subseteq X$, we obtain the induced submethod G_Y on Y. For any sequence $\mathbf{y} = (y_n) \in s(Y)$, we have $y_1 \in Y$ and therefore $G_Y(\mathbf{y}) = G_Y(\mathbf{y}) = y_1$ is well defined for all sequences in $s(Y)$, where domain of G_Y is $c_{G_Y}(Y) = s(Y)$.

The following counterexample shows that even when a G-method is defined on X it does not guarantee the existence of its submethod on a subset $Y \subseteq X$.

Example 3.2 *Consider a set X and choose a fixed element $x_0 \in X$. Define the constant method G such that $G(\mathbf{x}) = x_0$ for any sequence $\mathbf{x} = (x_n) \in s(X)$. The method G assigns the value x_0 to each sequence and it is defined for all the sequences in X. Hence the domain of G is all sequences and therefore $c_G(X) = s(X)$.*

Now let $Y \subseteq X$ be any nonempty subset such that $x_0 \notin Y$. For any sequence $\mathbf{y} = (y_n) \in s(Y)$ we have $G(\mathbf{y}) = x_0 \notin Y$. Hence $\mathbf{y} \notin c_G(Y)$, because no sequence in $s(Y)$ satisfies the requirement $G(\mathbf{y}) \in Y$, and therefore the domain $c_{G_Y}(Y) = \varnothing$. Since a method must have a nonempty domain, the induced submethod G_Y is not defined *on Y.*

However if $y_0 \in Y$, then the submethod G_Y is defined for all the sequences in Y

We now establish several essential properties of the G-submethod.

Definition 3.3 *For a set X and a method G on it, consider the submethod G_Y on the subset $Y \subseteq X$. Then for the induced G-submethod G_Y, we can state the following:*

1. *A subset $F \subseteq Y$ is* G_Y-closed *in Y if whenever any sequence $\boldsymbol{x} = (x_n)$ in F is G_Y-convergent to $l \in Y$, then $l \in F$.*

2. *A subset $U \subseteq Y$ is* G_Y-open *in Y if its complement $Y \setminus U$ is* G_Y-closed.

4. Results

Theorem 4.1 *Consider a method G on a set X, a G-closed subset $Y \subseteq X$ and $B \subseteq Y$. Then B is G-closed in X if and only if B is G_Y-closed in Y.*

Proof:

$\Rightarrow$: Assume that B is a G-closed subset in (X, G). We need to prove that it is G_Y-closed in Y. If $u \in [B]^{G_Y}$ then there exists a sequence $\mathbf{a} \in c_G(X) \cap s(B)$ with $G_Y(\mathbf{a}) = G(\mathbf{a}) = u$. Since B is G-closed in X, one has $u \in B$ and therefore B is G_Y-closed in Y.

$\Leftarrow$: Conversely, let B be G_Y-closed in Y. We must prove that B is G-closed in X. If $u \in [B]^G$, then there exists a sequence $\mathbf{a}$ with the terms in B such that $\mathbf{a} \in c_G(X)$ and $G(\mathbf{a}) = u$, where $u \in Y$ as Y is G-closed. Hence $\mathbf{a} \in c_G(X) \cap B$ and $G_Y(\mathbf{a}) = u$. Since B is G_Y-closed we have $u \in B$, which implies that B is G-closed. □

Note 1 *Unlike G-closed subsets, Theorem 4.1 does not generally hold for G-open subsets. If Y is a G-open subset of X, then a G_Y-open subset B of Y is not necessarily G-open in X.*

Example 4.1 *[4] Consider a method G on the set $X = \mathbb{R}$ defined by $G(\boldsymbol{x}) = \lim_{n\to\infty} \frac{x_n + x_{n+1}}{2}$ for some sequences $\boldsymbol{x} = (x_n)$, which is regular. Consider the subsets $Y = (0, +\infty)$ and $A = (0, 1)$. Now for the subset A in Y, we have that $Y \setminus A = [1, \infty)$ is G-closed in X. Hence by Theorem 4.1, $[1, \infty) \cap Y = [1, \infty)$ is G_Y-closed in Y and therefore $[Y \setminus A]^{G_Y} = [1, \infty)$ which proves that A is G_Y-open in Y.*

We show that A is not G-open in X. The complement $X \setminus A = (-\infty, 0] \cup [1, \infty)$, is not G-closed, since for any sequence (a_n) in $X \setminus A$, G-limit $G(a_n) = u$ may not be in $(-\infty, 0] \cup [1, \infty)$ and therefore, A is not G-open in X. For example the sequence $(a_n) = (1, 0, 1, 0, \dots)$ has the terms in $X \setminus A$ but $G(a_n) = \frac{1}{2} \notin X \setminus A$

Hence, even though Y is G-open in X and A is G_Y-open in Y, the set A is not G-open in X.

Theorem 4.2 *Consider a G-method on X, a G-closed subset $Y \subseteq X$ and $B \subseteq Y$. Then B is G_Y-closed in Y if and only if there exists a G-closed subset F in X such that $B = F \cap Y$.*

Proof:
$\Rightarrow$: Suppose B is a G_Y-closed subset of Y. Since Y is G-closed, by Theorem 4.1 we have that the subset B is G-closed in X and therefore $B = F \cap Y$ for $F = B$.

$\Leftarrow$: Conversely, if $B = F \cap Y$ for a G-closed subset F in X, we show that B is G_Y-closed in Y. Here $x \in [B]^{G_Y}$ means that there exists a sequence $\mathbf{a} = (a_n)$ in $B = F \cap Y$ with $G_Y(\mathbf{a}) = x \in Y$. Then $\mathbf{a}$ is a sequence in F and $G(\mathbf{a}) = G_Y(\mathbf{a}) = x \in Y$ and therefore $x \in [F]^G$. Since F is G-closed in X we have $x \in F$. By $B = F \cap Y$, it implies that $x \in B$ and therefore $B = F \cap Y$ is G_Y-closed in Y. □

Remark 4.1 *In classical topology, for a topological space (X, τ); and the subsets $Y \subseteq X$ and $U \subseteq Y$ we have the results $\overline{U}^Y = \overline{U}^X \cap Y$, and $U^{oY} = U^{oX} \cap Y$. Analogously, in the context of G-hulls under submethods, we have the following result.*

Proposition 4.1 *Suppose X is endowed with a method G; $Y \subseteq X$ and $B \subseteq Y$ are subsets. Then $[B]^{G_Y} = [B]^G \cap Y$.*

Proof:
If $l \in [B]^{G_Y}$, then there exists a sequence $\mathbf{x} = (x_n)$ in B such that $\mathbf{x}$ is G_Y-convergent to $l \in Y$. By $G(\mathbf{x}) = G_Y(\mathbf{x}) = G(\mathbf{x}) = l \in Y$, it implies that $\mathbf{x}$ is also G-convergent to l and therefore $l \in [B]^G \cap Y$.

Conversely if $l \in [B]^G \cap Y$, then a sequence $\mathbf{x} = (x_n)$ in B exists such that $\mathbf{x}$ is G-convergent to $l \in Y$. Since $\mathbf{x} \in s(Y)$ and $G(\mathbf{x}) = l \in Y$, this implies $\mathbf{x} \in c_G(Y)$ and $G_Y(\mathbf{x}) = l$ which means $l \in [B]^{G_Y}$. □

Theorem 4.3 *Consider a method (X, G) and a G-closed subset $Y \subseteq X$. Then $B \subseteq Y$ is G_Y-open subset in Y if and only if there exists a G-open subset U in X such that $B = U \cap Y$.*

Proof: $\Rightarrow$: Suppose $B \subseteq Y$ is G_Y-open in Y. By definition, $Y \setminus B$ is G_Y-closed in Y. Since Y is G-closed, by Theorem 4.2 we have a G-closed subset $F \subseteq X$ such that $Y \setminus B = F \cap Y$. Then we have

$$\begin{aligned} B &= Y \setminus (Y \setminus B) \\ &= Y \setminus (F \cap Y) \\ &= (X \setminus (F \cap Y)) \cap Y \\ &= ((X \setminus F) \cup (X \setminus Y)) \cap Y \\ &= ((X \setminus F) \cap Y) \cup ((X \setminus Y) \cap Y) \\ &= (X \setminus F) \cap Y \cup \emptyset \\ &= (X \setminus F) \cap Y. \end{aligned}$$

Since F is G-closed, it follows that $X \setminus F$ is G-open in X. Thus by assuming $U = X \setminus F$, we have $B = U \cap Y$, where U is a G-open subset of X.

$\Leftarrow$: Conversely, assume $B = U \cap Y$ for a G-open subset $U \subseteq X$. We must show that B is G_Y-open in Y, equivalently that $Y \setminus B$ is G_Y-closed in Y. Consider

$$\begin{aligned} Y \setminus B &= Y \setminus (U \cap Y) \\ &= (X \setminus (U \cap Y)) \cap Y \\ &= ((X \setminus U) \cup (X \setminus Y)) \cap Y \\ &= ((X \setminus U) \cap Y) \cup ((X \setminus Y) \cap Y) \\ &= (X \setminus U) \cap Y \cup \emptyset \\ &= (X \setminus U) \cap Y. \end{aligned}$$

Here $F = X \setminus U$ is G-closed and by Theorem 4.2, the subset $F \cap Y$ is G_Y-closed in Y, and therefore B is G_Y-open in Y. □

Proposition 4.2 *Consider a method (X, G), $Y \subseteq X$ a G-closed subset, and $B \subseteq Y$. For $B^{\circ G}$ and $B^{\circ G_Y}$ defined*

$$B^{\circ G} = \bigcup\{U \subseteq X : U \subseteq B \text{ and } U \text{ is } G\text{-open}\}$$

and similarly

$$B^{\circ G_Y} = \bigcup\{V \subseteq Y : V \subseteq B \text{ and } V \text{ is } G_Y\text{-open in } Y\}.$$

we have $B^{\circ G} \cap Y \subseteq B^{\circ G_Y}$.

Proof: Let $b \in B^{\circ G} \cap Y$. It implies that there exists a G-open subset $U \subseteq X$ such that $b \in U \subseteq B$. By Theorem 4.3, the subset $U \cap Y$ is G_Y-open and $b \in U \cap Y \subseteq B$. Hence, $b \in B^{\circ G_Y}$, and therefore

$$B^{\circ G} \cap Y \subseteq B^{\circ G_Y}.$$

□

Definition 4.1 *Let (X, G) be a method and $A \subseteq X$ a subset. We say that $x \in X$ is a G-accumulation point of A if there exists a sequence $\mathbf{a} = (a_n)$ with the terms in $A \setminus \{x\}$ such that $G(\mathbf{a}) = x$. We write A'^G for the set of all G-accumulation points of A.*

Proposition 4.3 Consider a method (X, G) with the subsets $Y \subseteq X$ and $B \subseteq Y$. Then $B'^{G_Y} = B'^G \cap Y$.

Proof: $\Rightarrow$: For $x \in B'^{G_Y}$, there exists a sequence $\mathbf{b} = (b_n)$ with the terms in $B \setminus \{x\}$ such that $G_Y(\mathbf{b}) = x$. By $B \subseteq Y$, the terms of the sequence $\mathbf{b} = (b_n)$ are in Y, i.e., $\mathbf{b} \in s(Y)$. By the definition of the submethod G_Y we have $G_Y(\mathbf{b}) = G(\mathbf{b}) = x$. Hence $\mathbf{b}$ is a sequence in $B \setminus \{x\}$ such that $G(\mathbf{b}) = x$, that means x is a G-accumulation point of B in X, i.e., $x \in B'^G$. Since $x \in Y$, it implies that $x \in B'^G \cap Y$.

$\Leftarrow$: Conversely, assume that $x \in B'^G \cap Y$. Then $x \in Y$ and by $x \in B'^G$, there exists a sequence $\mathbf{b} = (b_n)$ with the terms in $B \setminus \{x\}$ such that $G(\mathbf{b}) = x$. By $B \subseteq Y$ and $x \in Y$, it implies that $\mathbf{b} \in s(Y)$ and $G(\mathbf{b}) = x \in Y$. Hence $\mathbf{b} \in c_G(Y)$, and by using the definition of G_Y we have $G_Y(\mathbf{b}) = G(\mathbf{b}) = x$. It follows that, the sequence $\mathbf{b} = (b_n)$, which is in $B \setminus \{x\}$, is G_Y-convergent to x. Hence x is a G_Y-accumulation point of B in Y and therefore $x \in B'^{G_Y}$.

Combining both inclusions, we have

$$B'^{G_Y} = B'^G \cap Y.$$

□

Definition 4.2 *Assume that (X, G) is a method and $Y \subseteq X$. The method G is called* point-wise *if $G(\boldsymbol{x}) = x$ for any constant sequence $\boldsymbol{x} = (x, x, x, \dots)$. The submethod G_Y induced on Y is called a* point-wise submethod *if for each point $y \in Y$, the constant sequence $\boldsymbol{y} = (y, y, y, \dots)$ is* G_Y-convergent *to y.*

Proposition 4.4 *In a point-wise method (X, G) any singleton subset of X is G-closed.*

Proof: Assume that (X, G) is a point-wise method and prove that for any $a \in X$, the singleton set $A = \{a\}$ is G-closed. If $x \in [\{a\}]^G$, then we have a sequence $\mathbf{a} = (a_n)$ with the terms in the singleton set $\{a\}$, which is necessarily the constant sequence $\mathbf{a} = (a, a, \dots)$, such that $G(\mathbf{a}) = x$. As the method G is pointwise, it implies that $G(\mathbf{a}) = a$ and therefore $x = a \in A$ which means $[\{a\}]^G \subseteq \{a\}$. Hence the singleton $\{a\}$ is G-closed. Further since the method is piece-wise we have $\{a\} \subseteq [\{a\}]^G$ and therefore $[\{a\}]^G = \{a\}$.

□

Proposition 4.5 *A submethod of a point-wise method is also pointwise.*

Proof: Let (X, G) be a piecewise method and $Y \subseteq X$ a subset. Prove that the submethod G_Y induced on Y is piecewise. Let $y \in Y$ and $\mathbf{y}$ be the constant sequence $\mathbf{y} = (y, y, y, \dots)$. As the method G is piecewise one has $\mathbf{y} \in c_G(X)$ and $G(\mathbf{y}) = y$. Hence $\mathbf{y} \in c_G(X) \cap s(Y)$ and $G(\mathbf{y}) = y \in Y$ and therefore $\mathbf{y} \in c_{G_Y}(Y)$ and $G_Y(\mathbf{y}) = y$ which proves that the submethod G_Y is piecewise. □

Definition 4.3 *A method (X, G) is called G-T_0 if for any pair of distinct points $x, y \in X$, there exists a G-open neighbourhood U with $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.*

Proposition 4.6 *If (X, G) is a G-T_0 method and $Y \subseteq X$ a G-closed subset, then (Y, G_Y) is also G-T_0.*

Proof: Suppose (X, G) is G-T_0. We need to prove that (Y, G_Y) is also G_Y-T_0. Let $x, y \in Y$ with $x \neq y$. As (X, G) is G-T_0, then there exists a G-open subset U of X that distinguishes these points, that is, either $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.

Since Y is G-closed by Theorem 4.3 we have that $V = U \cap Y$ is G_Y-open in Y. If $x \in U$ and $y \notin U$, then $x \in U \cap Y$ and $y \notin U \cap Y$. Similarly, If $y \in U$ and $x \notin U$, then $y \in U \cap Y$ and $x \notin U \cap Y$.

Thus, in each case, $U \cap Y$ separates the points x, y. Hence, (Y, G_Y) is G-T_0. □

Definition 4.4 *A method (X, G) is called G-T_1 if for each pair of distinct points $x, y \in X$, there exist G-open subsets U, V with $x \in U, y \notin U$ and $y \in V, x \notin V$.*

Proposition 4.7 *A method (X, G) is a G-T_1 method if and only if every singleton subset $\{x\}$ is G-closed.*

Proof:

$\Rightarrow$: Suppose that the method (X, G) is a G-T_1, and $a \in X$. To prove that $\{a\}$ is G-closed, it is sufficient to show that its complement $\{a\}^c$ is G-open.

If $x \in \{a\}^c$, then x and a are distinct points and therefore there exist G-open subsets U_x and U_a such that $x \in U_x$, $a \notin U_x$ and $a \in U_a$, $x \notin U_a$. Therefore, $U_x \cap \{a\} = \emptyset$, so $U_x \subseteq \{a\}^c$. Hence $\{a\}^c$ is G-open, and therefore $\{a\}$ is G-closed.

$\Leftarrow$: Conversely, assume every singleton $\{a\}$ is G-closed. Let $x, y \in X$ with $x \neq y$. Since $\{x\}$ is G-closed, its complement $X \setminus \{x\}$ is explicitly G-open. Similarly, $\{y\}$ is also G-closed, hence $X \setminus \{y\}$ is G-open as well. Clearly

$$x \in X \setminus \{y\}, \quad y \notin X \setminus \{y\}, \quad \text{and} \quad y \in X \setminus \{x\}, \quad x \notin X \setminus \{x\}.$$

Therefore, we have G-open subsets $U = X \setminus \{y\}$ and $V = X \setminus \{x\}$ that separate x and y, proving (X, G) is G-T_1. □

Corollary 4.1 *As a result of Propositions 4.4 and 4.7 a piecewise method (X, G) is $G - T_1$.*

Proposition 4.8 *Assume that the method (X, G) is G-T_1, and $Y \subseteq X$ is a G-closed subset. Then the submethod (Y, G_Y) is also G_Y-T_1 method.*

Proof: By Proposition 4.7, to prove that the submethod (Y, G_Y) is G_Y-T_1, it is enough to show that every singleton subset $\{y\} \subseteq Y$ is G_Y-closed. Since (X, G) is a G-T_1 method, the singleton $\{y\}$ is G-closed in X. Now, by Theorem 4.2, it follows that $\{y\} \cap Y = \{y\}$ is G_Y-closed in Y. Since this holds for every $y \in Y$, we conclude that all singleton subsets of Y are G_Y-closed and therefore by Proposition 4.7 the submethod (Y, G_Y) is G_Y-T_1. □

Definition 4.5 *A method (X, G) is said to be a G-Hausdorff (or G-T_2) if for any two distinct points $x, y \in X$, there exist disjoint G-open subsets $U, V \subseteq X$ such that $x \in U, \quad y \in V, \quad$ and $\quad U \cap V = \emptyset$.*

Proposition 4.9 *Assume that (X, G) is a G-Hausdorff method. Then, for any G-closed subset $Y \subseteq X$, the induced submethod (Y, G_Y) is also G-Hausdorff.*

Proof: Suppose (X, G) is G-Hausdorff, and $Y \subseteq X$ a G-closed subset. Let $x, y \in Y$ be the points such that $x \neq y$. Since (X, G) is G-Hausdorff, there exist disjoint G-open neighbourhoods $U, V \subseteq X$ such that $x \in U, \quad y \in V, \quad$ and $\quad U \cap V = \emptyset$. Then by Theorem 4.3, $U' = U \cap Y$ and $V' = V \cap Y$ are G_Y-open subsets in G_Y and $x \in U'$, $y \in V'$ such that

$$U' \cap V' = (U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset \cap Y = \emptyset.$$

Hence, the induced submethod (Y, G_Y) is also G-Hausdorff. □

Proposition 4.10 *If the method (X, G) is G-Hausdorff, then it is G-T_1.*

Proof: Assume that the method (X, G) is G-Hausdorff. To show it is G-T_1, let $x, y \in X$, $x \neq y$. By Definition 4.5 of G-Hausdorff method, there exist disjoint G-open subsets $U, V \subseteq X$ such that $x \in U, \quad y \in V, \quad U \cap V = \emptyset$. Then by Definition 4.4 of $G - T_1$ method, the other details of the proof follow. □

Definition 4.6 *A method (X, G) is G-*compact *if any sequence $\boldsymbol{x} = (x_n)$ in X has a subsequence $\boldsymbol{x}_k = (x_{n_k})$ which is G-convergent.*

Theorem 4.4 Suppose that the method (X, G) is G-compact and $Y \subseteq X$ a G-closed subset, then (Y, G_Y) is G_Y-compact.

Proof: Let $\mathbf{y} = (y_n)$ be a sequence in the submethod (Y, G_Y). Since the method (X, G) is G-compact, the sequence $\mathbf{y} = (y_n)$ has a subsequent $\mathbf{y}_k = (y_{n_k})$ which is G-convergent, and therefore $\mathbf{y}_k \in c_G(X) \cap s(Y)$. Since Y is G-closed, we have $G(\mathbf{y}_k) = u \in Y$. Thus, $\mathbf{y}_k \in c_{G_Y}(Y)$, that means $\mathbf{y}_k$ is a subsequence of $\mathbf{y}$ and G_Y-convergent to $u \in Y$. Hence (Y, G_Y) is G_Y-compact. □

Definition 4.7 *A method (X, G) is said to be G-*countably compact *if every infinite subset $A \subseteq X$ has at least one G-*accumulation point.

Theorem 4.5 *If the method (X, G) is G-countably compact and $Y \subseteq X$ a G-closed subset, then the submethod (Y, G_Y) is G_Y-countably compact.*

Proof: Let $B \subseteq Y$ be an infinite subset. Since the method (X, G) is G-countably compact, at least a point $x \in B'^G$ exists. By Proposition 4.3 we have

$$B'^{G_Y} = B'^G \cap Y.$$

By $x \in B'^G$, there exists a sequence $\mathbf{b} = (b_n)$ with the term in $B \setminus \{x\}$ such that $G(\mathbf{b}) = x$. Here since $B \subseteq Y$, the terms of the sequence $\mathbf{b}$ are in Y. Since Y is G-closed it implies that $x \in Y$ and therefore $x \in B'^G \cap Y = B'^{G_Y}$. Thus, the submethod (Y, G_Y) is G_Y-countably compact. □

5. Conclusion

In this paper, we have systematically extended the notion of G-convergence methods to subsets via G-submethods. We studied G_Y-hulls, G_Y-closed, and G_Y-open subsets, and established their relationships with the corresponding G-method. We characterized conditions under which submethods inherit properties such as G-T_0, G-T_1, and G-Hausdorff from the parent method (X, G).

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