Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 10** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.80349

Integer-Order Disturbance Observer Based Hybrid Synchronization of SEIR Tuberculosis System Using Active Control

Ayub Khan, Roopesh Tehri, Neeti Goel *, Diya Baidya

ABSTRACT: This paper presents an investigation of chaos control and synchronization in an integer-order non-linear SEIR tuberculosis (TB) system using the active control method. The model describes the transmission dynamics of tuberculosis through four interacting compartments, that is, susceptible, exposed, infectious and recovered — whose non-linear coupling leads to chaotic oscillations under certain parameter conditions. Such chaotic behaviour reflects the inherent complexity and unpredictability of disease spread within a population. To suppress this chaotic behaviour and achieve synchronization, an active control approach is developed using master–slave framework. In this paper, the slave system is influenced by constant but bounded external disturbances, while suitable control inputs are applied to stabilize the dynamics. A Lyapunov-based stability analysis is performed to establish sufficient conditions for the asymptotic convergence of synchronization errors. The analysis confirms that the controlled system remains stable and robust even in the presence of constant disturbances. Numerical simulations further demonstrate that the designed controller effectively eliminates chaotic oscillations, ensures complete synchronization of the system states and maintains smooth and stable trajectories. Overall, the results validate that the active control method provides an efficient and reliable means of controlling chaos and achieving synchronization in non-linear integer-order tuberculosis system. This approach can be adapted to a broad class of biological and engineering systems exhibiting similar non-linear and chaotic characteristics.

Keywords: Tuberculosis, disturbance observer, chaos synchronization, active control, Lyapunov stability.

Contents

1. Introduction

Non-linear dynamical systems are fundamental to understanding a wide range of real-world phenomena such as weather patterns [12], electrical circuits [5], population dynamics [10] and biological interactions [6] which are governed by laws that often lead to complex and unpredictable behaviour. A notable characteristic of non-linear systems is their potential to exhibit chaos which is an unstable, aperiodic behaviour in deterministic systems that is highly sensitive to initial conditions, termed as "Butterfly Effect" by Lorenz in 1963 [11]. Even a minute change in the starting state of a system which is chaotic can result in vastly different trajectories over time. This extreme sensitivity, though initially seen as a source of unpredictability, has developed into one of non-linear science's most fascinating phenomena, as it shows the delicate balance between order and disorder inherent in many natural processes.

The study of chaos is not limited to theoretical interest but has found broad practical applications in fields such as secure communication [23], chemical reaction control [14], mechanical systems [20], neural

* Corresponding author.
2020 *Mathematics Subject Classification:* 34H10.
Submitted November 28, 2025. Published March 15, 2026

networks [3] and epidemiology [13]. In biological systems, chaotic behaviour frequently emerges due to the non-linear interactions between populations or physiological processes. Epidemiological systems, in particular, can exhibit chaotic oscillations as a result of feedback mechanisms, infection transmission, latency, immunity and environmental variability. These oscillations represent irregular fluctuations in disease incidence that often resemble the unpredictable nature of real epidemic patterns. Thus, understanding and controlling chaos in such systems hold significant importance for both theory and public health management.

In the context of infectious diseases, mathematical modelling serves as a crucial tool for describing the spread, persistence and control of infections within a population. Among the various compartmental models developed in epidemiology, the SEIR (Susceptible–Exposed–Infectious–Recovered) framework stands as one of the most widely used formulations. It divides a population into four interacting compartments and captures the transitions between them through non-linear differential equations [16]. When non-linearities in transmission, infection and recovery processes interact under specific parameter values, the system can display periodic, quasi-periodic or chaotic dynamics. The tuberculosis (TB) model, a variant of the SEIR structure, exemplifies this complexity. Tuberculosis, caused by *Mycobacterium tuberculosis*, is still a health concern globally since its progression is slow, has long latency period and possibility for recurrence [17]. Modelling the transmission dynamics of TB using non-linear systems allows researchers to understand its long-term behaviour but it also reveals the possibility of chaotic fluctuations that make long-term prediction difficult.

The emergence of chaos in epidemiological models poses both challenges and opportunities. From a control perspective, chaotic dynamics can amplify uncertainties, making prediction, prevention and containment strategies harder to design. However, chaos also offers an opportunity to apply sophisticated control techniques to stabilize the system and guide it toward predictable behaviour. As a result, various control methods have been developed in order to suppress or regulate chaos. Over the past few decades, numerous approaches have been proposed, including feedback control method [1], sliding-mode control [21], active control [18], adaptive control [7], backstepping [19] and much more.

Among these, the active control method has gained recognition for its simplicity, analytical clarity and effectiveness in stabilizing non-linear chaotic systems. The method of active control is based on the idea of constructing control inputs that directly compensate for the non-linear terms responsible for chaotic motion while adding suitable linear feedback terms that drive the system toward stability. By using the difference between the master system as well as the slave system states, active control will cancel the non-linear terms which will help in the convergence of the error dynamics as it will tend to zero.

Synchronization of chaotic systems is another fascinating phenomenon related to chaos control. When two or more chaotic systems interact or are properly coupled, they can exhibit synchronized behaviour despite their sensitive dependence on initial conditions. In synchronization, the state trajectories of one system, known as the slave system, follow the original system called the master system. It has wide range of applications including secure communications [22], image encryption [2], biological modelling [15] and many more. In the context of epidemiological models, synchronization can represent the alignment of epidemic patterns across different regions or populations or the convergence of a predictive model with real-world data. Achieving synchronization between two chaotic systems, particularly when there are external disturbances, is an important objective in non-linear control theory.

In many practical scenarios, biological systems are subject to external disturbances and uncertainties that can significantly affect their behaviour. For instance, changes in environmental conditions, migration or public health interventions can be viewed as external inputs that disturb the system. When designing control strategies for such systems, it is therefore essential to consider the effect of disturbances and ensure that the control law remains robust. In this paper, the disturbances are modelled as constant bounded parameters, representing steady external influences or uncertainties in the system. The primary aim of this paper is to design and use the active control law that can effectively suppress chaos and synchronize the master and slave tuberculosis systems. Lyapunov stability analysis validates the control design.

Therefore, the paper is framed as follows: Section 2 provides the mathematical preliminaries that are essential and used in this paper. Section 3 provides the model description of the integer-order SEIR Tuberculosis system. Section 4 shows the chaos synchronization of The Tuberculosis system using disturbance observer based active control method. Section 5 presents the numerical simulations of the proposed method. Section 6 finally concludes the paper.

2. Mathematical Preliminaries

In this section, the basic mathematical background necessary for developing and analyzing the active control approach used in this paper is presented. All systems considered here are assumed to be deterministic, continuous and differentiable, ensuring the existence and uniqueness of their solutions.

In a dynamical system, normally chaos arises when the non-linear system shows sensitivity to initial conditions or due to topological mixing or periodic orbits which are dense within a bounded phase space. Now, let us consider a general n-dimensional continuous-time non-linear dynamical system expressed as

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a non-linear differentiable function, and the overdot denotes the derivative with respect to time. Depending on the form of $f(x)$, the system may display stable, periodic or chaotic behaviour. When the trajectories of $x(t)$ remain bounded but never settle to equilibrium or periodic motion, the system is said to exhibit chaos.

In the context of control and synchronization, it is common to consider two similar systems — a master (drive) system and a slave (response) system — coupled in such a way that the slave system attempts to follow the trajectory of the master system. The master system is given by [9]:

$$\dot{M}_{m:i} = F(M(t)), \tag{2.2}$$

and the slave system is chosen to be as [9]:

$$\dot{S}_{s:i} = F(S(t)) + U_i(t) + D_i(t), \tag{2.3}$$

where $U_i(t) \in \mathbb{R}^n$ denotes the control inputs that will be designed and $D_i(t) \in \mathbb{R}^n$ are the constant external disturbances. The main objective of synchronization is to determine a proper control law $U_i(t)$ so that the slave system's states asymptotically track those of the master system, despite possible disturbances or uncertainties.

Error synchronization is defined in this paper as:

$$\mathscr{E}_i = S_{s:i} - M_{m:i} \tag{2.4}$$

If a control input $u(t)$ can be found so that [8]

$$\lim_{t\to\infty} \|\mathscr{E}(t)\| = 0, \tag{2.5}$$

then complete synchronization is achieved. In many practical cases, the exact non-linear function $f(\cdot)$ is known, and therefore a control law can be designed to cancel the non-linear mismatch $f(y) - f(x)$, leaving a linear error system whose stability can be analyzed using classical Lyapunov methods.

We use the Lyapunov direct method to find the error system's stability since it provides a systematic way to determine whether a given equilibrium point is stable without solving the system explicitly. For this, we consider a scalar function which is continuous as well as differentiable $\mathscr{V}_l : \mathbb{R}^n \to \mathbb{R}$, called the Lyapunov function, that satisfies the properties that $\mathscr{V}_l$ should be positive definite and $\dot{\mathscr{V}}_l$ should be negative semi-definite [8].

If these conditions hold, then the equilibrium point will become stable.

In control design, the Lyapunov function is typically chosen as a quadratic form, such as

$$V_l(t) = \frac{1}{2}\mathscr{E}^T\mathscr{E}, \tag{2.6}$$

Its time derivative is given by:

$$\dot{V}_l(t) = \mathscr{E}^T\dot{\mathscr{E}}. \tag{2.7}$$

These theoretical results form the foundation for the subsequent sections, where the integer-order SEIR-type tuberculosis model is described and controlled using the active control approach. The Lyapunov-based design framework ensures that the proposed controller can stabilize chaotic dynamics and achieve robust synchronization efficiently.

3. Non-linear Tuberculosis Model Description

Although recent studies in biological systems often use fractional-order derivatives to include memory and hereditary effects, this paper use an integer-order model for the SEIR-TB system because it gives simple framework for designing and verifying control laws and it also helps to make mathematical interpretation more clearly because even though fractional-order systems gives us more realistic description but it often requires complex numerical methods and additional parameters that may not clear physical interpretations. Hence, this paper will only focus on integer-order Tuberculosis model which can later be extended to fractional-order synchronization in future research.
In this paper, we will consider the SEIR Tuberculosis system from [4] and re-parametrize it which will act as the master system and is as follows:

$$\begin{aligned}
\dot{M}_{m:1} &= A - aM_1 - BM_1M_3 \\
\dot{M}_{m:2} &= BM_1M_3 - AM_2 - CM_2 \\
\dot{M}_{m:3} &= CM_2 - AM_3 - DM_3 \\
\dot{M}_{m:4} &= DM_3 - AM_4
\end{aligned} \tag{3.1}$$

Where, M_1, M_2, M_3 and M_4 are susceptible, exposed, infected and recovered classes respectively.
A, B, C and D are birth-death rate, contact rate, disease rate and recovery rate respectively and are all non-negative.
Here, the sum of all these classes make up the total population.
Some of the assumptions for the model (3.1) taken in [4] are as follows:

1. Simply by touching individuals who are infected, a person can become infected.
2. Age, sex, societal position, race, climate, etc. doesn't influence the likelihood of an individual being infected.
3. The interactions between individuals are same with the same degree.
4. The death rate A is assumed to be constant across all population compartments and is balanced by an equal rate of new individuals joining the susceptible population maintaining a stable total population size.
5. In the population, no immigration, emigration, birth or death is considered. The disease transmits only in a close area which in turn keeps total population constant.
6. There is equal natural death rate across all compartments. [4]

The system (3.1) exhibits chaos which can be seen from phase portraits given in Figure 1. A phase portrait plots the trajectories of the system's state variables over time, revealing its stability, periodicity, and potential for chaotic behaviour. For the figure, the parameter values are taken to be as: $A = 1, B = 12, C = 15, D = 10$ and the initial conditions are taken to be as: $M(0) = (1.2, 0.1, 1.3, -0.2)$
Figure 1(a) shows a collection of trajectories that appear to be converging on a complex, three-dimensional structure. The trajectories are tightly interwoven, suggesting that despite starting from different initial conditions, the system's state is being drawn towards a specific, bounded region of the phase space. This indicates the presence of an attractor. Figure 1(b) shows that the trajectories form a double-winged or butterfly-like shape, which is a classic signature of a chaotic strange attractor which reveals that the system's state is confined to a finite region of the phase space, yet the trajectories never repeat. This non-repeating behaviour, combined with the sensitive dependence on initial conditions that is inherent to chaos, means that even a tiny change in the starting point will lead to a dramatically different path over time, though all paths remain within the bounds of the attractor.

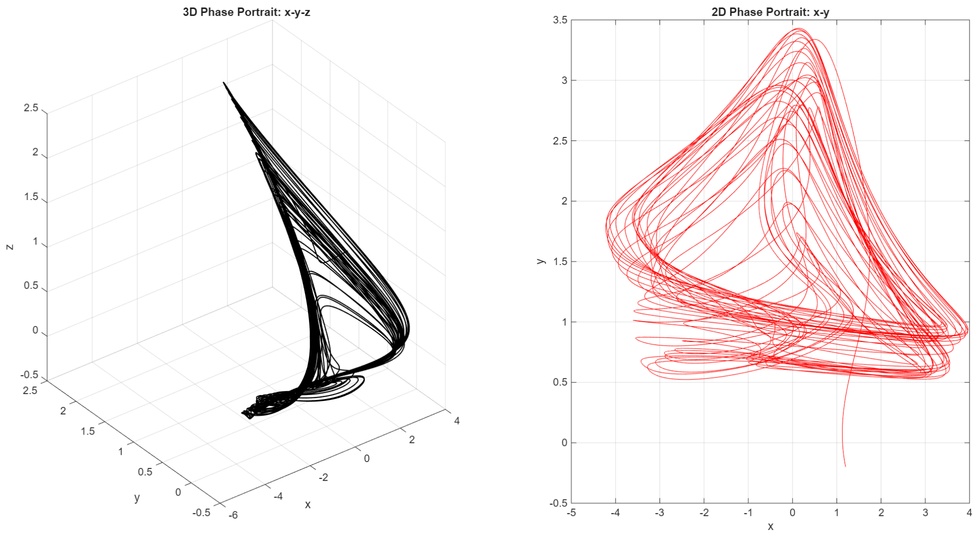


Figure 1: Phase portraits of Tuberculosis system (3.1) in (a) x-y-z plane (b) x-y plane

Also, figure 2, we see that the values of the computed lyapunov exponents are:
$\lambda_1 = 6.0035, \lambda_2 = -1.7049, \lambda_3 = -6.5862, \lambda_4 = -7.3619$
For this simulation, the parameter values are taken as : $A = 0.0012, B = 0.2568, C = 0.148, D = 0.104$, constant disturbances are taken as: $D_i = 10 \; \forall i = 1, 2, 3, 4$ and initial condition are chosen as: $M(0) = (-1500, 200, 50, -200)$
The most significant finding from all the values of the lyapunov exponents is that there exists a positive lyapunov exponent, that is, $\lambda_1 = 6.0035$ This is an indicator of a chaotic system. The positive lyapunov exponent also validates the visual evidence of the strange attractor and the system's complex dynamics observed in the phase portraits.

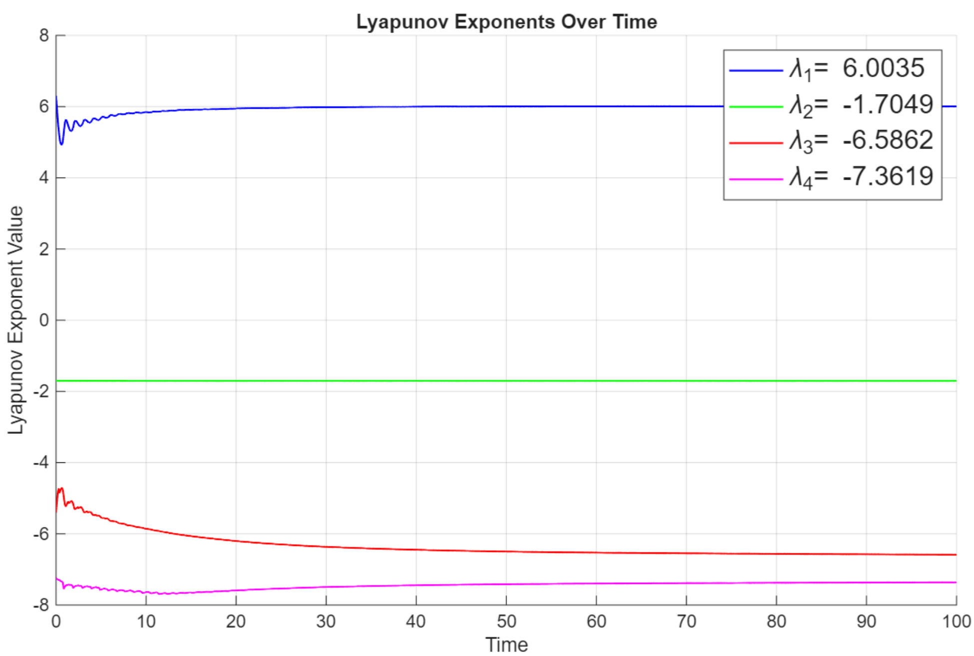
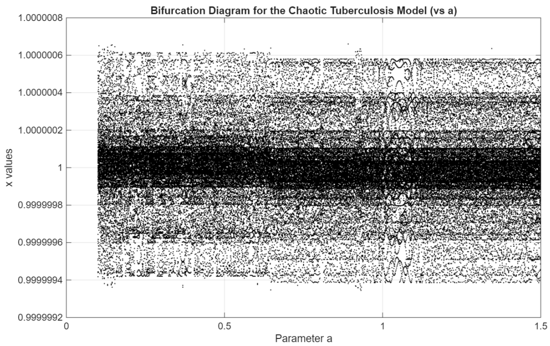


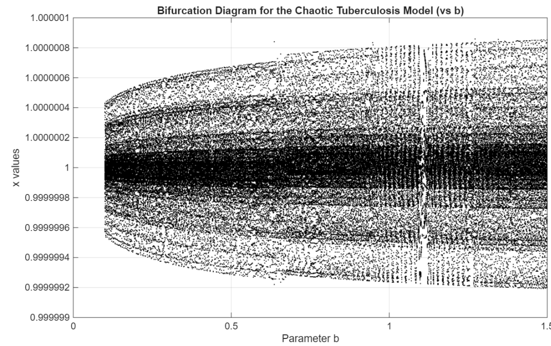
Figure 2: Lyapunov exponents VS time graph of Tuberculosis system

Now, bifurcation diagrams for parameters, that is, A, B, C, D are presented in Figure 3 to illustrate how variations in system parameters influence the dynamic behaviour of the tuberculosis model. These diagrams provide visual evidence of the transition between periodic and chaotic regimes as each parameter is varied within its respective range, while all other parameters remain fixed. As shown in Figure 3(a), when the parameter A varies from 0 to approximately 1.5, the system initially exhibits regular periodic oscillations characterized by distinct, well-separated trajectories in the bifurcation diagram. However, as A increases beyond approximately 0.4, the trajectories become denser and the system transitions into a chaotic regime where the x-values are distributed across a wide band. This dense, irregular distribution indicates the presence of sustained chaotic oscillations. The coexistence of multiple dense clusters also suggests intermittent chaos, where short windows of periodicity are interrupted by bursts of chaotic fluctuations. For the variation of parameter B, as illustrated in Figure 3(b), the system displays persistent chaotic behaviour over almost the entire range of (b) from 0 to 1.5. The dense, overlapping bands of x-values throughout the bifurcation plot indicate that the non-linear coupling between the susceptible and infectious populations, governed by parameter B, is the dominant source of irregular oscillations. The narrow oscillatory zones near the middle range of B suggest transient stabilization but the system quickly re-enters a chaotic state, confirming that parameter B plays an important role in maintaining the complex, unpredictable evolution of the model. In Figure 3(c), we can see that for small values of the parameter C (approximately 0 to 0.15), the system is stable, as indicated by a single, thin line of x values, suggesting it converges to a fixed point. As the parameter C increases, the diagram shows a series of bifurcations which is the classic signature of a period-doubling cascade, which leads the system from a stable state to a periodic state, and then eventually into a chaotic regime. After a certain value of C in the Figure, the lines start merging into a wide and dense band of x-values which shows that for a single value of C, the system no longer settles to a periodic cycle but instead explores a continuous range of values, which is the hallmark of chaos. Now, in Figure 3(d), we can see the effect of parameter D on the system dynamics. When D increases from 0 to about 1.5, the bifurcation diagram shows a gradual contraction of the chaotic bands. At lower values of D, we can see wide chaotic oscillations which signifies intense non-linear feedback from the recovery process to the other compartments. As the value of D increases, we can see that the chaotic lines narrows and the oscillations begin to gather together which implies that with higher values of D, the system stabilizes. Around the middle range, faint periodic windows emerge, representing temporary regularity in the system's evolution. However, as D continues to increase beyond approximately 1.2, these windows close and the trajectories merge again into a single thick chaotic band, indicating a reappearance of irregular motion. This alternating pattern of stabilization and chaos with increasing D reflects a classic route to chaos through period-doubling and subsequent band merging, characteristic of non-linear epidemiological models.

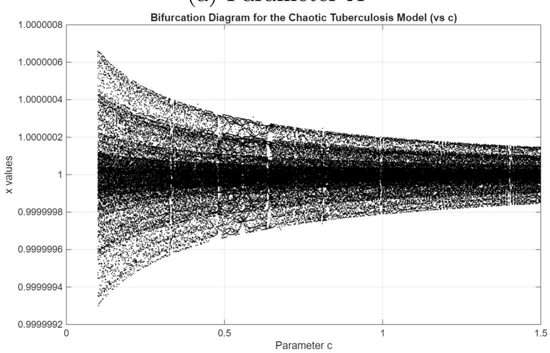
Overall, the bifurcation analyses demonstrate that the tuberculosis system (3.1) exhibits strong dependence on its parameters and can easily transition between periodic and chaotic states under slight parameter variations.



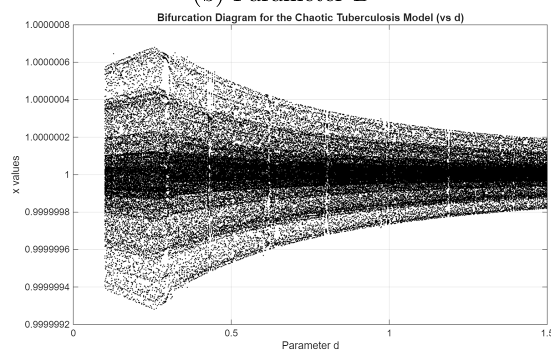
(a) Parameter A



(b) Parameter B



(c) Parameter C



(d) Parameter D

Figure 3: Bifurcation Diagrams of the parameters of the Tuberculosis system (3.1) (a)Parameter A (b)Parameter B (c)Parameter C (d)Parameter D

Now, our goal is to design control inputs such that the master (3.1) and slave (4.1) systems gets synchronized even in the presence of external disturbances via active control method such that $lim_{t\to\infty}\mathscr{E}_i(t) = 0,\ i = 1, 2, 3, 4$
which will be done in the subsequent section.

4. Integer-Order Disturbance Observer Based Hybrid Synchronization of the Tuberculosis Model via Active Control Method

In this section, an active control strategy is developed to achieve complete synchronization between the master and slave tuberculosis systems in the presence of constant bounded disturbances. The design objective is to construct appropriate control functions that will stabilize the resulting error dynamics and ensure asymptotic convergence of synchronization errors to zero, that is,

$$lim_{t\to\infty}||\mathscr{E}_i(t)|| = 0 \; for \; i = 1, 2, 3, 4$$

For this purpose, the corresponding slave system for (3.1) is chosen as follows:

$$\begin{aligned}
\dot{S}_{s:1} &= A - AS_1 - BS_1S_3 + D_1 + U_1(t)\\
\dot{S}_{s:2} &= BS_1S_3 - AS_2 - CS_2 + D_2 + U_2(t)\\
\dot{S}_{s:3} &= CS_2 - AS_3 - DS_3 + D_3 + U_3(t)\\
\dot{S}_{s:4} &= DS_3 - AS_4 + D_4 + U_4(t)
\end{aligned} \tag{4.1}$$

Where, S_1, S_2, S_3 and S_4 are susceptible, exposed, infected and recovered classes respectively.
A, B, C and D are birth-death rate, contact rate, disease rate and recovery rate respectively and are all non-negative.

$D_i^{\prime}s,\ i=1,2,3,4$ are constant bounded external disturbances satisfying $|D_i| \leq D_{i,max},\ i=1,2,3,4$ and $U_i(t)^{\prime}s,\ i=1,2,3,4$ are control inputs that need to be designed later.
We define the synchronization error functions as:

$$\begin{aligned} \mathscr{E}_1 &= S_{s:1} - M_{m:1} \\ \mathscr{E}_2 &= S_{s:2} - M_{m:2} \\ \mathscr{E}_3 &= S_{s:3} - M_{m:3} \\ \mathscr{E}_4 &= S_{s:4} - M_{m:4} \end{aligned} \tag{4.2}$$

So, the derivative of state errors (4.2) are:

$$\begin{aligned} \dot{\mathscr{E}_1} &= \dot{S_{s:1}} - \dot{M_{m:1}} \\ \dot{\mathscr{E}_2} &= \dot{S_{s:2}} - \dot{M_{m:2}} \\ \dot{\mathscr{E}_3} &= \dot{S_{s:3}} - \dot{M_{m:3}} \\ \dot{\mathscr{E}_4} &= \dot{S_{s:4}} - \dot{M_{m:4}} \end{aligned} \tag{4.3}$$

Now, substituting (3.1) and (4.1) in (4.3),the resulting dynamics for error is:

$$\begin{aligned} \dot{\mathscr{E}_1} &= A - AS_1 - BS_1S_3 + D_1 + U_1(t) - A - AM_1 - bM_1M_3 \\ &= -A\mathscr{E}_1 - BS_1S_3 + BM_1M_3 + D_1 + U_1(t) \\ \dot{\mathscr{E}_2} &= BS_1S_3 - AS_2 - CS_2 + D_2 + U_2(t) - BM_1M_3 + AM_2 + CM_2 \\ &= -A\mathscr{E}_2 - C\mathscr{E}_2 + BS_1S_3 - BM_1M_3 + D_2 + U_2(t) \\ \dot{\mathscr{E}_3} &= CS_2 - AS_3 - DS_3 + D_3 + U_3(t) - CM_2 + AM_3 + DM_3 \\ &= C\mathscr{E}_2 - A\mathscr{E}_3 - D\mathscr{E}_3 + D_3 + U_3(t) \\ \dot{\mathscr{E}_4} &= DS_3 - AS_4 + D_4 + U_4(t) - DM_3 + AM_4 \\ &= D\mathscr{E}_3 - A\mathscr{E}_4 + D_4 + U_4(t) \end{aligned} \tag{4.4}$$

Now, from (4.4), the active controllers are as follows:

$$\begin{aligned} U_1(t) &= A\mathscr{E}_1 + BS_1S_3 - BM_1M_3 - D_1 - \mathscr{K}_1\mathscr{E}_1 \\ U_2(t) &= A\mathscr{E}_2 + C\mathscr{E}_2 - BS_1S_3 + BM_1M_3 - D_2 - \mathscr{K}_2\mathscr{E}_2 \\ U_3(t) &= -C\mathscr{E}_2 + A\mathscr{E}_3 + D\mathscr{E}_3 - D_3 - \mathscr{K}_3\mathscr{E}_3 \\ U_4(t) &= -D\mathscr{E}_3 + A\mathscr{E}_4 - D_4 - \mathscr{K}_4\mathscr{E}_4 \end{aligned} \tag{4.5}$$

Where, $\mathscr{K}_i^{\prime}s > 0,\ i=1,2,3,4$ are some positive gain constants.
Substituting equation (4.5) in (4.4), the error dynamics are:

$$\begin{aligned} \dot{\mathscr{E}_1} &= -\mathscr{K}_1\mathscr{E}_1 \\ \dot{\mathscr{E}_2} &= -\mathscr{K}_2\mathscr{E}_2 \\ \dot{\mathscr{E}_3} &= -\mathscr{K}_3\mathscr{E}_3 \\ \dot{\mathscr{E}_4} &= -\mathscr{K}_4\mathscr{E}_4 \end{aligned} \tag{4.6}$$

Next, we define the lyapunov functions as:

$$V(t) = \frac{1}{2}(\mathscr{E}_1^2 + \mathscr{E}_2^2 + \mathscr{E}_3^2 + \mathscr{E}_4^2) \tag{4.7}$$

From (4.7), we can clearly see that $V(t)$ is definitely positive.
Now, taking the derivative of (4.7), we get:

$$\dot{V(t)} = \mathscr{E}_1\dot{\mathscr{E}_1} + \mathscr{E}_2\dot{\mathscr{E}_2} + \mathscr{E}_3\dot{\mathscr{E}_3} + \mathscr{E}_4\dot{\mathscr{E}_4} \tag{4.8}$$

Substituting (4.6) in (4.8), we get:

$$\begin{aligned} \dot{V(t)} &= \mathscr{E}_1(-\mathscr{K}_1\mathscr{E}_1) + \mathscr{E}_2(-\mathscr{K}_2\mathscr{E}_2) + \mathscr{E}_3(-\mathscr{K}_3\mathscr{E}_3) + \mathscr{E}_4(-\mathscr{K}_4\mathscr{E}_4) \\ &= -\mathscr{K}_1\mathscr{E}_1^2 - \mathscr{K}_2\mathscr{E}_2^2 - \mathscr{K}_3\mathscr{E}_3^2 - \mathscr{K}_4\mathscr{E}_4^2 \\ &=< 0 \end{aligned} \tag{4.9}$$

Therefore, by lyapunov stability theory, we can see that the chaotic tuberculosis systems (3.1) and (4.1) are asymptotically hybrid projective synchronized globally by the designed active controller.

5. Numerical Simulations

This section provides critical validation for the proposed active control strategy, demonstrating its effectiveness in synchronizing the chaotic tuberculosis system despite the presence of constant bounded disturbances. For this purpose, we conducted some simulation in MATLAB.
The parameter values are chosen as:
$A = 0.0012, B = 0.2568, C = 0.148, D = 0.104$
The constant disturbances are taken as:
$D_1 = 0.05, D_2 = -0.03, D_3 = 0.04, D_4 = -0.02$
The initial conditions that we have taken for master (3.1) and slave (4.1) are:
$M(0) = (3, 1, 2, -5)$ and $S(0) = (5, 3, -6, 6)$
The positive control gains are: $\mathscr{K}_i = 50 \forall i = 1, 2, 3, 4$
From Figure 4 we can see that the graph shows a period of rapid fluctuations in the synchronization error, particularly within the first half-second. The errors start with a large magnitude, reaching upto 10 on the y-axis. After the initial transient phase, the graph shows a dramatic and rapid convergence of all four synchronization error signals toward zero. The convergence occurs within approximately 0.5 seconds. This rapid and sustained convergence is a clear indicator that the active control strategy is highly effective. The control mechanism successfully overcomes the system's intrinsic chaotic tendencies and the constant disturbances to force the systems into a synchronized state.

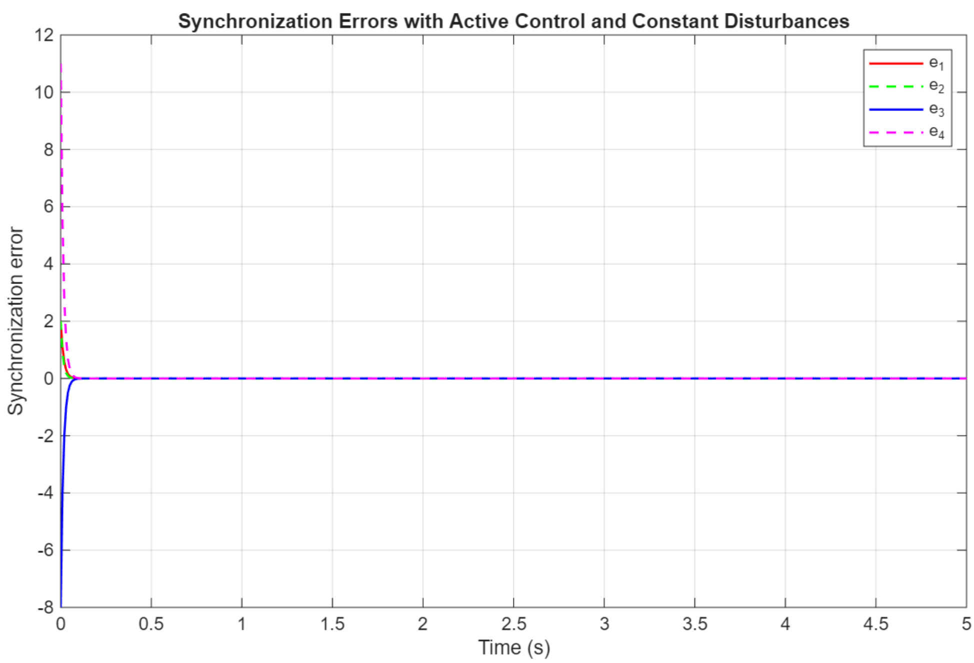


Figure 4: Error synchronization graph of master (3.1) and slave (4.1) tuberculosis system using the active control method in the presence of constant disturbances

6. Conclusion

This paper has focused on the analysis, control and synchronization of an integer-order non-linear tuberculosis (TB) model exhibiting chaotic behaviour. The Tuberculosis model which is structured as an SEIR-type dynamical system has four interacting compartments which captures the complexity of the model. Then, this paper showed that the Tuberculosis system with constant disturbances exhibits chaos under certain parameter values through bifurcation diagrams, phase portraits and positive lyapunov exponents.

After this, to suppress the inherent chaos and achieve synchronization between the master and slave Tuberculosis systems, this paper used active control method. The master system represented the uncontrolled chaotic Tuberculosis model while the slave system included constant bounded disturbances to mimic external influences such as environmental fluctuations, demographic variability or inaccuracies in parameter estimation as well as control inputs. This ensured that the error synchronization asymptotically converged to zero guaranteeing complete synchronization between the master and slave systems by lyapunov stability theory.This also helped in verifying the robustness of the active control method against constant bounded disturbances.

Then, numerical simulations were conducted using MATLAB which helped in validating the theoretical findings and provided visual confirmation of the system's dynamic transitions since it showed that the synchronization errors between corresponding state variables rapidly decreased to zero and the trajectories of the master and slave systems became identical indicating smooth and steady behaviour after synchronization and hence chaos was successfully suppressed.

This confirms that the proposed control scheme is not only theoretically sound but also practically viable for real-world systems that are inevitably affected by noise, external perturbations or parameter variations. Therefore, the results open new directions for extending the methodology to fractional-order systems, hybrid synchronization schemes and adaptive control scenarios where system parameters may vary with time.

Future Work

1. We can extend this integer-order Tuberculosis system to fractional-order Tuberculosis system since it will also include memory and hereditary effects.
2. Instead of constant disturbances, we can use time-varying disturbances and use Fractional order disturbance observer with parameter uncertainties.
3. We can also use unbounded disturbances, since in real-life, we don't know which type of disturbances are exactly affecting the disease and what is their bound.
4. We can also try to incorporate real-time data of Tuberculosis and make effective preventive strategies using this or any other control framework to see the effectiveness of the proposed method.

Conflicts of Interest

All the authors of this paper declare that there are no conflict of interest regarding the publication of this paper.

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Ayub Khan,
Basic Sciences Research Centre (BSRC),
Imam Mohammad Ibn Saud Islamic University,
Kingdom of Saudi Arabia.
E-mail address: ayubkdu@gmail.com

and

Roopesh Tehri,
Department of Mathematics,
Acharya Narendra Dev College,
University of Delhi, Delhi, India - 110019.
E-mail address: roopeshtehri@andc.du.ac.in

and

Neeti Goel,
Department of Mathematics,
Sri Guru Nanak Dev Khalsa College,

University of Delhi, Delhi, India - 110005.
E-mail address: neeti.goel@sgndkc.du.ac.in

and

Diya Baidya,
Department of Mathematics,
Sri Guru Nanak Dev Khalsa College,
University of Delhi, Delhi, India - 110005.
E-mail address: mat22.1302.diya@sgndkc.du.ac.in