Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 2** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.80359

Solutions to the Matrix Yang-Baxter Equation

Askar Ali M[1] *, and Himadri Mukherjee[2]

ABSTRACT: This article explores various classes of solutions to the Yang-Baxter type matrix equation, $AXA = XAX$, over the field $K = \mathbb{C}$ or $\mathbb{R}$, by means of the spectral properties of the solution space. We first prove several global constraints on any solution X, including a spectral inclusion $\sigma(X) \subseteq \sigma(A) \cup \{0\}$ and A-invariance of $\ker X$. Specializing to the key case where A is a single Jordan block, we completely characterised solutions for the case where A is equivalent to a Jordan block. We extend our findings to encompass the general scenario of multiple Jordan blocks under specific conditions. Additionally, novel tools from commutative algebra, such as the Gröbner basis, were also applied to arrive at solutions. We construct pencils of solutions that generate new families from known ones under clear algebraic conditions. Together, these results supply a toolbox for analyzing $AXA = XAX$ beyond diagonalizable A, clarifying how spectral data and Jordan structure govern the solution set.

Keywords: Yang-Baxter equation, matrix equation, Jordan canonical form, Jordan block, matrix pencils.

Contents

1. Introduction

The Yang-Baxter equation, in its various forms, has a long and fascinating history spanning various subjects and fields of science. It was first introduced, in a general form, in the physics of statistical mechanics among the works of Baxter [3]. Baxter introduced the equations in the context of the so-called eight-vertex model. It was named the "star triangle" equation, acknowledging that the equation was also known in the electrical engineering community since 1899 as the "star-triangle" transformation [22]. In the context of electrical engineering, under certain conditions, one can show that it is equivalent to arranging three registers or three coils in the shape of "Y" or the shape of a Δ.

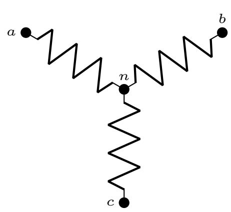

Figure 1: Star

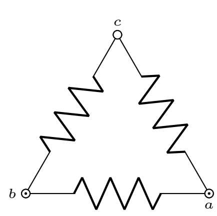

Figure 2: Triangle

* Corresponding author.

2020 *Mathematics Subject Classification:* Primary: 15A24, 16T25; Secondary: 15A22.
Submitted November 25, 2025. Published March 14, 2026

Later, in the context of quantum mechanics, Prof. Yang reintroduced the equation in the context of the spin model of quantum mechanics [Chapter 2, [21]]. The physical approach to this equation is to consider a vector space (finite or infinite) V, together with an operator R on $V \otimes V$, and a set of operators R_{12}, R_{13}, R_{23} on the vector space $V \otimes V \otimes V$, where R_{ij} is the operator defined as $R \otimes I_V$, where R acts on the i and jth entries and the remaining component is acted on by the identity operator of V. The Yang-Baxter equation for this R matrix is then written as,

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

The Yang-Baxter equation also arises in several mathematical areas, one of the earliest ones being in the context of the braid groups. These relations are realized as a relation of order of composition between different braid generators [1]. In the context of quantization of Lie groups, Drinfeld [14,19] introduced the study of solving the Yang-Baxter equation as stated above in the category of sets where the R is a set-theoretic map from the set $X \times X$ to itself and R_{ij}s as maps on $X \times X \times X$ are defined as R acting on the components i and j and as identity on the remaining component. In the field of topological quantum computing, the YBE finds its way as the following equation. For R, a linear operator on $V \otimes V$,

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R).$$

In this area, a quantum gate is used to entangle the states satisfy the braiding condition, thus solving the above equation. In the area of quantum information theory, the matrix R is assumed to be a unitary matrix, thus giving a quantum gate [15,16,7]. This equation is notoriously difficult to solve owing to the large number of equations involved. There are complete unitary solutions only till the dimension of V is six. So naturally, recent efforts have been to generalize the equation to make it more approachable to several ways of solving it. Once we generalize the matrix equation, we obtain,

$$YXY = XYX,$$

where $X = R \otimes I$ and $Y = I \otimes R$ relates this equation back to the original quest.
In this paper, we have considered the following equation, for a given $A \in M_n(K)$, where $K = \mathbb{R}$ or $\mathbb{C}$.

$$AXA = XAX. \tag{1.1}$$

In the remaining portion of the paper, this equation will be called as the Yang-Baxter equation (YBE).
There have been several interesting and impactful articles published by many authors related to the YB matrix equation. A number of interesting works have come up since the question posed by Drinfeld in [14], namely the works in the YBE in set-theoretic set up [4] exploring the equation and solutions in the category of $Sets$. The complexity of having n^2 non-linear equation in n^2 variable to solve makes it challenging to find a set of complete solutions for an arbitrary coefficient matrix. This compelled many researchers to find solutions by restricting the coefficient matrix A to a class of tractable or nice sets of matrices, especially having algebraic properties like diagonalizability or commutativity with the solution matrix.
The solution sets when the coefficient matrix is diagonalizable have been studied in detail in [10,6,17,5,29]. The article [23], discusses all commuting solutions for A in Jordan canonical form, and corresponding to each Jordan block, the authors in loc. cit. described solutions in Toeplitz forms. Several other papers also found commuting solutions in various cases, namely in [11,26,25,9]. Contrasting with the commuting solutions the authors of [28] have studied non-commuting solutions when A satisfies $A^4 = A$, by exploring various possibilities for its minimal polynomials. Irrespective of its simple appearance, when the coefficient matrix is nilpotent poses additional challenges to solving it. In [13], the authors have found all commuting solutions for A being a nilpotent matrix. Later, one of the same authors, along with others, gave an equivalent system for finding commuting solutions to the YBE, when A is nilpotent and of index 3 [27]. Recently, authors in [2] generalized the YBE to a system of Yang-Baxter matrix equations as $AXA = XBX$ and $BXB = XAX$, and studied for various classes of solutions.
Non-linearity of the questions paves way for geometric (or commutative algebraic) techniques such as manifold theory or algebraic techniques like Gröbner basis to be applied to get further insight into the

space of solutions. A new horizon has been given for the discussion of the solution to the YBE in [12], by following the path-connected subsets of the solution. Also, the authors explain techniques to generate infinitely many solutions from existing solutions using generalized inverse. Recently, Saijie Chen et al. have found solutions to YBE for matrices with non-singular Jordan blocks, using generalized inverse in [8]. In [20], the author has demonstrated a manifold structure for the solution set of matrix YBE when the coefficient matrix has rank 1.
The assumptions on coefficient matrices naturally restrict the applicability of the techniques developed in these papers to a much larger class owing to exploiting the properties of the special class to solve the equation. As a result, we have taken a spectral theory approach to break the equation into sub-classes that have a given number of Jordan blocks in A. This way, we hope to arrive at a general technique to solve the equation completely. In this paper, we have completely solved the case of A having one Jordan blocks; we also have a large class of solutions when A has any number of Jordan blocks. Together with these, we have also obtained a number of very interesting general results that hold for any coefficient matrix A. An understanding of the solution in the linear algebra setting will definitely give important insight into the solution of YBE in general.
In our paper, the results are organized in a manner that there are global results, i.e., results without any assumption on the coefficient matrix, or the type of solutions and special results that come with assumptions on the coefficient matrix. Before every result, we have mentioned whether it is a global or a local result to facilitate the reader. There are a few minor overlaps in the results, especially with the newest articles, but each time we have presented a new way of looking at it or have extended the results. Which is why we have not removed those results from our paper, but have mentioned whenever such overlaps occurred.

2. Preliminaries

Let us recall the following notations, which are used throughout the discussion.

- Let K be a field. For a coefficient matrix $A \in M_n(K)$, the set of all solutions to the Yang-Baxter equation $AXA = XAX$ is denoted by $Sol_A := \{X \in M_n(K) | AXA = XAX\}$.
- For a matrix $M \in M_n(K)$, $\sigma(M)$ denotes the spectrum of M.
- For a matrix $M \in M_n(K)$, and $\lambda \in \sigma(M)$, $E_\lambda(M)$ (or simply E_λ) represent the eigenspace of M corresponding to λ, and $P_\lambda(M)$ (or P_λ) represents the generalized eigenspace of M corresponding to λ.
- Let V be a vector space, then we say $A \in M_n(K)$ preserves V, if $A(V) \subseteq V$.
- For a $M \in M_n(K)$, min_M represents the minimal polynomial of M.
- $K[A]$ represent the polynomial ring on $A \in M_n(K)$, over K.

Theorem 2.1 **([24])** *Given matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$, the Sylvester equation $AX + XB = C$ has a unique solution $X \in \mathbb{C}^{n\times m}$ for any $C \in \mathbb{C}^{n\times m}$, if and only if A and $-B$ do not share any eigenvalue.*

Lemma 2.2 *For any $g \in GL_n(K)$, $gSol_Ag^{-1} = Sol_{gAg^{-1}}$.*

Proof: Let $B = gAg^{-1}$, and $x \in gSol_Ag^{-1}$. Then $x = gXg^{-1}$, for some $X \in Sol_A$. Now for $y = BxB = gAg^{-1}xgAg^{-1}$, that is $g^{-1}yg = A(g^{-1}xg)A = AXA = XAX = g^{-1}xgAg^{-1}xg = g^{-1}xBxg$ that is, $y = xBx$. which implies $BxB = xBx$ □

As an immediate consequence of the above Lemma 2.2, without loss of generality, we can assume A to be in the Jordan-canonical form.

Corollary 2.2.1 *Let $X \in Sol_A$, then for any $g \in G_A = \{g \in GL_n(K)/gAg^{-1} = A\}$, gXg^{-1} is a solution to the YBE, $AXA = XAX$.*

Proof: Let $X \in Sol_A$. For a $g \in G_A$, we have $gSol_Ag^{-1} = Sol_{gAg^{-1}} = Sol_A$. $\square$

Proposition 2.3 *Let the coefficient matrix* $A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & \dots & A_m \end{pmatrix}$, *where A_i's are square matrices, then* $X = \begin{pmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \dots & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & \dots & X_m \end{pmatrix} \in Sol_A$, *where* $X_i \in Sol_{A_i}$.

Through the above Lemmas and Proposition, it make ample sense to study the solution set to (1.1), when the coefficient matrix is a Jordan block. Whenever we have solutions to Jordan blocks, Proposition 2.3 guarantees solutions to an arbitrary coefficient matrix A, having multiple Jordan blocks. This way, we are more closer to solving the open problem of finding solutions to YBE, for an arbitrary coefficient matrix A.

3. Coefficient Matrix Being Single Jordan Block

In this section, we characterize the solution X to (1.1) through the spectral properties, particularly when A is a Jordan block. Though, many results are global in nature.

Lemma 3.1 *Let $X \in Sol_A$ for the YBE with coefficient matrix A, and λ be an eigenvalue of A with eigenvector v, then either λ is an eigenvalue of X or $AXv = 0$.*

Proof: We have, $Av = \lambda v$, thus $XAv = \lambda Xv$ or $AXAv = \lambda AXv$. By utilizing the YBE, we get $XAXv = \lambda AXv$. Thus, if $AXv \neq 0$, then λ is an eigenvalue of X. Further, by the symmetry of the YBE, we can see, if λ is an eigenvalue of X with eigenvector v, then either λ is an eigenvalue of A or $XAv = 0$. $\square$

Theorem 3.2 *Let coefficient matrix A be invertible and, $X \in Sol_A$. Then the following are true.*

(a) $\sigma(X) \subseteq \sigma(A) \cup \{0\}$.

(b) A preserves the kernel of X.

Proof:

(a) From the above Lemma 3.1, we know that if λ is an eigenvalue of X, then either it is an eigenvalue of A or $XAv = 0$, where v is an arbitrary eigenvector of X, with respect to λ. Now, since A is invertible, we get $Av \neq 0$. Thus, $XAv = 0$, implies Av is an eigenvector of X with eigenvalue 0.

(b) Let $v \in Ker(X)$, we have

$$\begin{aligned} Xv &= 0 \\ \implies AXv &= 0 \\ \implies XAXv &= 0 \\ \implies AXAv &= 0 \\ \implies XAv &= 0 \\ \implies Av &\in Ker(X). \end{aligned}$$

This gives, $A(Ker(X)) \subseteq Ker(X)$. Since A is invertible, we get $dim(A(Ker(X))) = dim(Ker(X))$, gives, $A(Ker(X)) = Ker(X)$.

□

Theorem 3.3 *Let the coefficient matrix A be invertible, and if $\forall\, \lambda \in \sigma(A)$, $dim(E_\lambda) = 1$.*

(a) If $Ker(X) = E_\lambda$, then $\lambda \notin \sigma(X)$.

(b) If $X(E_\lambda) \neq 0$, then $\sigma(A) \subseteq \sigma(X)$.

(c) If $\lambda \notin \sigma(X)$, then $X(P_\lambda) = 0$.

Proof:

(a) If possible, let us assume $\lambda \in \sigma(X)$. Let w be an eigenvector for λ, thus we have $Xw = \lambda w$. Now since $\lambda \in \sigma(A)$, let us choose an eigenvector for it, say v, we have $Av = \lambda v$. Since we have $E_\lambda = Ker(X)$ we obtain $Xv = 0$. We also have $XAXw = \lambda XAw$, or $AXAw = \lambda XAw$, using the YBE. Which gives us $XAw \in E_\lambda$, now since the eigenspace is one dimensional (generated by v as a basis), we have $XAw = \mu v$ for some μ (could also be zero). This gives us $X^2AXw = 0$ or $X(XAX)w = 0$. Or $XAX(Aw) = 0$, which gives us $AXA^2w = 0$, thus $XA^2w = 0$ (as A is invertible). So we have $A^2w \in ker(X)$. Now since $ker(X) = E_\lambda$ we have $A^2w = lv$ for some $l \in K$. So we have:

$$\begin{aligned} A^2w &= (lv = l/\lambda)Av \\ Aw &= (l/\lambda)v \\ w &= (l/\lambda)A^{-1}v \\ w &= (l/\lambda^2)v \end{aligned} \tag{3.1}$$

Thus $w \in E_\lambda$ but then $Xw = 0 = \lambda w$ giving us $w = 0$ a contradiction.

(b) By the Lemma 3.1, if $X(E_\lambda) \neq 0$, then $\lambda \in \sigma(X)$. If this is true for all $\lambda \in \sigma(A)$, then we have $\sigma(A) \subseteq \sigma(X)$.

(c) Since $\lambda \notin \sigma(X)$, we know that X annihilates the eigenspace of λ so $E_\lambda \subset Ker(X)$. Let $\{v_1, \ldots, v_r\}$ be the canonical basis for the generalized eigen space P_λ, such that $Av_1 = \lambda v_1$, and $Av_i = v_{i-1} + \lambda v_i$, for $i = 2, \ldots, r$. Then we have, $Xv_1 = 0$. Now, $XAv_2 = Xv_1 + \lambda Xv_2$, or $AXAv_2 = \lambda AXv_2$. Then by YBE, $XAXv_2 = \lambda AXv_2$. This gives, if $AXv_2 \neq 0$, then $\lambda \in \sigma(X)$, which is not true. Hence $Xv_2 = 0$. Continuing this process, by induction we get $Xv_i = 0$, for $i = 1, 2, \ldots, r$. Which implies $XP_\lambda = 0$.

□

Corollary 3.3.1 *Let A be an invertible coefficient matrix for the YBE, and for a $\lambda \in \sigma(A)$, the Jordan block corresponding to λ, J_λ has size strictly greater than 1 and has geometric multiplicity 1, then for any $X \in Sol_A$, such that $\lambda \notin \sigma(X)$, $Ker(X) \neq E_\lambda$.*

Proof: If, for a $\lambda \in \sigma(A)$, $\lambda \notin \sigma(X)$, with $dim(E_\lambda) = 1$, then, by Theorem 3.3, we have X annihilate any cyclic subspace of generalized eigenspace of A, corresponding to λ. Let P_λ be the maximum cyclic subspace of such generalized eigenspace. Then, $P_\lambda \subseteq Ker(X)$. Since, $dim(P_\lambda)$ is same as Jordan block corresponding to λ, which is strictly greater than one, and E_λ has dimension one, we have $Ker(X) \neq E_\lambda$.
□

Lemma 3.4 *Let A be an invertible coefficient matrix, and $X \in Sol_A$ is also invertible, then $A \simeq X$. In particular, they have the same Jordan form.*

Proof: The result follows from the fact,

$$AXA = XAX \iff X^{-1}AX = AXA^{-1}$$

□

Lemma 3.5 *If the coefficient matrix A is a Jordan block, with eigenvalue λ, such that $\lambda \neq 0$, and if a solution of the YB equation X is not the zero matrix, then $\sigma(X) = \{\lambda\}$.*

Proof: We have $\sigma(X) \subseteq \sigma(A) \cup \{0\}$, and $\sigma(A) = \{\lambda\}$. Also, we have $A(Ker(X)) \subset Ker(X)$, by Theorem 3.2. So, $Ker(X)$ is an invariant subspace of A. i.e., $Ker(X) = E_\lambda$ or K^n, as λ is the only eigenvalue of A, $P_\lambda = K^n$. Now, if $Ker(X) = K^n$, then $X = 0$. If $Ker(X) = E_\lambda$, then by Theorem 3.3, we have $\lambda \notin \sigma(X)$; which gives $\sigma(X) \subset \{0\}$. i.e., $X = 0$. As a result, if $X \neq 0$, then $\sigma(X) = \{\lambda\}$. □

Lemma 3.6 *$AXA = XAX$ if and only if $AXA^n = X^nAX$ for all $n \in \mathbb{N}$. Similarly, $AXA = XAX$ if and only if $A^nXA = XAX^n$, $\forall\ n \in \mathbb{N}$.*

Proof:

$$AXA = XAX$$
$$\implies AXA^2 = XAXA = X^2AX.$$

Then by induction, we have, $\forall\ n \in \mathbb{N}$, $AXA^n = X^nAX$. Conversely, if $AXA^n = X^nAX$, is true for any $n \in \mathbb{N}$, then in particular, it is true for $n = 1$. i.e., $AXA = XAX$. Similar way, we can prove, $AXA = XAX \iff A^nXA = XAX^n$, $\forall\ n \in \mathbb{N}$ □

Lemma 3.7 *If $XAX = AXA$, and X and A commute, then $(I - e^{t(A-X)})XA = 0$, for any $t \in \mathbb{R}$.*

Proof: If $XAX = AXA$, then as a consequence of above lemma, we have $XAe^{tX} = e^{tA}XA$ for any $t \in \mathbb{R}$.
Which gives

$$\begin{aligned} XAe^{tX}X &= e^{tA}XAX \\ XAXe^{tX} &= e^{tA}XAX \\ AXAe^{tX} &= e^{tA}AXA \\ &= Ae^{tA}XA \end{aligned}$$

This gives $XA = e^{t(A-X)}XA$, implies $(I - e^{t(A-X)})XA = 0$. □

Lemma 3.8 *Let $X \in Sol_A$, be a commuting solution, such that $A - X$ is invertible, then $AX = 0$.*

Proof: If X is a commuting solution in Sol_A, then we have, $A^2X = AX^2$. By taking $Y = AX$, we have, $AY - YX = 0$. By Sylvester's Theorem 2.1, $Y = 0$ if and only if, $\sigma(A) \cap \sigma(X) = \emptyset$, i.e., $A - X$ is invertible. □

Theorem 3.9 *Let $\phi_A(x)$ be the characteristic polynomial of A (respectively, $\phi_X(x)$ be the characteristic polynomial of X). Then the following is true,*

(a) $XA\phi_A(X) = 0$

(b) $\phi_A(X)AX = 0$

Proof: Let

$$\phi_A(x) = a_0 + a_1x + \cdots + a_nx^n.$$

Then we have,

$$\begin{aligned} XA\phi_A(X) &= a_0XA + a_1XAX + a_2XAX^2 + \cdots + a_nXAX^n \\ &= a_0XA + a_1AXA + a_2A^2XA + \cdots + a_nA^nXA \\ &= (a_0I + a_1A + a_2A^2 + \cdots + a_nA^n)XA \\ &= \phi_A(A)XA = 0 \end{aligned}$$

Similarly, we have $\phi_A(X)AX = 0$. □

Corollary 3.9.1 *If $A^n = 0$, then $XAX^m = X^mAX = 0$, for every $m \geq n$.*

Lemma 3.10 *Let A, X be two square matrices of the same dimension. $\phi_A(X)$ is invertible if and only if $\sigma(A) \cap \sigma(X) = \emptyset$.*

Proof: Let $\phi_A(X) = (X - \lambda_1 I) \ldots (X - \lambda_n I)$. Then $\phi_A(X)$ is invertible if and only if for any i, $\lambda_i \notin \sigma(X)$. □

Theorem 3.11 *Let $AXA = XAX$, if $\sigma(A) \cap \sigma(X) = \emptyset$, then either*

(a) $A = 0$ and X is invertible
or

(b) A is invertible and $X = 0$.

Proof: As $\sigma(A) \cap \sigma(X) = \emptyset$, only atmost one of $\sigma(A), \sigma(X)$ can contain 0. If none of them contains 0, then both A, X are invertible and hence $X \simeq A$, which implies $\sigma(A) = \sigma(X)$, which is not possible. Therefore, one must contain 0. If $0 \in \sigma(A)$, then $0 \notin \sigma(X)$ and X is invertible. Also, since $\sigma(A) \cap \sigma(X) = \emptyset$, we have min_A, min_X are co-prime. As a result, $\phi_A(X)$ is invertible. Then

$$XA\phi_A(X) = \phi_A(X)AX = 0 \Longrightarrow XA = 0.$$

Since X is invertible, A must be 0. Similarly, if $0 \in \sigma(X)$, then $X = 0$ and A is invertible. □

Theorem 3.12 *If A is a Jordan block, with eigenvalue λ, then for the YBE, $AXA = XAX$, the following are true.*

(a) If $\lambda \neq 0$ and $det(X) \neq 0$ then $X \simeq A$.

(b) If $\lambda \neq 0$ and $det(X) = 0$ then $X = 0$.

(c) If $\lambda = 0$, then the YBE does not have any invertible solution X.

Proof: The first case follows from Lemma 3.4. Now, by Lemma 3.5, if A is invertible and $X \neq 0$, the $\sigma(X) = \{\lambda\}$, i.e X is invertible. Thus, if X is singular, X must be 0 matrix. Let $\lambda = 0$, and assume if possible, X is invertible. Then, $\sigma(A) \cap \sigma(X) = \emptyset$. Thus, by Theorem 3.11, A must be 0, which is not true. Hence, X must be singular in this case. □

3.1. Some Special Solutions

Example 3.12.1 *When A is a 2×2 Jordan block, with eigenvalue λ, for some $\lambda \neq 0$, non-trivial solution X has one of the following form:*
$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$, $\begin{pmatrix}\lambda+\lambda\sqrt{a} & a\\ -\lambda^2 & \lambda-\lambda\sqrt{a}\end{pmatrix}$ *or* $\begin{pmatrix}\lambda-\lambda\sqrt{a} & a\\ -\lambda^2 & \lambda+\lambda\sqrt{a}\end{pmatrix}$ *where $a \in K$*

Proof: Let $X = \begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}$, then for $A = \lambda I_2 + B$

$$\begin{aligned}
AXA &= \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\\
&= \begin{pmatrix}\lambda^2 x_1 + \lambda x_3 & \lambda x_1 + x_3 + \lambda^2 x_2 + \lambda x_4\\ \lambda^2 x_3 & \lambda x_3 + \lambda^2 x_4\end{pmatrix}\\
XAX &= \begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\\
&= \begin{pmatrix}\lambda x_1^2 + x_1x_3 + \lambda x_3x_2 & \lambda x_1x_2 + x_1x_4 + \lambda x_4x_2\\ \lambda x_1x_3 + x_3^2 + \lambda x_3x_4 & \lambda x_2x_3 + x_4x_3 + \lambda x_4^2\end{pmatrix}
\end{aligned} \tag{3.2}$$

Upon equating and solving, we have, $x_3 = 0$ or $-\lambda^2$. Now if $x_3 = 0$, as a consequence from above equations, we get $x_1 = x_4 = 0$ or λ. But, $x_1, x_4 = 0$ is not possible, since by Theorem 3.12, when $\lambda \neq 0$, non-trivial solution X is similar to A, hence trace of A and X must be equal, which is 2λ. As a result, we have $x_1, x_4 = \lambda$, also which leads to $x_2 = 1$.
Now, if $x_3 = -\lambda^2$, then we have $x_1, x_4 = \lambda \pm \lambda\sqrt{x_2}$. But again, since A and X are similar, their trace must be equal. As a result, x_1 should be conjugate of x_4, and x_2 has a free choice.
Note that, when the dimension of the coefficient matrix increases, there will be more free variable as a in this case, which hardly follows any pattern w.r.t the dimension. □

Example 3.12.2 *Let A be the Jordan block of size 2, with eigenvalue $\lambda = 0$, we have the following.*

$$\begin{aligned}
AXA &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & x_3\\ 0 & 0\end{pmatrix}\\
XAX &= \begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix} = \begin{pmatrix}x_1x_3 & x_1x_4\\ x_3^2 & x_4x_3\end{pmatrix}
\end{aligned} \tag{3.3}$$

This gives $x_3 = 0, x_1x_4 = 0$ and x_2 has free choice over K.
Hence, $Sol_A = \left\{\begin{pmatrix}a & \alpha\\ 0 & b\end{pmatrix} \mid a, b, \alpha \in K,\ ab = 0\right\}$.

3.2. Some General Solutions for the Nilpotent Block

Example 3.12.3 *Let A be the 3×3 Jordan block with eigenvalue 0, and the variable matrix be $X = \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$. Upon equating $AXA = XAX$, we have the following polynomials, whose zeros give Sol_A.*

$$\begin{array}{lll}
f_1 = ad + bg & f_2 = ae + bh - d & f_3 = af + bi - e\\
f_4 = d^2 + eg & f_5 = de + eh - g & f_6 = df + ei - h\\
f_7 = gd + hg & f_8 = ge + h^2 & f_9 = gf + hi
\end{array} \tag{3.4}$$

Let us find a Gröbner basis for the ideal $\langle f_1, f_2, \ldots, f_9\rangle$ in the ring $\mathbb{C}[a,b,c,d,e,f,g,h,i]$, with respect to the lexicographic order for $a > b > \cdots > i$. On simplifying from the Gröbner basis, we get $g, h, d, e = 0$ and $af + bi = 0$, and other variables have free choice. Hence,

$$Sol_A = \left\{\begin{pmatrix}a & b & c\\ 0 & 0 & f\\ 0 & 0 & i\end{pmatrix} \mid af + bi = 0\right\}.$$

Example 3.12.4 *For the coefficient matrix A being a Jordan block of size n with eigenvalue 0, one can generalize the solutions in Example 3.12.3, to the following solutions class. But note that, for $n > 3$, this does not give the whole set of solutions.*

$$X = \begin{pmatrix} 0 & a_1 & a_2 & .... & a_{n-2} & \alpha \\ 0 & 0 & 0 & \dots & \sum_{i=1}^{n-3} a_i b_{i+1} & b_1 \\ 0 & 0 & 0 & \dots & 0 & b_2 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 0 & b_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

We can directly verify the above matrix belongs Sol_A.

Theorem 3.13 *[Theorem 3.2.4.2, [18]] Let $A \in M_n(K)$ be in its Jordan-canonical form, then for a matrix, $M \in M_n(K)$, $AM = MA$ implies $M \in K[A]$.*

Theorem 3.14 *Let A be the Jordan block of size $n+1$, $n \geq 3$, with eigenvalue 0. Any commuting solution to the YBE, with the coefficient matrix A, have the form,*

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & \alpha & \beta \\ 0 & 0 & 1 & 0 & \dots & 0 & \alpha \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} 0 & 0 & \dots & \alpha & \beta \\ 0 & 0 & \dots & 0 & \alpha \\ 0 & 0 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \textit{where } \alpha, \beta \in K$$

Proof: If X is a commuting solution to the YB equation, then by Theorem 3.13, X must be a polynomial in A. Let, $X = a_0 I + a_1 A + ... + a_{n-1}A^{n-1} + a_n A^n$ (note $A^{n+1} = 0$). Then,

$$\begin{aligned} AXA &= A(a_0 I + a_1 A + \cdots + a_{n-1}A^{n-1} + a_n A^n)A \\ &= a_0 A^2 + a_1 A^3 + \cdots + a_{n-2}A^n \\ &= \sum_{i=2}^{n} a_{i-2}A^i \\ XAX &= (a_0 I + a_1 A + \cdots + a_n A^n)A(a_0 I + a_1 A + \cdots + a_n A^n) \\ &= (a_{n-1}a_0 + a_{n-2}a_1 + \cdots + a_0 a_{n-1})A^n + \cdots + (a_1 a_0 + a_0 a_1)A^2 + a_0^2 A \\ &= \sum_{i=1}^{n} \sum_{j+k=i-1} a_j a_k A^i \end{aligned} \tag{3.5}$$

Upon equating, we get

$$\begin{aligned} & a_0 = 0, \ 2a_0 a_1 = a_0 \\ & a_1 = 2a_2 a_0 + a_1^2, (\implies a_1 = 0 \text{ or } 1) \\ & a_2 = a_3 a_0 + a_2 a_1 + a_1 a_2 + a_0 a_3, (\implies a_2 = 0) \\ & a_3 = a_4 a_0 + a_3 a_1 + a_2^2 + a_1 a_3 + a_0 a_4, (\implies a_3 = 0) \end{aligned} \tag{3.6}$$

Continuing this way, we obtain, in general,

$$a_{n-2} = a_{n-1}a_0 + a_{n-2}a_1 + ... + a_1 a_{n-2} + a_0 a_{n-1}, (\implies a_{n-2} = 0)$$

and a_{n-1}, a_n has free choice over K. Hence X has a general form as stated above when $a_1 = 1$, and $a_1 = 0$ respectively. □

4. Pencils of Solutions

Let $A, B \in M_n(K)$. The degree one matrix pencil is defined as $A + \lambda B$, where λ is indeterminate. In this section, we describe a few pencils of solutions, which essentially generate more solution classes from the known solutions.

Lemma 4.1 *Let $X \in Sol_A$, then for a matrix M with $MA = 0 = AM$, for any $\alpha \in K$, $X + \alpha M \in Sol_A$.*

Proof: We have $A(X + \alpha M)A = AXA + \alpha AMA = AXA$

$$(X + \alpha M)A(X + \alpha M) = XAX + \alpha XAM + \alpha MAX + \alpha^2 MAM = XAX$$

Note that in this case, M itself forms a solution. □

Lemma 4.2 *Let $X \in Sol_A$, for any $\alpha, \beta \in K$, $\alpha X + \beta M \in Sol_A$, if $MA = 0 = AM$ and $AXA = 0$.*

Proof: If $X \in Sol_A$, then for $\alpha \neq 0, 1$, αX is a solution if $AXA = 0$ [12]. Then the rest follows directly from the above Lemma 4.1. □

When A is non-singular, $MA = 0 = AM$ demands M to be 0. This discussion is interesting when A is singular. Let A be a singular matrix in its Jordan form. Say $A = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$ where $J_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $J_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \neq 0 \in K$. Then we can see that M has the form $\begin{pmatrix} 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, where $\alpha \in K$. When J_2 is Jordan block of 0, that is $J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, M have the following form. $M = \begin{pmatrix} 0 & 0 & a_1 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & a_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, where $a_i \in K$. We can observe that when there is a Jordan block of non-zero eigenvalue, the corresponding rows and columns in M are annihilated. Also, corresponding blocks in M are solutions to respective YBEs $J_i X J_i = X J_i X$.

Theorem 4.3 *Let $X_0, X_1 \in Sol_A$, for any $\lambda(\neq 0) \in K$, $X_0 + \lambda X_1 \in Sol_A$ if and only if $AX_1A = 0 = X_1AX_1$ and $X_0AX_1 + X_1AX_0 = 0$.*

Proof: ($\Rightarrow$) We have $X_0AX_0 = AX_0A, X_1AX_1 = AX_1A$. Which implies $A(X_0 + \lambda X_1)A = AX_0A + \lambda AX_1A = AX_0A$, as $AX_1A = 0$. Also, we have $(X_0 + \lambda X_1)A(X_0 + \lambda X_1) = X_0AX_0$.
($\Leftarrow$) For any $\lambda(\neq 0) \in K$, if $X_0 + \lambda X_1$ is a solution, then,

$$\begin{aligned} A(X_0 + \lambda X_1)A &= AX_0A + \lambda AX_1A \\ &= (X_0 + \lambda X_1)A(X_0 + \lambda X_1) \\ &= X_0AX_0 + \lambda X_0AX_1 + \lambda X_1AX_0 + \lambda^2 X_1AX_1. \end{aligned} \tag{4.1}$$

Then $\lambda AX_1A = \lambda X_0AX_1 + \lambda X_1AX_0 + \lambda^2 X_1AX_1$. Since it's true for every λ, it's true for $\lambda = 1$ also, which gives $X_0AX_1 + X_1AX_0 = 0$. This implies $\lambda AX_1A = 0 = \lambda^2 X_1AX_1$ which is true for any λ, gives $AX_1A = 0 = X_1AX_1$. □

Example 4.3.1 *When A is the 3×3 Jordan block, with eigenvalue zero, in Example 3.12.3 of special solutions, we have seen that X must have the form* $\begin{pmatrix} a & b & c \\ 0 & 0 & f \\ 0 & 0 & i \end{pmatrix}$, *such that $af+bi=0$. Then, by taking* $X_j = \begin{pmatrix} a_j & b_j & c_j \\ 0 & 0 & f_j \\ 0 & 0 & i_j \end{pmatrix} \in Sol_A$, *for $j=0,1$, we can easily verify the condition for $X_0+\lambda X_1 \in Sol_A$. Note that $AX_jA=0$. Now,*

$X_1AX_0 + X_0AX_1 = \begin{pmatrix} 0 & 0 & a_1f_0+a_0f_1+b_1i_0+b_0i_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, *if $\lambda(a_1f_0+a_0f_1+b_1i_0+b_0i_1)=0$. This agrees with the general form of solution when*

$$\begin{pmatrix} a_0+\lambda a_1 & b_0+\lambda b_1 & c_0+\lambda c_1 \\ 0 & 0 & f_0+\lambda f_1 \\ 0 & 0 & i_0+\lambda i_1 \end{pmatrix} \in Sol_A.$$

5. Conclusion

We analyzed the matrix equation $AXA = XAX$ by combining spectral theory with elementary algebraic geometry. Globally, we established structural restrictions on solutions X (spectral inclusion and A-invariance of $\ker X$) and showed that simultaneous invertibility forces A and X to be similar. Focusing on the central case of a single Jordan block, we obtained a sharp dichotomy for $\lambda \neq 0$: either $X=0$ or X is similar to A. In the nilpotent regime ($\lambda = 0$), we proved there are no invertible solutions and described rich commuting families (with explicit canonical forms) together with concrete low-dimensional classifications, supported by Gröbner-basis computations in 3×3. We also demonstrated how block-diagonal coefficient matrices inherit solutions from their Jordan constituents and introduced solution pencils that systematically produce new families when certain mixed terms vanish. These contributions move the program toward a general classification beyond diagonalizable A and indicate several directions for further work: (i) extending the single-block classification to complete multi-block interaction rules (including coupling constraints across blocks), (ii) developing Gröbner-based algorithms that scale to higher indices and sizes to map the geometry (dimension, components) of solution varieties, and (iii) exploring related quadratic equations such as $\Phi_A(X)X = 0$ and $\Phi_A(X)X = \Phi_X(A)A$ within the same framework. We expect the spectral/Jordan approach and pencil constructions to be equally useful for neighboring matrix identities arising in algebra, integrable systems, and quantum information.

References

1. E. Artin, *Theory of braids*, Ann. of Math. (2) **48**, 101–126, (1947).

2. A. M. Askar, H. Mukherjee, and B. D. Djordjević, *Solutions to a system of Yang–Baxter matrix equations*, Filomat **38** (29), 10169–10192, (2024).

3. R. J. Baxter, *Solvable eight-vertex model on an arbitrary planar lattice*, Philos. Trans. Roy. Soc. London Ser. A **289** (1359), 315–346, (1978).

4. M. Castelli, G. Pinto, and W. Rump, *On the indecomposable involutive set-theoretic solutions of the Yang–Baxter equation of prime-power size*, Comm. Algebra **48** (5), 1941–1955, (2020).

5. D. Chen, Z. Chen, and X. Yong, *Explicit solutions of the Yang–Baxter-like matrix equation for a diagonalizable matrix with spectrum contained in* $1, \alpha, 0$, Appl. Math. Comput. **348**, 523–530, (2019).

6. D. Chen and X. Yong, *Finding solutions to the Yang–Baxter-like matrix equation for diagonalizable coefficient matrix*, Symmetry **14** (8), (2022).

7. R. S. Chen, *Generalized Yang–Baxter equations and braiding quantum gates*, J. Knot Theory Ramifications **21** (9), 1250087, 18, (2012).

8. S. Chen, X. Li, L. Zhu, and Q. Huang, *Solutions of the Yang–Baxter-like matrix equation for the matrix with non-singular Jordan blocks*, J. Appl. Anal. Comput. **13** (2), 986–999, (2023).

9. J. Ding and C. Zhang, *On the structure of the spectral solutions of the Yang–Baxter matrix equation*, Appl. Math. Lett. **35**, 86–89, (2014).

10. J. Ding and N. H. Rhee, *Computing solutions of the Yang–Baxter-like matrix equation for diagonalisable matrices*, East Asian J. Appl. Math. **5** (1), 75–84, (2015).

11. J. Ding, C. Zhang, and N. Rhee, *Commuting solutions of the Yang–Baxter matrix equation*, Appl. Math. Lett. **44**, 1–4, (2015).

12. N. v. Dinvcić and B. D. Djordjević, *On the intrinsic structure of the solution set to the Yang–Baxter-like matrix equation*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **116** (2), Paper No. 73, 24, (2022).

13. Q. Dong, J. Ding, and Q. Huang, *Commuting solutions of a quadratic matrix equation for nilpotent matrices*, Algebra Colloq. **25** (1) , 31–44, (2018).

14. V. G. Drinfeld, *On some unsolved problems in quantum group theory*, in *Quantum groups* (Leningrad, 1990), Springer, Berlin, 1–8, (1992).

15. H. A. Dye, *Unitary solutions to the Yang–Baxter equation in dimension four*, Quantum Inf. Process. **2** (1–2), 117–151, (2002).

16. M. L. Ge, K. Xue, R. Y. Zhang, and Q. Zhao, *Yang–Baxter equations and quantum entanglements*, Quantum Inf. Process. **15** (12), 5211–5242, (2016).

17. Q. Huang, M. Saeed Ibrahim Adam, J. Ding, and L. Zhu, *All non-commuting solutions of the Yang–Baxter matrix equation for a class of diagonalizable matrices*, Oper. Matrices **13** (1), 187–195, (2019).

18. R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, (2013).

19. A. P. Isaev, *Lectures on quantum groups and Yang–Baxter equations*, (2022).

20. L. Lu, *Manifold expressions of all solutions of the Yang–Baxter-like matrix equation for rank-one matrices*, Appl. Math. Lett. **132**, 108175, (2022).

21. Z. Q. Ma, *Yang–Baxter equation and quantum enveloping algebras*, World Scientific Publishing Co., Inc., River Edge, NJ; Institute of Theoretical Physics, Beijing, (1993).

22. J. H. H. Perk and H. Au-Yang, *Yang–Baxter equations*, (2006).

23. D. Shen, M. Wei, and Z. Jia, *On commuting solutions of the Yang–Baxter-like matrix equation*, J. Math. Anal. Appl. **462** (1), 665–696, (2018).

24. J. Sylvester, *Sur l'équations en matrices $px = xq$*, C. R. Acad. Sci. Paris **99**, no. 2, 67–71, (1884).

25. H. Tian, *All solutions of the Yang–Baxter-like matrix equation for rank-one matrices*, Appl. Math. Lett. **51**, 55–59, (2016).

26. H. H. Yin, X. Wang, X. B. Tang, and L. Chen, *On the commuting solutions to the Yang–Baxter-like matrix equation for identity matrix minus special rank-two matrices*, Filomat **32** (13), 4591–4609, (2018).

27. D. Zhou and J. Ding, *Solving the Yang–Baxter-like matrix equation for nilpotent matrices of index three*, Int. J. Comput. Math. **95** (2), 303–315, (2018).

28. D. M. Zhou and H. Q. Vu, *Some non-commuting solutions of the Yang–Baxter-like matrix equation*, Open Math. **18** (1), 948–969, (2020).

29. D. M. Zhou, X. X. Ye, Q. W. Wang, J. W. Ding, and W. Y. Hu, *Explicit solutions of the Yang–Baxter-like matrix equation for a singular diagonalizable matrix with three distinct eigenvalues*, Filomat **35** (12), 3971–3982, (2021).

[1] *Department of Mathematics, IISER Thiruvananthapuram, India*

[2] *Department of Mathematics, BITS Pilani K. K. Birla Goa Campus, India*

E-mail address: `askaralim@iisertvm.ac.in` (Askar†), `himadrim@goa.bits-pilani.ac.in` (Himadri)